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A RESULT ON THE INCOMPARABILITY OF LINKED PRIME IDEALS

Karl A. Kosler

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Abstract. It is shown that linked prime ideals in certain fully semiprimary Noetherian ring are incomparable.

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1. Introduction

If P and Q are prime ideals of a two-sided Noetherian ring R, then there is a *link* from Q to P, denoted $Q \rightsquigarrow P$, provided there exists an ideal A such that $QP \subseteq A \subset Q \cap P$ and $(Q \cap P)/A$ is torsion-free as a left R/Q-module and as a right R/P-module. A long-standing conjecture in noncommutative ring theory is that no link can exist when P and Q are comparable. From [3, Theorem 1.8] this is true if R satisfies the second layer condition. More recently, Vyas in [8] showed that this is true if R is a Noetherian ring with global dimension 1. In this note, we use a result of Vyas' from [8] to prove that this result holds if R is a fully semiprimary Noetherian ring lacking certain pairs of annihilator primes. Some examples are given to illustrate that the second layer condition still fails in this case.

A Noetherian bimodule ${}_SB_R$ is called *right fully semiprimary* or *right FSN* provided every subbimodule C has a right primary decomposition i.e. there are subbimodules C_i of B, $i = 1, \ldots, n$, such that $C = C_1 \cap \ldots \cap C_n$ and each $(B/C_i)_R$ has a unique associated prime ideal. The left-hand versions of these terms are similarly defined if S is a two-sided Noetherian ring. As usual, terms unmodifed by 'left' or 'right' are meant to hold on both sides. Thus, 'FSN' means right and left FSN, and R is (right) FSN if the bimodule ${}_RR_R$ is (right) FSN.

By [4, Lemma 8.3.6], a Noetherian bimodule ${}_SB_R$ is right FSN if and only if every biuniform factor B/C has a unique right associated prime. (A bimodule is called *biuniform* if the intersection of any two nonzero subbimodules is nonzero.)

Furthermore, by [4, Theorem 8.3.9], if R is a Noetherian ring satisfying the right second layer condition, then R is right FSN. In fact, from [2, Theorem 3.4], the right second layer condition insures that every Noetherian S-R bimodule is right FSN. However, the domain in [6, Example 4.3.15] is FSN but fails to satisfy the right or left second layer condition. More examples of these sort of rings appear in [7].

2. Notation and definitions

Throughout, R will always be a Noetherian ring. We will write M_R to indicate that M is a right R-module. Similarly, ${}_SM$ and ${}_SM_R$ indicate that M is a left S-module and an S-R bimodule respectively. A Noetherian bimodule is a *nonzero* bimodule ${}_SM_R$ where both ${}_SM$ and M_R are Noetherian. We use $N \leq M_R$ to indicate that N is a right R-submodule of M. Likewise, $N \leq {}_SM$ ($N \leq {}_SM_R$) indicates that N is a left S-submodule (S-R subbimodule) of M. If $N \leq M_R$ is essential in M, then we will write $N \leq_e M$.

For a nonempty subset X of a right R-module M, $r_R(X) = \{r \in R \mid xr = 0$ for all $x \in X\}$. If Y is a nonempty subset of R, then $\operatorname{ann}_M(Y) = \{m \in M \mid my = 0$ for all $y \in Y\}$. In case X is a nonempty subset of a left R-module, then $l_R(X) = \{r \in R \mid rx = 0$ for all $x \in X\}$.

See [1], [2] and [4] for the definition of the second layer condition and its role in the structure of Noetherian rings.

3. Preliminaries

In this section, we collect the results that we need to prove the main result of the paper.

A Noetherian bimodule ${}_SC_R$ is called a *right cell* provided $r_R(C)$ is a prime ideal, C is torsion-free as a right $R/r_R(C)$-module and for all $0 \neq C' \leq {}_SC_R$, C/C' is torsion over $R/r_R(C)$. A *left cell* is defined likewise and, of course, *cell* means right and left cell.

The next result follows from [2, Proposition 2.1] and its proof.

Proposition 3.1. (1) *Every Noetherian bimodule contains a (right) cell.*

(2) *If ${}_SB_R$ is a biuniform Noetherian bimodule, then the sum of all the (right) cells in B is the unique largest (right) cell in B.*

Definition 3.2. [2, page 383]. A biuniform bimodule ${}_SB_R$ is called *right uneven* provided there exists a cell $C \subset B$ such that B/C is a cell and the following statements are true:

(1) $r_R(B/C) \subset r_R(C)$.
(2) $C \subset \mathrm{ann}_B(Q)$ where $Q = r_R(B/C)$.

In this situation, it is easy to see that C is the unique largest cell in B.

The first two statements in the next result are parts of [2, Proposition 2.2] and [2, Proposition 2.4] respectively. The third statement is a slight restatement of [2, Theorem 3.2]. The last statement is contained in the proof of [5, Proposition 3.1].

Proposition 3.3. *Let ${}_SB_R$ be a biuniform Noetherian bimodule with unique largest cell C such that B/C is a cell. Set $P = r_R(C)$ and $Q = r_R(B/C)$.*

(1) *B is right uneven if and only if $P \neq Q$ and C is not an essential submodule of B_R.*
(2) *If $C \leq_e B_R$, then $C = \mathrm{ann}_B(P)$.*
(3) *B is right FSN iff no subfactor bimodule of B is right uneven.*
(4) *If P is a maximal associated prime ideal of B_R and $\mathrm{ann}_B(P) \leq_e B_R$, then any associated prime ideal of $B/\mathrm{ann}_B(P)$ is linked to P.*

If R is a Noetherian prime ring, then the torsion submodule of a right R-module M is denoted by $\mathrm{T}(M_R)$ and similarly for left modules. If M is an S-R bimodule, then $\mathrm{T}(M_R)$ is a subbimodule of M. When M is a Noetherian S-R bimodule, then $\mathrm{T}(M_R)$ is annihilated by a regular element of R cf. [2, Lemma 1.1].

The next result is the special case of [8, Corollary 3.8] for a Noetherian ring R.

Proposition 3.4. *Let Q be a prime ideal of a Noetherian ring R. No prime ideal $P \supset Q$ has a link $P \rightsquigarrow Q$ iff* $\mathrm{T}({}_{R/Q}(Q/Q^2)) \subseteq \mathrm{T}((Q/Q^2)_{R/Q})$.

Corollary 3.5. *Let R be a Noetherian ring and let Q be a prime ideal. Then no prime ideal $P \supset Q$ has $P \rightsquigarrow Q$ or $Q \rightsquigarrow P$ iff* $\mathrm{T}({}_{R/Q}(Q/Q^2)) = \mathrm{T}((Q/Q^2)_{R/Q})$.

Proof. Note that $Q \rightsquigarrow P$ if and only if $P \rightsquigarrow Q$ in R^{op}, the opposite ring of R. It then follows from Proposition 3.4 that no $P \supset Q$ has $Q \rightsquigarrow P$ if and only if $\mathrm{T}((Q/Q^2)_{R/Q}) \subseteq \mathrm{T}({}_{R/Q}(Q/Q^2))$. □

4. Incomparability and (†)

Let R and S be Noetherian rings. A biuniform Noetherian bimodule ${}_SB_R$ is said to satisfy $(\ddagger)_r$ provided there exists a cell $C \subset B$ such that B/C is a cell and

$r_R(B/C) \subset r_R(C)$. We say that a Noetherian bimodule ${}_SB_R$ satisfies $(\dagger)_r$ provided *no* biuniform subfactor bimodule of B satisfies $(\ddagger)_r$. The left-hand versions $(\ddagger)_l$ and $(\dagger)_l$ are defined similarly. Finally, B satisfies $(\dagger)$ (resp. $(\ddagger)$) provided it satisfies both $(\dagger)_r$ and $(\dagger)_l$ (resp. $(\ddagger)_r$ and $(\ddagger)_l$). A ring R satisfies any of these conditions provided the same is true of the bimodule ${}_RR_R$.

By Proposition 3.3(3), if B satisfies $(\dagger)$, then B is FSN. As mentioned earlier, the domain R in [6, Example 4.3.15] is an FSN ring that does not satisfy the right or left second layer condition. Since the only proper ideals of R are 0 and a single nonzero prime ideal, R satisfies $(\dagger)$. For the domain R constructed in [7], if $C \subset B$ are ideals where both C and B/C are cells, then $(\ddagger)$ fails for B because the right and left annihilator of C is 0. $(\ddagger)$ fails if B and C are ideals of R/A, where A is an ideal containing P_0 the unique minimal (prime) ideal of R, since in this case, R/A is a commutative ring (see Proposition 4.1 below). Thus, R satisfies $(\dagger)$.

We do not know of an example of an FSN ring that does not satisfies $(\dagger)$. However, as a trivial consequence of the next result, a ring satisfying the second layer condition satisfies $(\dagger)$.

Proposition 4.1. *Let R and S be Noetherian rings that satisfy the second layer condition. Then every Noetherian bimodule ${}_SB_R$ satisfies* $(\dagger)$.

Proof. It suffices to show that no biuniform Noetherian bimodule satisfies $(\ddagger)_r$. Suppose, then, that there exists a biuniform Noetherian bimodule ${}_SB_R$ with a cell $C \subset B$ such that B/C is a cell and $Q = r_R(B/C) \subset r_R(C) = P$. By [2, Theorem 3.4], the second layer condition implies that B is right FSN. By Proposition 3.3(3), B is not right uneven, and so by Proposition 3.3(1), $C \leq_e B_R$. Thus, by Proposition 3.3(2), $C = \operatorname{ann}_B(P)$. Since C is both torsion-free as a right R/P-module and essential in B_R, P is a maximal associated prime ideal of B. It follows from Proposition 3.3(4) that $Q \rightsquigarrow P$ contradicting the incomparability of linked prime ideals of R from [3, Theorem 1.8]. The corresponding left-hand result follows by symmetry. □

Theorem 4.2. *If R is a Noetherian ring that satisfies* $(\dagger)$*, then R is an FSN ring where no two distinct comparable prime ideals are linked.*

Proof. Since R satisfies $(\dagger)$, the same is true of all factor rings of R. Thus, we can proceed by Noetherian induction: Assume that the result fails for R but holds true for all proper factors of R. Let Q and P be the prime ideals of R where $Q \rightsquigarrow P$ and $P \supset Q$. From the definition of link, there is an ideal A with $QP \subseteq A \subset Q \cap P$ such that $(Q \cap P)/A$ is a torsion-free as a left R/Q-module and as a right R/P-module.

It follows that $Q/A \rightsquigarrow P/A$ which is contrary to the induction hypothesis if $A \neq 0$. Thus, $Q^2 = QP = A = 0$, ${}_{R/Q}Q$ is torsion-free and $Q_{R/P}$ is torsion-free. Note that since $P \supset Q$, Q is torsion as a *right* R/Q-module. Also, since $Q \subseteq l_R(Q)$ and ${}_{R/Q}Q$ is torsion-free, $Q = l_R(Q)$. Similarly, $P = r_R(Q)$.

Let $0 \neq X \subset Q$ be an ideal. As a right R/Q-module, Q/X is torsion. Also, $(Q/X)^2 = 0$. Then by the induction hypothesis together with Corollary 3.5,
$\mathrm{T}({}_{R/Q}(Q/X)) = \mathrm{T}((Q/X)_{R/Q}) = Q/X$ whence Q is a left cell. In particular, $X \leq_e {}_RQ$. Since ${}_{R/Q}(Q/X)$ is torsion, there exists a regular element $c+Q$ of R/Q with $cQ \subset X$. Define a right R-homomorphism $\phi : Q \to X$ via $\phi(r) = cr$. Since ${}_{R/Q}Q$ is torsion-free, ϕ is a monomorphism and so the right R-modules X and Q have equal uniform dimension. It follows that $X \leq_e Q_{R/P}$. Thus, Q/X is torsion as a right R/P-module. Therefore, Q is also a right cell.

Suppose then that A, B are ideals of R with $A \cap B = 0$. Then $AB = 0$ and so one of A or B is contained in Q. If $A \subseteq Q$, then since $A \leq_e {}_RQ$, $Q \cap B = 0$. Thus, $BQ = 0$ forcing $B \subseteq l_R(Q) = Q$ which is impossible. Therefore, R is biuniform. In particular, P is biuniform.

Consider the bimodules $Q \subset P$. From above, Q is a cell. Trivially, P/Q is cell. Finally, $r_R(P/Q) = Q \subset r_R(Q) = P$. Thus, P is an R-R bimodule that satisfies $(\ddagger)_r$ contradicting the standing hypothesis.

If $P \rightsquigarrow Q$ where $P \supset Q$, then the symmetric argument yields a contradiction to $(\ddagger)_l$. □

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Karl A. Kosler
Department of Mathematics
University of Wisconsin-Waukesha
1500 University Drive, Waukesha
Wisconsin 53188-2799
e-mail: karl.kosler@uwc.edu