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A NOTE ON SIMPLE MODULES OVER QUASI-LOCAL RINGS

Paula A. A. B. Carvalho, Christian Lomp and Patrick F. Smith

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Dedicated to the memory of Professor John Clark

Abstract. Matlis showed that the injective hull of a simple module over a commutative Noetherian ring is Artinian. In several recent papers, non-commutative Noetherian rings whose injective hulls of simple modules are locally Artinian have been studied. This property had been denoted by property $(\diamond)$. In this paper we investigate, which non-Noetherian semiprimary commutative quasi-local rings $(R, \mathfrak{m})$ satisfy property $(\diamond)$. For quasi-local rings $(R, \mathfrak{m})$ with $\mathfrak{m}^3 = 0$, we prove a characterization of this property in terms of the dual space of $\mathrm{Soc}(R)$. Furthermore, we show that $(R, \mathfrak{m})$ satisfies $(\diamond)$ if and only if its associated graded ring $\mathrm{gr}(R)$ does.

Given a field F and vector spaces V and W and a symmetric bilinear map $\beta : V \times V \to W$ we consider commutative quasi-local rings of the form $F \times V \times W$, whose product is given by

$$(\lambda_1, v_1, w_1)(\lambda_2, v_2, w_2) = (\lambda_1\lambda_2, \lambda_1 v_2 + \lambda_2 v_1, \lambda_1 w_2 + \lambda_2 w_1 + \beta(v_1, v_2))$$

in order to build new examples and to illustrate our theory. In particular we prove that a quasi-local commutative ring with radical cube-zero does not satisfy $(\diamond)$ if and only if it has a factor, whose associated graded ring is of the form $F \times V \times F$ with V infinite dimensional and β non-degenerated.

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1. Introduction

The structure and in particular finiteness conditions of injective hulls of simple modules have been widely studied. Rosenberg and Zelinsky's work [15] is one of the earliest studies of finiteness conditions on the injective hull of a simple module. Matlis showed in his seminal paper [13] that any injective hull of a simple module

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over a commutative Noetherian module is Artinian. Jans in [11] has termed a ring R to be left co-Noetherian if every simple left R-module has an Artinian injective hull. Vamos showed in [20] that a commutative ring R is co-Noetherian if and only if $R_{\mathfrak{m}}$ is Noetherian for any maximal ideal $\mathfrak{m} \in \mathrm{Max}(R)$ - generalising in this way Matlis' result. In connection with the Jacobson Conjecture for non-commutative Noetherian rings, Jategaonkar showed in [12] (see also [6,17]) that the injective hulls of simple modules are locally Artinian, i.e. any finitely generated submodule is Artinian, provided the ring R is fully bounded Noetherian. We say that a ring R satisfies condition $(\diamond)$ if

$$\text{Injective hulls of simple left } R\text{-modules are locally Artinian.} \qquad (\diamond)$$

In this paper we study $(\diamond)$ for, not necessarily Noetherian, quasi-local commutative rings R with maximal ideal $\mathfrak{m}$ such that $\mathfrak{m}^3 = 0$. A description of such rings is given in terms of the dual space of $\mathrm{Soc}(R)$ seen as a vector space over $R/\mathfrak{m}$ (Theorem 4.4). Furthermore, we relate property $(\diamond)$ of $(R, \mathfrak{m})$ with its associated graded ring $\mathrm{gr}(R) = R/\mathfrak{m} \oplus \mathfrak{m}/\mathfrak{m}^2 \oplus \mathfrak{m}^2$ in Corollary 4.6. Given a field F and vector spaces V and W and a symmetric bilinear map $\beta : V \times V \to W$ we consider commutative quasi-local rings of the form $F \times V \times W$, whose product is given by

$$(\lambda_1, v_1, w_1)(\lambda_2, v_2, w_2) = (\lambda_1\lambda_2, \lambda_1 v_2 + \lambda_2 v_1, \lambda_1 w_2 + \lambda_2 w_1 + \beta(v_1, v_2))$$

to build new examples and to illustrate our theory. In particular we prove in Proposition 5.3 that a quasi-local commutative ring with radical cube-zero does not satisfy $(\diamond)$ if and only if it has a factor whose associated graded ring is of the form $F \times V \times F$ with V of infinite dimension and β non-degenerate.

2. Preliminaries

The following lemma shows that condition $(\diamond)$ is intrinsically linked to Krull's intersection Theorem:

Lemma 2.1. *Let R be a (not necessarily commutative) ring with Jacobson radical J, such that finitely generated Artinian modules have finite length. If R has property $(\diamond)$, then for any left ideal I of R one has*

$$\bigcap_{n=0}^{\infty} (I + J^n) = I.$$

Proof. Let I be any left ideal of R. Then by Birkhoff's theorem R/I embeds into a product of cyclic modules R/K_i with essential simple socle, where $I \subseteq K_i$ and $\bigcap K_i = I$. By hypothesis each of these modules R/K_i is Artinian and hence has finite length. Thus there exists a number $n_i \geq 1$ such that $J^{n_i}R/K_i = 0$, or equivalently $J^{n_i} \subseteq K_i$. Hence $I + J^{n_i} \subseteq K_i$ for all i and as the intersection of the K_i's is I, we have

$$I = \bigcap_i K_i \supseteq \bigcap_i (I + J^{n_i}) \supseteq \bigcap_n (I + J^n) \supseteq I.$$

□

Remark 2.2. (1) *Assuming the hypotheses of Lemma 2.1, one can easily adapt the above proof to show that* $\bigcap_{n=0}^{\infty} (J^n M) = 0$, *for any finitely generated left* R*-module* M*. Furthermore, if* M *is a finitely generated essential extension of a simple left* R*-module, then there exists* $n > 0$ *such that* $J^n M = 0$.

(2) *Finitely generated Artinian left* R*-modules have finite length if for example* R *is left Noetherian or if* R *is commutative. For the latter case let* M *be an Artinian module over a commutative ring* R *generated by* $x_1, \ldots, x_k$*. Then*

$$R/Ann(M) \to R(x_1, ..., x_k) \subseteq M^k$$

is an embedding. Since M^k *is Artinian,* $R/Ann(M)$ *is Artinian and by the Hopkins-Levitzki's Theorem* $R/Ann(M)$ *is Noetherian. As* M *is finitely generated over* $R/Ann(M)$*,* M *is also Noetherian, i.e. has finite length.*

(3) *We follow the terminology of commutative ring theory and call a commutative ring* R *quasi-local if it has a unique maximal ideal* $\mathfrak{m}$*. A local ring is a commutative Noetherian quasi-local ring. From Lemma 2.1 we see that any commutative quasi-local ring* $(R, \mathfrak{m})$*, that satisfies* $(\diamond)$*, is separated in the* $\mathfrak{m}$*-adic topology. Moreover if* $\mathfrak{m}^n$ *is idempotent, for some* $n \geq 1$*, then* $\mathfrak{m}^n = 0$.

Recall that a ring R with Jacobson radical J is called *semilocal* if R/J is semisimple. A semilocal ring with nilpotent Jacobson radical is called *semiprimary.* The second socle of a module M is the submodule $\mathrm{Soc}_2(M)$ of M with $\mathrm{Soc}(M/\mathrm{Soc}(M)) = \mathrm{Soc}_2(M)/\mathrm{Soc}(M)$. For an ideal K of R, denote by $\mathrm{Ann}_M(K)$ the set of elements $m \in M$ such that $Km = 0$. For a semilocal ring R, it is well-known that $\mathrm{Ann}_M(J) = \mathrm{Soc}(M)$ and that $\mathrm{Ann}_M(J^2) = \mathrm{Soc}_2(M)$. For a left ideal I set $(I : K) = \{r \in R \mid Kr \subseteq I\}$.

Proposition 2.3. *The following statements are equivalent, for a semiprimary ring* R *with Jacobson radical* J.

(a) R *has property* $(\diamond)$.

(b) $\mathrm{Soc}_2(M)$ *has finite length, for any finitely generated left R-module M with* $\mathrm{Soc}(M)$ *finitely generated.*

(c) $(I : J^2)/I$ *is finitely generated, for any left ideal I of R with $(I : J)/I$ finitely generated.*

Proof. Note first that since R is semiprimary, there exists $n \geq 0$ such that $J^n = 0$. Moreover since R is semilocal, $\mathrm{Soc}(M) = \mathrm{Ann}_M(J) = \{m \in M \mid Jm = 0\}$ for any left R-module M. In particular any left R-module has an essential socle, since any left R-module M has a finite socle series:

$$0 = J^n M \subseteq J^{n-1} M \subseteq \cdots \subseteq JM \subseteq M.$$

$(a) \Rightarrow (b)$ If R satisfies $(\diamond)$ then any finitely generated module with finitely generated (essential) socle is Artinian. Hence if $\mathrm{Soc}(M)$ is finitely generated, M must be Artinian and hence $M/\mathrm{Soc}(M)$ is Artinian.

$(b) \Rightarrow (c)$ for $M = R/I$ one has $\mathrm{Soc}(R/I) = (I : J)/I$ as mentioned above. Moreover, $(I : J^2) = ((I : J) : J)$ and therefore $(I : J^2)/(I : J) = \mathrm{Soc}(R/(I : J)) = \mathrm{Soc}(M/\mathrm{Soc}(M))$. Thus the statement follows from (b).

$(c) \Rightarrow (a)$ is clear since if I is a left ideal such that $M = R/I$ is a cyclic essential extension of a simple left R-module, then $(I : J)/I$ is cyclic and by assumption $(I : J^2)/I$ is finitely generated. Hence $(I : J^2)/I$ has finite length. Applying our hypothesis to $I' = (I : J)$, we can conclude that $(I : J^3)/I$ has finite length. Continuing we have also that $R/I = (I : J^n)/I$ has finite length. □

A sufficient condition for a ring to satisfy $(\diamond)$ is given by the following lemma.

Lemma 2.4. *Any ring R, with $R/\mathrm{Soc}({}_R R)$ being left Artinian, satisfies $(\diamond)$.*

Proof. Suppose $I \subset K \subseteq R$ are left ideals such that K/I is a simple left R-module and essential in R/I. If $\mathrm{Soc}({}_R R) \subseteq I$, then R/I is a factor of $R/\mathrm{Soc}({}_R R)$ and hence Artinian. If $\mathrm{Soc}({}_R R) \nsubseteq I$, then $(\mathrm{Soc}({}_R R) + I)/I$ is a semisimple submodule of R/I and hence must equal K/I, i.e. $\mathrm{Soc}({}_R R) + I = K$. As a quotient of $R/\mathrm{Soc}({}_R R)$, the module $R/K = R/(\mathrm{Soc}({}_R R) + I)$ is Artinian and so is R/I. □

Clearly it is not necessary for a ring R with $(\diamond)$ to satisfy $R/\mathrm{Soc}({}_R R)$ being Artinian. Moreover, Example 5.2 shows that there are commutative rings R such that $R/\mathrm{Soc}(R)$ satisfies $(\diamond)$, but R does not.

In recent papers [2,3,4,5,8,9,14,16], several non-commutative Noetherian rings have been shown to satisfy $(\diamond)$. In this note we intend to study condition $(\diamond)$ for commutative not necessarily Noetherian rings.

3. Local-global argument

Jans in [11] defined a ring R to be left co-Noetherian if for every simple left R-module its injective hull is Artinian. Vamos has shown in [20] that a commutative ring R is co-Noetherian if and only if $R_{\mathfrak{m}}$ is Noetherian for all $\mathfrak{m} \in \operatorname{Max}(R)$.

The following lemma shows the relation between co-Noetherianess and condition $(\diamond)$ for commutative quasi-local rings. The proof follows the ideas of [19, Theorem 1.8].

Lemma 3.1. *The following statements are equivalent for a commutative quasi-local ring R with maximal ideal $\mathfrak{m}$.*

(a) *R is Noetherian.*
(b) *R is co-Noetherian.*
(c) *R satisfies $(\diamond)$ and $\mathfrak{m}/\mathfrak{m}^2$ is finitely generated.*
(d) *R satisfies $\bigcap_{n=0}^{\infty}(I + \mathfrak{m}^n) = I$ for all (finitely generated) ideals I of R and $\mathfrak{m}/\mathfrak{m}^2$ is finitely generated.*

Proof. $(a) \Leftrightarrow (b)$ follows from Vamos' result [20, Theorem 2].

$(a) \Rightarrow (c)$ is clear and $(c) \Rightarrow (d)$ follows from Lemma 2.1.

$(d) \Rightarrow (a)$ There exists a finitely generated ideal B of R such that $\mathfrak{m} = B + \mathfrak{m}^2$. Then

$$\mathfrak{m} = B + \mathfrak{m}^2 = B + (B + \mathfrak{m}^2)^2 \subseteq B + \mathfrak{m}^3 \subseteq \mathfrak{m},$$

and in general $\mathfrak{m} = B + \mathfrak{m}^n$ for every positive integer n. It follows from (d) that $\mathfrak{m} = B$ and hence $\mathfrak{m}$ is finitely generated. Suppose that R is not Noetherian. Let Q be maximal among the ideals C of R such that C is not finitely generated. Then Q is a prime ideal of R by a standard argument (see [7, Theorem 2]). Clearly $Q \neq \mathfrak{m}$. Let $p \in \mathfrak{m}$ with $p \notin Q$. By the choice of Q the ideal $Q + Rp$ is finitely generated, say

$$Q + Rp = R(q_1 + r_1 p) + \cdots + R(q_k + r_k p), \tag{1}$$

for some positive integer k, $q_i \in Q (1 \leq i \leq k)$, $r_i \in R (1 \leq i \leq k)$. Let $D = Rq_1 + \cdots + Rq_k \subseteq Q$. Let $q \in Q \setminus D$. Then by equation (1) there exist $s_1, \ldots, s_k \in R$ and $d \in D$ such that $q - d = (s_1 r_1 + \cdots + s_k r_k)p \in Q$. Since Q is prime and $p \notin Q$, $(s_1 r_1 + \cdots + s_k r_k) \in Q$, i.e. $Q = D + Qp$. Now $Q = D + Qp^t$ for every positive integer t and hence

$$Q = \bigcap_{s=1}^{\infty}(D + Qp^s) \subseteq \bigcap_{s=1}^{\infty}(D + \mathfrak{m}^s) = D.$$

It follows that $Q = D$ and hence Q is finitely generated, a contradiction. Thus R is Noetherian. $\square$

The diamond condition is equivalent to the condition that any injective hull of a simple R-module is locally Artinian. By Remark 2.2, a commutative ring satisfies ($\diamond$) if and only if any injective hull E of a simple R-module is locally of finite length, i.e. any finitely generated submodule of E has finite length.

The last result of this section gives us a local-global argument for condition ($\diamond$) similar to the one of Vamos for co-Noetherian rings. We start by restating a well-known result that can be found for example in [18, Proposition 5.6].

Proposition 3.2. *Let R be commutative ring, $\mathfrak{m}$ a maximal ideal of R and denote by $R_\mathfrak{m}$ the localization of R at $\mathfrak{m}$. Then the injective hull $E = E(R/\mathfrak{m})$ of $R/\mathfrak{m}$ as R-module is also the injective hull of $R_\mathfrak{m}/\mathfrak{m}R_\mathfrak{m}$ as $R_\mathfrak{m}$-module.*

Making use of Proposition 3.2 we can now prove the

Theorem 3.3. *A commutative ring R satisfies ($\diamond$) if and only if $R_\mathfrak{m}$ satisfies ($\diamond$), for all $\mathfrak{m} \in \mathrm{Max}(R)$.*

For the proof of Theorem 3.3 we need the following two lemmas. In what follows $\mathrm{Ann}_S(-)$ denotes the annihilator of an S-module.

Lemma 3.4. *Let R be a commutative ring satisfying ($\diamond$), $\mathfrak{m}$ a maximal ideal of R and $x \in E(R/\mathfrak{m}) \setminus \{0\}$. Then there exists $k \geq 1$ such that $\mathfrak{m}^k x = 0$.*

Proof. Since R satisfies ($\diamond$), Rx has a finite composition series (see Remark 2.2(2)). Hence there exist maximal ideals $\mathfrak{m}_1, \ldots, \mathfrak{m}_n$ of R such that $\mathrm{Ann}_R(x) \subseteq \mathfrak{m}_i$, for all i and

$$\mathfrak{m}_1\mathfrak{m}_2 \cdots \mathfrak{m}_n \subseteq \mathrm{Ann}_R(x).$$

Write $\mathfrak{m}^k I = \mathfrak{m}_1\mathfrak{m}_2 \cdots \mathfrak{m}_n$ for some $k \geq 0$ and I the product of maximal ideals $\mathfrak{m}_i \neq \mathfrak{m}$. Now since $R_\mathfrak{m}\mathfrak{m}_i = R_\mathfrak{m}$ if $\mathfrak{m} \neq \mathfrak{m}_i$, it follows

$$R_\mathfrak{m}\mathfrak{m}^k = R_\mathfrak{m}\mathfrak{m}^k I \subseteq R_\mathfrak{m}\mathrm{Ann}_R(x) = \mathrm{Ann}_{R_\mathfrak{m}}(R_\mathfrak{m}x),$$

where $\mathrm{Ann}_{R_\mathfrak{m}}(R_\mathfrak{m}x) = R_\mathfrak{m}\mathrm{Ann}_R(x)$ follows from [1, Proposition 3.14]. As $x \neq 0$, $k \geq 1$ and the result follows. □

Lemma 3.5. *Let R be a commutative ring, $\mathfrak{m}$ a maximal ideal of R, $R_\mathfrak{m}$ the localization of R at $\mathfrak{m}$ and $x \in E(R/\mathfrak{m}) \setminus \{0\}$. If there exists $k \geq 1$ such that $\mathfrak{m}^k x = 0$, then $R_\mathfrak{m}x = Rx$ and the R-submodules lattice of Rx coincides with the $R_\mathfrak{m}$-submodule lattice of $R_\mathfrak{m}x$.*

Proof. For any $a \in R \setminus \mathfrak{m}$ we have $R = Ra + \mathfrak{m}$, which implies also $R = Ra + \mathfrak{m}^k$. Thus there exists $b \in R$ such that $1 - ab \in \mathfrak{m}^k$, i.e. $x = abx$. Hence $a^{-1}x = bx \in Rx$

shows that $R_{\mathfrak{m}}x = Rx$. We have just shown that any principal R-submodule of $E = E(R/\mathfrak{m})$ is also a principal $R_{\mathfrak{m}}$-submodule of E. Hence the R-submodule generated by any set of elements of E coincides with the $R_{\mathfrak{m}}$-submodule generated by that set. □

Proof of Theorem 3.3. Let $\mathfrak{m} \in \mathrm{Max}(R)$. By Proposition 3.2, $E = E(R/\mathfrak{m})$ is also the injective hull of the unique simple $R_{\mathfrak{m}}$-module $R_{\mathfrak{m}}/\mathfrak{m}R_{\mathfrak{m}}$.

Assume that R satisfies ($\diamond$). For $x \in E \setminus \{0\}$, Rx is an Artinian R-module and by Lemmas 3.4 and 3.5 also $R_{\mathfrak{m}}x = Rx$ is Artinian as $R_{\mathfrak{m}}$-module, so $R_{\mathfrak{m}}$ satisfies ($\diamond$).

Now assume that $R_{\mathfrak{m}}$ satisfies ($\diamond$). By hypothesis E is a locally Artinian $R_{\mathfrak{m}}$-module and hence has finite length by Remark 2.2. Hence there exist $k \geq 1$ such that $(\mathfrak{m}R_{\mathfrak{m}})^k x = 0$, for any $x \in E$. In particular $\mathfrak{m}^k x = 0$ and by Lemma 3.5 it follows that $Rx = R_{\mathfrak{m}}x$ has finite length as R-module. □

Theorem 3.3 shows that in order to characterize commutative rings satisfying ($\diamond$), we need only to focus on quasi-local commutative rings.

4. Commutative semiprimary quasi-local rings

By Lemma 3.1, a quasi-local commutative ring is co-Noetherian if and only if it is Noetherian. Recall, that an ideal I of a ring R is called *subdirectly irreducible* if R/I has an essential simple socle. Clearly a ring R satisfies ($\diamond$) if and only if R/I is Artinian, for all subdirectly irreducible ideals I of R.

4.1. Commutative quasi-local rings with square-zero maximal ideal. Given any vector space V over a field F, the trivial extension (or idealization) is defined on the vector space $R = F \times V$ with multiplication given by $(a, v)(b, w) = (ab, aw + vb)$, for all $a, b \in F$ and $v, w \in V$. Any such trivial extension R is a commutative quasi-local ring that satisfies ($\diamond$). However R is Noetherian if and only if V is finite dimensional.

Lemma 4.1. *Any commutative quasi-local ring with square-zero maximal ideal satisfies* ($\diamond$).

Proof. Let $(R, \mathfrak{m})$ be a commutative quasi-local ring with $\mathfrak{m}^2 = 0$. Since $\mathfrak{m}$ is a vector space over $R/\mathfrak{m}$, it is semisimple. Let K be any subdirectly irreducible ideal of R. If $K = \mathfrak{m}$, then $R/\mathfrak{m}$ is simple. So assume $K \subset \mathfrak{m}$. Then there exists a complement L such that $\mathfrak{m} = L \oplus K$ and $\mathrm{Soc}(R/K) = \mathfrak{m}/K \simeq L$ is simple.

$$0 \longrightarrow \mathfrak{m}/K \longrightarrow R/K \longrightarrow R/\mathfrak{m} \longrightarrow 0$$

is a short exact sequence. Hence R/K has length 2. □

4.2. Commutative quasi-local rings with cube-zero maximal ideal. In this section we will characterize commutative quasi-local rings $(R, \mathfrak{m})$ with $\mathfrak{m}^3 = 0$ that satisfy $(\diamond)$. Recall that $\mathrm{Soc}(R) = \mathrm{Ann}(\mathfrak{m}) = \{r \in R : r\mathfrak{m} = 0\}$. Hence $\mathfrak{m}^2 \subseteq \mathrm{Soc}(R)$. We start with a simple observation.

Lemma 4.2. *Let $(R, \mathfrak{m})$ be a commutative quasi-local ring with $\mathfrak{m}^3 = 0$.*

(1) *If $\mathfrak{m}/\mathrm{Soc}(R)$ is finitely generated, then R satisfies $(\diamond)$.*
(2) *If $\mathrm{Soc}(R)$ is finitely generated, then R satisfies $(\diamond)$ if and only if $\mathfrak{m}/\mathrm{Soc}(R)$ is finitely generated.*

Proof. (1) If $\mathfrak{m}/\mathrm{Soc}(R)$ is finitely generated, then $R/\mathrm{Soc}(R)$ is Artinian and by Lemma 2.4, R satisfies $(\diamond)$.

(2) If $\mathrm{Soc}(R)$ is finitely generated and R satisfies $(\diamond)$, then $\mathrm{Soc}_2(R) = \mathfrak{m}$ is finitely generated, by Proposition 2.3 and in particular $\mathfrak{m}/\mathrm{Soc}(R)$ is finitely generated. Conversely if $\mathfrak{m}/\mathrm{Soc}(R)$ is finitely generated and $\mathrm{Soc}(R)$ is finitely generated, then $\mathfrak{m}$, and so R, has finite length. In particular, R satisfies $(\diamond)$. □

The last lemma raises the question, whether the reverse conclusion of (1) holds. That is, whether $\mathfrak{m}/\mathrm{Soc}(R)$ needs to be finitely generated for a commutative quasi-local ring R with $\mathfrak{m}^3 = 0$ and satisfying property $(\diamond)$. As we will see in Example 5.4, this need not be the case.

Lemma 4.3. *Let $(R, \mathfrak{m})$ be a commutative quasi-local ring with residue field $F = R/\mathfrak{m}$. Suppose $\mathfrak{m}^3 = 0$. Then there exists a correspondence between subdirectly irreducible ideals of R that do not contain $\mathrm{Soc}(R)$ and non-zero linear maps $f : \mathrm{Soc}(R) \to F$. Each corresponding pair (I, f) satisfies*

$$\mathrm{Soc}(R) \cap I = \ker(f) \qquad \text{and} \qquad \mathrm{Soc}(R) + I = V_f := \{a \in \mathfrak{m} \mid f(\mathfrak{m}a) = 0\}.$$

Proof. Let I be a subdirectly irreducible ideal that does not contain $\mathrm{Soc}(R)$, then $\mathrm{Soc}(R/I) = (\mathrm{Soc}(R) + I)/I$ is simple. Thus $\mathrm{Soc}(R) = Fx \oplus (\mathrm{Soc}(R) \cap I)$, for a non-zero element $x \in \mathrm{Soc}(R)$. Let $f : \mathrm{Soc}(R) \to F$ be the linear map such that $\ker(f) = \mathrm{Soc}(R) \cap I$ and $f(x) = 1$. Clearly $\mathrm{Soc}(R) + I \subseteq V_f$, because

$$\mathfrak{m}(\mathrm{Soc}(R) + I) = \mathfrak{m}I \subseteq \mathfrak{m}^2 \cap I \subseteq \mathrm{Soc}(R) \cap I = \ker(f).$$

To show that $V_f = \mathrm{Soc}(R) + I$ we use the essentiality of $\mathrm{Soc}(R/I) = (\mathrm{Soc}(R) + I)/I$ in R/I as follows: For any $a \in V_f \setminus I$, there exists $r \in R$ such that $ra + I$ is a non-zero element of $\mathrm{Soc}(R/I) = (\mathrm{Soc}(R) + I)/I$. Note that $r \notin \mathfrak{m}$ since otherwise

$f(ra) = 0$ and hence $ra \in \ker(f) \subseteq I$. Therefore r is invertible and $a + I = r^{-1}ra + I \in \mathrm{Soc}(R/I)$, i.e. $V_f = \mathrm{Soc}(R) + I$.

On the contrary, let f be any non-zero element of $\mathrm{Soc}(R)^* = \mathrm{Hom}_F(\mathrm{Soc}(R), F)$. Then there exists an element $x \in \mathrm{Soc}(R)$ with $f(x) = 1$, such that $\mathrm{Soc}(R) = Fx \oplus \ker(f)$. Let I be an ideal of R that contains $\ker(f)$ and that is maximal with respect to $x \notin I$. Thus I is subdirectly irreducible and $\mathrm{Soc}(R/I) = (\mathrm{Soc}(R)+I)/I = (Fx \oplus I)/I$ is simple and essential in R/I. By construction $\ker(f) = I \cap \mathrm{Soc}(R)$.

Note that $\mathfrak{m}(\mathrm{Soc}(R)+I) = \mathfrak{m}I \subseteq \mathfrak{m}^2 \cap I \subseteq \mathrm{Soc}(R) \cap I = \ker(f)$, i.e. $\mathrm{Soc}(R)+I \subseteq V_f$. To show the reverse inclusion, let $a \in V_f \setminus I$, then by essentiality there exists $r \in R$ with $ra = u + v \in \mathrm{Soc}(R) + I$, $u \in \mathrm{Soc}(R)$, $v \in I$ and $ra \notin I$. If $r \in \mathfrak{m}$, then $ra \in \ker(f) \subseteq I$, contradicting essentiality. Hence $r \notin \mathfrak{m}$ and $a \in \mathrm{Soc}(R) + I$, i.e. $V_f = \mathrm{Soc}(R) + I$. □

Theorem 4.4. *Retain the notation V_f from Lemma 4.3. Let $(R, \mathfrak{m})$ be a commutative quasi-local ring with residue field F and $\mathfrak{m}^3 = 0$. Then R satisfies $(\diamond)$ if and only if $\mathfrak{m}/V_f$ is finite dimensional over F, for any $f \in \mathrm{Soc}(R)^*$.*

Proof. Suppose that R satisfies $(\diamond)$ and let $f \in \mathrm{Hom}(\mathrm{Soc}(R), F)$. If $f = 0$, then $V_f = \mathfrak{m}$ and $\mathfrak{m}/V_f$ has dimension zero. If $f \neq 0$, then by Lemma 4.3, there exists a subdirectly irreducible ideal I with $V_f = \mathrm{Soc}(R) + I$ and I not containing $\mathrm{Soc}(R)$. As R satisfies $(\diamond)$, R/I is Artinian and as a subquotient $\mathfrak{m}/V_f = (\mathfrak{m}/I)/(V_f/I)$ is also Artinian. The R-module $\mathfrak{m}/V_f$ is semisimple since $\mathfrak{m}^2 \subseteq V_f$. Hence $\mathfrak{m}/V_f$ must be finite dimensional as vector space over F.

Suppose $\mathfrak{m}/V_f$ is finite dimensional for any $f \in \mathrm{Hom}(\mathrm{Soc}(R), F)$. Let I be a subdirectly irreducible ideal of R. If $\mathrm{Soc}(R) \subseteq I$, then R/I is an $R/\mathfrak{m}^2$-module. Since $R/\mathfrak{m}^2$ is a quasi-local ring with square-zero radical, we have by Lemma 4.1, that $R/\mathfrak{m}^2$ satisfies $(\diamond)$. Hence R/I must be Artinian. If $\mathrm{Soc}(R) \not\subseteq I$, then by Lemma 4.3, there exists a non-zero map $f : \mathrm{Soc}(R) \to F$ such that $\mathrm{Soc}(R)+I = V_f$. By hypothesis $\mathfrak{m}/V_f$ is finite dimensional and is therefore Artinian as R-module. As $R/\mathfrak{m}$ and V_f/I are simple modules, also R/I is Artinian, proving that R/I is Artinian for any subdirectly irreducible ideal I of R, i.e. R satisfies $(\diamond)$. □

Let $(R, \mathfrak{m})$ be any commutative quasi-local ring. The *associated graded ring* of R with respect to the $\mathfrak{m}$-filtration is the commutative ring $\mathrm{gr}(R) = \bigoplus_{n \geq 0} \mathfrak{m}^n/\mathfrak{m}^{n+1}$ with multiplication given by

$$(a + \mathfrak{m}^{i+1})(b + \mathfrak{m}^{j+1}) = ab + \mathfrak{m}^{i+j+1}, \qquad \forall a \in \mathfrak{m}^i, b \in \mathfrak{m}^j, \text{ and } i, j \geq 0.$$

For any ideal I of R, the associated graded ideal is $\mathrm{gr}(I) = \bigoplus_{n \geq 0}(I \cap \mathfrak{m}^n + \mathfrak{m}^{n+1})/\mathfrak{m}^{n+1}$. In particular $\mathrm{gr}(\mathfrak{m}) = \bigoplus_{n \geq 1} \mathfrak{m}^n/\mathfrak{m}^{n+1}$ is the unique maximal ideal of $\mathrm{gr}(R)$. Hence $(\mathrm{gr}(R), \mathrm{gr}(\mathfrak{m}))$ is a commutative quasi-local ring with residue field $F = R/\mathfrak{m}$. Furthermore, $\mathrm{gr}(\mathrm{Soc}(R))$ is contained in $\mathrm{Soc}(\mathrm{gr}(R))$.

In case $\mathfrak{m}^3 = 0$, the associated graded ring of $(R, \mathfrak{m})$ is $grR = F \times \mathfrak{m}/\mathfrak{m}^2 \times \mathfrak{m}^2$, where $F = R/\mathfrak{m}$ is the residue field of R. For any proper ideal I of R one has

$$\mathrm{gr}(I) = 0 \times (I + \mathfrak{m}^2)/\mathfrak{m}^2 \times (I \cap \mathfrak{m}^2)$$

and in particular $\mathrm{gr}(\mathrm{Soc}(R)) = 0 \times \mathrm{Soc}(R)/\mathfrak{m}^2 \times \mathfrak{m}^2 = \mathrm{Soc}(\mathrm{gr}(R))$, because for $(0, a + \mathfrak{m}^2, b) \in \mathrm{Soc}(\mathrm{gr}(R))$ we have that $(0, a + \mathfrak{m}^2, b)(0, x + \mathfrak{m}^2, 0) = (0, 0, ax) = (0, 0, 0)$, for all $x \in \mathfrak{m}$ if and only if $a \in \mathrm{Soc}(R)$. To shorten notation we will write the elements of $\mathrm{gr}(R)$ as (a_0, a_1, a_2) for $a_i \in \mathfrak{m}^i$, where the ith component is understood to be modulo $\mathfrak{m}^{i+1}$.

Lemma 4.5. *Let $(R, \mathfrak{m})$ be a commutative quasi-local ring with residue field F and $\mathfrak{m}^3 = 0$. For any $f \in \mathrm{Soc}(\mathrm{gr}(R))^* = \mathrm{gr}(\mathrm{Soc}(R))^*$ there exists $g \in \mathrm{Soc}(R)^*$ such that*

$$V_f = \{a \in \mathrm{gr}(\mathfrak{m}) : f(\mathrm{gr}(\mathfrak{m})a) = 0\} = \mathrm{gr}(V_g),$$

where V_g is the ideal of R defined in Lemma 4.3.

Proof. Let $f : \mathrm{gr}(\mathrm{Soc}(R)) \to F$ and denote by $\pi : \mathrm{Soc}(R) \to \mathfrak{m}^2$ the projection onto $\mathfrak{m}^2$, since $\mathfrak{m}^2$ is a direct summand of $\mathrm{Soc}(R)$. Define $g : \mathrm{Soc}(R) \to F$ by $g(a) = f(0, a, \pi(a))$, for all $a \in \mathrm{Soc}(R)$. Then $(0, x, y) \in V_f$ if and only if $f(0, 0, tx) = 0$, for all $t \in \mathfrak{m}$. Since $tx = \pi(tx) \in \mathfrak{m}^2$, the latter is equivalent to $g(tx) = 0$ for all $t \in \mathfrak{m}$, i.e. $x \in V_g$ (in the ring R). Hence $V_f = \{(0, x, y) \in \mathrm{gr}(\mathfrak{m}) \mid x \in V_g,\ y \in \mathfrak{m}^2\} = \mathrm{gr}(V_g)$. □

Corollary 4.6. *Let $(R, \mathfrak{m})$ be a quasi-local ring with $\mathfrak{m}^3 = 0$. Then R satisfies $(\diamond)$ if and only if its associated graded ring $\mathrm{gr}(R)$ does.*

Proof. If R satisfies $(\diamond)$ and $f : \mathrm{Soc}(\mathrm{gr}(R)) \to F$ is a non-zero map, then by Lemma 4.5 there exists $g : \mathrm{Soc}(R) \to F$ such that $V_f = \mathrm{gr}(V_g)$. Since V_g contains $\mathrm{Soc}(R)$ and hence $\mathfrak{m}^2$, we have $V_f = \mathrm{gr}(V_g) = 0 \times V_g/\mathfrak{m}^2 \times \mathfrak{m}^2$. Thus, $\mathrm{gr}(\mathfrak{m})/V_f \simeq \mathfrak{m}/V_g$. By Theorem 4.4, $\mathfrak{m}/V_g$ is finite dimensional as R satisfies $(\diamond)$. Hence $\mathrm{gr}(\mathfrak{m})/V_f$ is finite dimensional for all $f \in \mathrm{Soc}(\mathrm{gr}(R))^*$. Again by Theorem 4.4, $\mathrm{gr}(R)$ satisfies $(\diamond)$.

Let $f : \mathrm{Soc}(R) \to F$ be any non-zero linear map and let $V_f = \{a \in \mathfrak{m} \mid f(\mathfrak{m}a) = 0\}$. If $\mathfrak{m}^2 \subseteq \ker(f)$, then $V_f = \mathfrak{m}$. If $\mathfrak{m}^2 \not\subseteq \ker(f)$, then there exists $x \in \mathfrak{m}^2$ with $f(x) = 1$. Note that $V_f^2 \subseteq \ker(f) \cap \mathfrak{m}^2$, hence $I = 0 \times \left(V_f/\mathfrak{m}^2\right) \times \left(\ker(f) \cap \mathfrak{m}^2\right)$ is a subdirectly irreducible ideal of $\mathrm{gr}(R)$. To see this note that $E = 0 \times 0 \times Fx$ is a simple submodule of $\mathrm{gr}(R)/I \simeq F \times \mathfrak{m}/V_f \times Fx$. We will show that E is essential in $\mathrm{gr}(R)/I$. Let $(0,\overline{a},\overline{b}) \in \mathrm{gr}(R)/I$. If $a \notin V_f$, there exists $c \in \mathfrak{m}$ such that $f(ac) \neq 0$ and $ac - f(ac)x \in \ker(f) \cap \mathfrak{m}^2$. Hence $(0,\overline{a},\overline{b})(0,\overline{c},0) = (0,0,f(ac)x) \in E$ is non-zero. If $a \in V_f$, i.e. $\overline{a} = 0$, and $\overline{b} \neq 0$, then $(0,0,\overline{b})$ is a non-zero element of E. Hence E is an essential simple submodule of $\mathrm{gr}(R)/I$ and if $\mathrm{gr}(R)$ satisfies $(\diamond)$, the quotient $\mathrm{gr}(R)/I$ and therefore also $\mathfrak{m}/V_f$ must be Artinian, thus finite dimensional. By Theorem 4.4, R satisfies $(\diamond)$. □

5. Examples

Let $(R, \mathfrak{m})$ be a commutative quasi-local ring with $\mathfrak{m}^3 = 0$. The associated graded ring $\mathrm{gr}(R)$ is of the form $\mathrm{gr}(R) = F \oplus V \oplus W$ where $F = R/\mathfrak{m}$ and $V = \mathfrak{m}/\mathfrak{m}^2$ and $W = \mathfrak{m}^2$ are vector spaces over F. Moreover, the multiplication of R induces a symmetric bilinear map $\beta : V \times V \to W$. Hence $\mathrm{gr}(R)$ is uniquely determined by (F, V, W, β) and its multiplication can be identified with the multiplication of a generalised matrix ring. Writing the elements of $S = F \times V \times W$ as 3-tuples (λ, v, w) we have that the multiplication is given by

$$(\lambda_1, v_1, w_1)(\lambda_2, v_2, w_2) = (\lambda_1\lambda_2, \lambda_1 v_2 + \lambda_2 v_1, \lambda_1 w_2 + \lambda_2 w_1 + \beta(v_1, v_2)).$$

The units are precisely the elements (λ, v, w) with $\lambda \neq 0$ and the unique maximal ideal of S is given by $\mathrm{Jac}(S) = 0 \times V \times W$. Let

$$V_\beta^\perp = \{a \in V \mid \beta(V, a) = 0\},$$

then $\mathrm{Soc}(S) = 0 \times V_\beta^\perp \times W$, while $\mathrm{Jac}(S)^2 = 0 \times 0 \times \mathrm{Im}(\beta)$.

Recall, that β is called non-degenerate or non-singular if $V_\beta^\perp = 0$. In general β need not be non-degenerate:

Example 5.1. *Let F be any field and V be any vector space over F with countably infinite basis $\{v_0, v_1, v_2, \ldots\}$. Define the symmetric bilinear form $\beta : V \times V \to F$ with $\beta(v_0, v_0) = 1$ and $\beta(v_i, v_j) = 0$ for any $(i, j) \neq (0, 0)$. Then $S = F \times V \times F$ is a commutative quasi-local ring that satisfies $(\diamond)$, because $V_\beta^\perp = \mathrm{span}(v_i \mid i > 0)$ and hence*

$$\mathrm{Jac}(S)/\mathrm{Soc}(S) = (0 \times V \times F)/(0 \times V_\beta^\perp \times F) \simeq F$$

is one-dimensional. By Lemma 4.2, S satisfies $(\diamond)$. Note that S is not Artinian. Moreover, $S = \mathrm{gr}(R)$, where $R = F[x_0, x_1, x_2 \ldots]/\langle x_0^3,\ x_i x_j \mid (i, j) \neq (0, 0)\rangle$.

The bilinear form of the last example was not non-degenerate. Since $0 \times V_\beta^\perp \times 0$ is always an ideal, we can pass to $F \times V/V_\beta^\perp \times W$ where the bilinear form β is now non-degenerate. The following is a natural example of such a ring with non-degenerate bilinear form:

Example 5.2. *Let $F = \mathbb{R}$ and let $V = C([0,1])$ be the space of continuous real valued functions on $[0,1]$. Set*

$$\langle f, g\rangle = \int_0^1 f(x)g(x)dx, \qquad \forall f, g \in C([0,1]).$$

Then $\langle\, ,\rangle : V \times V \to \mathbb{R}$ is a non-degenerate symmetric bilinear form on V. Hence, by Lemma 4.2, $R = \mathbb{R} \times V \times \mathbb{R}$ is a commutative quasi-local ring with cube-zero radical, that does not satisfy $(\diamond)$, because its socle $\mathrm{Soc}(R) = 0 \times 0 \times \mathbb{R}$ is one-dimensional, hence finitely generated, but $\mathrm{Jac}(R)/\mathrm{Soc}(R)$ is infinite dimensional, hence not finitely generated as R-module.

These kind of rings must occur as the associated graded ring of a quotient of a commutative quasi-local ring $(R, \mathfrak{m})$ with $\mathfrak{m}^3 = 0$ that does not satisfy $(\diamond)$.

Proposition 5.3. *A commutative quasi-local ring $(R, \mathfrak{m})$ with $\mathfrak{m}^3 = 0$ and residue field F does not satisfy $(\diamond)$ if and only if it has a factor R/I whose associated graded ring $\mathrm{gr}(R/I)$ is of the form $F \times V \times F$ with $\dim(V) = \infty$ and non-degenerate bilinear form $\beta : V \times V \to F$.*

Proof. By Theorem 4.4, R does not satisfy $(\diamond)$ if and only if there exists $f \in \mathrm{Soc}(R)^*$ such that $\mathfrak{m}/V_f$ has infinite dimension. Suppose that R does not satisfy $(\diamond)$ and choose such f. Note that $f \neq 0$ since $V_f \neq \mathfrak{m}$. By Lemma 4.3 there exists a subdirectly irreducible ideal I of R such that $V_f = \mathrm{Soc}(R) + I$ and $\ker(f) = \mathrm{Soc}(R) \cap I$. In particular $(\mathfrak{m}/I)^2 = V_f/I = \mathrm{Soc}(R/I) \simeq F$, because if $\mathfrak{m}^2 \subseteq I$, then $\mathfrak{m}/I \subseteq \mathrm{Soc}(R/I) = V_f/I$ and hence $\mathfrak{m} = V_f$, contradicting $V_f \neq \mathfrak{m}$. Hence $\mathfrak{m}^2 \not\subseteq I$ and $(\mathfrak{m}/I)^2 = \mathrm{Soc}(R/I)$ as R/I has a simple socle. Moreover, $\mathrm{gr}(R/I) = F \times \mathfrak{m}/V_f \times F$, with bilinear form $\beta : \mathfrak{m}/V_f \times \mathfrak{m}/V_f \to F$, which is non-degenerate by the definition of V_f.

On the other hand if R has a factor R/I whose associated graded ring $\mathrm{gr}(R/I)$ is of the form $F \times V \times F$ with infinite dimensional vector space V and non-degenerate bilinear form $\beta : V \times V \to F$, then $\mathrm{Soc}(\mathrm{gr}(R/I)) = 0 \times 0 \times F$ is a simple submodule of $\mathrm{gr}(R/I)$, which is essential since β is non-degenerate. As V is infinite dimensional, the semisimple $\mathrm{gr}(R/I)$-module $(0 \times V \times F)/(0 \times 0 \times F)$ is not artinian and hence $\mathrm{gr}(R/I)$ does not satisfy $(\diamond)$. By Corollary 4.6, R/I does not satisfy $(\diamond)$ and therefore also R does not. $\square$

Let $V = A$ be any unital commutative F-algebra. Consider the multiplication of A as a symmetric non-degenerate bilinear map $\mu : A \times A \to A$ and form the ring $S = F \times A \times A$ as before. In order to apply Theorem 4.4 recall that $\mathfrak{m} = 0 \times A \times A$ and $\mathrm{Soc}(S) = 0 \times 0 \times A$, as the multiplication of A is non-degenerate. Hence elements of $\mathrm{Soc}(S)^*$ can be identified with elements of A^*. For any $f \in A^*$ we defined

$$V_f = \{(0,a,b) \in \mathfrak{m} \mid f(Aa) = 0\} = 0 \times I(f) \times A,$$

where $I(f)$ is the largest ideal of A that is contained in $\ker(f)$. Theorem 4.4 says that S satisfies ($\diamond$) if and only if $\mathfrak{m}/V_f \simeq A/I(f)$ is finite dimensional for any $f \in A^*$. From the theory of coalgebras, we borrow the notion of the *finite dual* A° of an algebra, which is the subspace of A^* consisting of the elements $f \in A^*$ that contain an ideal of finite codimension in their kernel. Hence S satisfies ($\diamond$) if and only if $A^\circ = A^*$.

Example 5.4. *The trivial extension $A = F \times V$ of a vector space V (see 4.1) is an example of an algebra A satisfying $A^\circ = A^*$. To see this, note that for any linear subspace U of A, $U \cap V$ is an ideal of A. Thus, if $f \in A^*$, then* $\ker(f) \cap V$ *is an ideal of codimension less or equal to* 2 *and $f \in A^\circ$. In particular for such A, $S = F \times A \times A$ satisfies ($\diamond$). However if V is infinite dimensional, then* $\mathfrak{m}/\mathrm{Soc}(S) \simeq V$ *is not finitely generated as S-module, which shows that the converse of Lemma 4.2(1) does not hold.*

However, it might happen that the kernel of an element of A^* does not contain an ideal of finite codimension as the following example shows.

Example 5.5. *Let A be a commutative unital F-algebra with multiplication μ and $f \in A^*$. Note that the composition $\beta = f \circ \mu$ is a non-degenerate bilinear form if and only if $f(Aa) \neq 0$, for all non-zero $a \in A$. Or, in other words, β is non-degenerate if and only if* $\ker(f)$ *does not contain any non-zero ideal of A. Such a map f can be constructed in case $A = F[x]$ and F has characteristic zero. Suppose $f : F[x] \to F$ is a linear map and let $0 \neq a = \sum_{i=0}^n \lambda_i x^i \in F[x]$ be such that $f(F[x]a) = 0$. For any $m \geq 0$, we have*

$$f(x^m a) = \sum_{i=0}^n \lambda_i f(x^{m+i}) = 0.$$

Thus $v = (\lambda_0, \lambda_1, \ldots, \lambda_n)$ is in the kernel of the linear map given by the matrix:

$$B_n = \begin{pmatrix} f(1) & f(x) & f(x^2) & \cdots & f(x^n) \\ f(x) & f(x^2) & f(x^3) & \cdots & f(x^{n+1}) \\ f(x^2) & f(x^3) & f(x^4) & \cdots & f(x^{n+2}) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ f(x^n) & f(x^{n+1}) & f(x^{n+2}) & \cdots & f(x^{2n+1}) \end{pmatrix}$$

In particular $\det(B_n) = 0$. *Hence if the sequence* $(f(x^n))_{n\in\mathbb{N}}$ *produces a sequence of matrices* (B_n) *that have all non-zero determinant, then for each* $0 \neq a = \sum_{i=0}^{n} \lambda_i x^i \in A$, *there exists* $0 \leq m \leq n$ *such that* $f(x^m a) \neq 0$, *i.e.* $\beta = f \circ \beta$ *is non-degenerate.*

Matrices of the form of B_n *are called Toeplitz or Hankel matrices. A particular example of such a matrix is the Hilbert matrix, which is the matrix*

$$B_{n-1} = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{n} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \cdots & \frac{1}{n+1} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \cdots & \frac{1}{n+2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{n} & \frac{1}{n+1} & \frac{1}{n+2} & \cdots & \frac{1}{2n-1} \end{pmatrix}$$

in case F *has characteristic* 0. *In 1894, Hilbert computed that* $\det(B_{n-1}) = \frac{c_n^4}{c_{2n}}$, *where* $c_n = \prod_{i=1}^{n-1} i^{n-i}$ *(see* [10]*). Hence if we define* $f(x^n) = \frac{1}{n+1}$ *for any* $n \geq 0$, *then the kernel of* f *does not contain any non-zero ideal and the bilinear form* $\beta = f \circ \mu$ *is non-degenerate. Hence* $S = F \times F[x] \times F[x]$ *does not satisfy* $(\diamond)$ *by Proposition 5.3.*

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Paula A. A. B. Carvalho and **Christian Lomp** (Corresponding Author)
Department of Mathematics
Faculty of Science
University of Porto
Rua Campo Alegre, 687
4169-007 Porto, Portugal
e-mails: pbcarval@fc.up.pt (P. A. A. B. Carvalho)
clomp@fc.up.pt (C. Lomp)

Patrick F. Smith
School of Mathematics and Statistics
University of Glasgow
Glasgow G12 8SQ, Scotland
e-mail: patrick.smith@talktalk.net