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NOTE ON THE DW RINGS

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Abstract. In this paper we are mainly concerned with DW rings, i.e., rings in which every ideal is a w-ideal. We give some new classes of DW rings and we show how the concept of DW domains is used to characterize Prüfer domains and Dedekind domains. Namely, we prove that a ring is a Prüfer domain (resp., Dedekind domain) if and only if it a coherent (resp., Noetherian) DW domain with finite weak global dimension.

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1. Introduction

Let R be a domain with quotient field K, and let $\mathfrak{F}(R)$ denote the set of nonzero fractional ideals of R. A map $\star : \mathfrak{F}(R) \to \mathfrak{F}(R)$, $I \mapsto I_\star$, is said to be a star operation on R if the following conditions hold for every nonzero $a \in K$ and $I, J \in \mathfrak{F}(R)$: (1) $(aI)_\star = aI_\star$ and $R_\star = R$; (2) $I \subseteq J$ implies $I_\star \subseteq J_\star$; and (3) $I \subseteq I_\star$ and $(I_\star)_\star = I_\star$. It is common to denote the trivial star operation ($I \mapsto I$) by "d". For any fractional ideal I of R, I is called a fractional $\star$-ideal if $I_\star = I$ and I is called a $\star$-ideal of R if I is an ideal of R and $I_\star = I$.

For $I \in \mathfrak{F}(R)$, set $I^{-1} = \{x \in K \mid xI \subseteq R\}$. An ideal J of R is called a GV-ideal if J is a finitely generated nonzero fractional ideal of R and $J^{-1} = R$. The set of all GV-ideals of R is denoted by $GV(R)$. The w-operation on R is defined by $I_w = \{x \in K \mid \text{ there exists } J \in GV(R) \text{ such that } xJ \subseteq I\}$. One can see that the notion of a w-ideal coincides with the notion of a semi-divisorial ideal introduced by Glaz and Vasconcelos in 1977 [4] which may have some far reaching effects on the theory of star operations. As a star operation, the w-operation was briefly but effectively touched on by Hedstrom and Houston in 1980 under the name of F_∞-operation [5]. Later, this star operation was intensely studied by Wang and McCasland in a more general setting. In particular, Wang and McCasland showed that the w-envelope notion is a very useful tool in studying strong Mori domains [21,22].

For a domain R and a nonzero fractional ideal I of R, the v- and t-closures of I are defined, respectively, by $I_v := (I^{-1})^{-1}$ and $I_t := \cup J_v$, where J ranges over the set of nonzero finitely generated subideals of I. The t- and v-operations are also examples of star operations. It is well-known that for a domain R, $d \leq w \leq t \leq v$ in the sense that for each nonzero fractional ideal I of R, $I = I_d \subseteq I_w \subseteq I_t \subseteq I_v$, and the inclusions may be strict [15]. In [6], Heinzer has initiated the study of domains in which each ideal is divisorial (i.e. each ideal is a v-ideal, or $d = v$) and called them divisorial domains. Inspired by this work, Houston and Zafrullah studied the so-called TV domains, i.e. domains in which each t-ideal is a v-ideal (or, $t = v$, see [8]). Mimouni has studied the TW domains, i.e., domains in which each w-ideal is a t-ideal, or $w = t$ (see [15]) and DW domains, or domains in which each ideal is a w-ideal, i.e. the $d = w$ (see [16]).

In [25], the authors extend the notion of the w-operation to commutative rings with zero-divisors. Let R be a commutative ring (not necessary a domain) and J an ideal of R. Following [25], J is called a GV-ideal if J is finitely generated and the natural homomorphism $\varphi : R \to J^* = \mathrm{Hom}_R(J, R)$ is an isomorphism. Let M be an R-module, and define

$$\mathrm{tor}_{GV}(M) = \{x \in M \mid Jx = 0 \text{ for some} J \in GV(R)\}$$

where $GV(R)$ is the set of GV-ideals of R. It is clear that $\mathrm{tor}_{GV}(M)$ is a submodule of M. Now, M is said to be GV-torsion (resp., GV-torsion-free) if $\mathrm{tor}_{GV}(M) = M$ (resp., $\mathrm{tor}_{GV}(M) = 0$). A GV-torsion-free module M is called a w-module if $\mathrm{Ext}^1_R(R/J, M) = 0$ for any $J \in GV(R)$. Projective modules and reflexive modules are w-modules. In [27], it was shown that flat modules are w-modules.

A commutative ring is called a DW ring if every ideal of R is a w-ideal. Over a domain this last definition coincides with the definition of DW domain in [16].

In Section 2, we give some new classes of DW rings. Section 3 gives new characterizations of Krull domains, Dedekind domains and PvMDs.

Throughout, all rings considered are commutative with unity and all modules are unital. Let R be a ring and M be an R-module. As usual, we use $\mathrm{pd}_R(M)$, $\mathrm{id}_R(M)$, and $\mathrm{fd}_R(M)$ to denote, respectively, the classical projective dimension, injective dimension, and flat dimension of M, and $\mathrm{wdim}(R)$ and $\mathrm{gldim}(R)$ to denote, respectively, the weak and global homological dimensions of R.

2. On DW rings

Let w-Max(R) denote the set of w-ideals of R maximal among proper integral w-ideals of R (maximal w-ideals). By [25, Proposition 3.8], every maximal w-ideal is prime. Let M and N be R-modules and let $f : M \to N$ be a homomorphism.

Following [18], f is called a *w-monomorphism* if $f_{\mathfrak{m}} : M_{\mathfrak{m}} \to N_{\mathfrak{m}}$ is a monomorphism for all $\mathfrak{m} \in w-\mathrm{Max}(R)$. An R-module M is called a w-flat module if the induced map $1 \otimes f : M \otimes A \to M \otimes B$ is a w-monomorphism for any w-monomorphism $f : A \to B$. Certainly, flat modules are w-flat. The notion of w-flat modules appeared first in [17] over a domain and was extended to arbitrary commutative rings in [12]. Recently, modules of this type have received attention in several papers in the literature (see for example [12,19,23]).

By [23, Proposition 1.1], it is clear that over a DW ring, w-flat modules coincide with flat modules. The next result shows that DW rings are the only rings with this property.

Proposition 2.1. *Let R be a ring. The following are equivalent:*

(1) *R is a DW ring.*
(2) *Every w-flat module is flat.*
(3) *Every finitely presented w-flat module is projective.*
(4) *Every GV-torsion module is flat.*
(5) *Every finitely presented GV-torsion module is projective.*
(6) $\mathrm{fd}_R(F) \leq 1$ *for every w-flat module F.*
(7) $\mathrm{fd}_R(F) \leq 1$ *for every finitely presented w-flat module F.*
(8) $\mathrm{fd}_R(F) \leq 1$ *for every GV-torsion module F.*
(9) $\mathrm{fd}_R(F) \leq 1$ *for every finitely presented GV-torsion module F.*

Proof. It is proved in [24, Theorem 2.7] that an R-module M is GV-torsion if and only if $M_{\mathfrak{m}} = 0$ for all maximal w-ideals $\mathfrak{m}$ of R. Hence, by [23, Proposition 1.1], it is clear that a GV-torsion R-module is necessary a w-flat R-module. Hence, the implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (7) $\Rightarrow$ (9), (2) $\Rightarrow$ (4) $\Rightarrow$ (5) $\Rightarrow$ (9), and (2) $\Rightarrow$ (6) $\Rightarrow$ (8) $\Rightarrow$ (9) hold. So, we have only to prove the implication (9) $\Rightarrow$ (1). So, let $J \in GV(R)$. The R-module R/J is a finitely presented GV-torsion module, and so $\mathrm{fd}_R(R/J) \leq 1$. Then, J is a flat R-module, and so a w-ideal. Thus, $J = J_w$. On the other hand, by [25, Proposition 3.5], $J_w = R$. Thus, $GV(R) = \{R\}$, which means that R is a DW ring (by [18, Theorem 3.8]). □

The next proposition gives a new class of DW rings.

Proposition 2.2. *Let R be a ring such that* $\mathrm{fd}_R(I) \leq 1$ *for any injective R-module I. Then R is a DW ring. In particular, if* $\mathrm{wdim}(R) \leq 1$*, then R is a DW ring.*

Proof. Let J be a GV ideal of R and let $E(R/J)$ denote the the injective hull of R/J. Pick a short exact sequence $0 \to K \to F \to R/J \to 0$ where F is a flat R-module. By hypothesis, K is a flat R-module. Then, by [20, Theorem 6.1.17],

$E(R/J)$ is a GV-torsion free R-module. Hence, R/J is a GV-torsion free R-module (as a submodule of $E(R/J)$). Then, $R/J = \{0\}$ (since R/J is also a GV-torsion R-module). Thus, $GV(R) = \{R\}$, which means that R is a DW ring (by [18, Theorem 3.8]). □

Remark 2.3. *Let R be a ring. An R-module M is called Gorenstein flat, if there exists an exact sequence of flat R-modules $\mathbf{F} : ... \to F_1 \to F_0 \to F^0 \to F^1 \to ...$ such that $M \cong \mathrm{Im}(F_0 \to F^0)$ and such that the functor $- \otimes_R I$ leaves $\mathbf{F}$ exact whenever I is an injective R-module. The Gorenstein flat dimension is defined in terms of Gorenstein flat resolutions and denoted by* $\mathrm{Gfd}(-)$ *(see* [7]*). The weak Gorenstein global dimension of R is defined by*

$$\mathrm{wGgldim}(R) = \sup\{\mathrm{Gfd}(M) \mid M \textit{ is an } R\textit{-module}\}.$$

The class of rings indicated in Proposition 2.2 is exactly the class of rings with $\mathrm{wGgldim}(R) \leq 1$ *(by* [13, Theorem 2.12]*).*

Example 2.4. *Let n be a positive integer and set $R := \mathbb{Z}/n\mathbb{Z}$. It is well known that R is a quasi-Frobenius ring. Hence, every injective module is projective (and so flat). Thus, R is a DW ring. Moreover, by* [13, Theorem 3.2], $\mathrm{wGgldim}(R[X]) = 1$ *since* $\mathrm{wGgldim}(R) = 0$ *(by* [13, Theorem 2.12]*). Thus, $R[X]$ is also a DW ring. Moreover, R (and so $R[X]$) has an infinite weak global dimension when n is not square-free.*

Proposition 2.5. *Let R_1 and R_2 be two rings. Then $R_1 \times R_2$ is a DW ring if and only if R_1 and R_2 are DW rings.*

Proof. Follows immediately from [25, Proposition1.2(5)] and [18, Theorem3.8]. □

The next example shows that, for a positive integer $n > 1$, we can always find an example of a ring R with $\sup\{\mathrm{fd}_R(E) \mid E$ is an injective R-module$\} = n$ (that is $\mathrm{wGgldim}(R) = n$ by [13, Theorem 2.12]) and R is not a DW ring.

Example 2.6.
(1) *Let $(R, \mathfrak{m})$ be a regular local ring with* $\mathrm{gldim}(R) = n \geq 2$. *By* [11, 13, Exercise 2, p. 102], $\mathfrak{m}^{-1} = R$ *(since* $\mathrm{grad}(\mathfrak{m}) = n \geq 2$ *where* $\mathrm{grad}(\mathfrak{m})$ *is the grade of* $\mathfrak{m}$*). Thus, $\mathfrak{m} \in GV(R)$, and so R is not DW (by* [18, Theorem 3.8]*). By* [1, Corollary 3.3] *and* [13, Theorems 2.12]*, we have* $\mathrm{wGgldim}(R) = n$.
(2) *Let $T := \mathbb{Z}/4\mathbb{Z}$ which is a quasi-Frobenius ring with infinite weak global dimension. Then,* $\mathrm{wdim}(R \times T) = \infty$, $\mathrm{wGgldim}(R \times T) = n$ *(by* [13, Theorem 3.1] *since* $\mathrm{wGgldim}(T) = 0$*), and $R \times T$ is not a DW ring by Proposition 2.5.*

In [16, Proposition 2.12], Mimouni proved that, for an integral domain R, the polynomial ring $R[X]$ is a DW domain if and only if R is a field. However, even outside the context of integral domains, the ring $R[X]$ may be a DW ring. For example, the ring $\mathbb{Z}/4\mathbb{Z}[X]$ is a DW ring which is not a domain (by Example 2.4).

Proposition 2.7. *Let R be a ring and let X be an indeterminate over R. If the ring $R[X]$ is DW then:*

(1) *every non-zero-divisor element of R is unit (that is $R = T(R)$ where $T(R)$ denotes the total ring of fractions of R).*
(2) ([20, Corollary 6.3.15]) *R is a DW ring.*

Proof. (1) Let a be a non-zero-divisor element of R and set $J = (a, X)$. Then J is a finitely generated regular ideal of $R[X]$. Thus, by [20, Corollary 6.6.9], $J \in GV(R[X])$ if and only if $J^{-1} = R[X]$ (with $J^{-1} := \{u \in T(R[X]) \mid uJ \subseteq R[X]\}$). Let $u \in J^{-1}$. Then $au \in R[X]$, and so $u \in T(R)[X]$. Moreover, $uX \in R[X]$ implies that $u \in R[X]$. Hence, $J^{-1} = R[X]$, and since $R[X]$ is a DW ring, $J = R[X]$. Thus, a is a unit.
(2) Let $J \in GV(R)$, then $J[X] \in GV(R[X]) = \{R[X]\}$. Thus, $J = R$, and so R is DW. □

Recall that a ring R is called Gorenstein Von Neumann regular [14] if $\mathrm{wGgldim}(R) = 0$ (that is every R-module is Gorenstein flat).

Corollary 2.8. *If R is a Gorenstein Von Neumann regular ring, then $T(R) = R$.*

Proof. By [13, Theorem 3.2], we have $\mathrm{wGgldim}(R[X]) = 1$. Hence, by Proposition 2.2 and Remark 2.3, $R[X]$ is a DW ring. Accordingly, by Proposition 2.7, $T(R) = R$. □

3. On DW domains

Let $\star$ be a star operation on a domain R. A fractional ideals I of R is said to be $\star$-invertible if $(II^{-1})_\star = R$. A domain R is called a Krull domain if it satisfies the following three conditions:

(1) for every prime ideal p of R of height one, R_p is a discrete valuation ring;
(2) $R = \cap R_p$, where p ranges over all prime ideals of R of height one;
(3) any nonzero element of R lies in only a finite number of prime ideals of height one.

It is proved that a ring R is a Krull domain if and only if R is a domain over which every nonzero w-ideal is w-invertible (see [20]).

Let I be a nonzero fractional ideal of R. Recall that I is a t-finite (or v-finite) ideal if there exists a finitely generated fractional ideal J of R such that $I = J_t = J_v$; and R is called a Prüfer v-multiplication domain ($PvMD$) if the set of its t-finite t-ideals forms a group under ideal t-multiplication ((($I, J) \mapsto (IJ)_t$). A useful characterizations is that R is a $PvMD$ if and only if each localization at a maximal t-ideal is a valuation domain if and only if every nonzero finitely generated ideal of R is t-invertible if and only if every nonzero finitely generated ideal of R is w-invertible. The class of $PvMD$'s includes Krull domains. A domain R is a v-domain if each nonzero finitely generated ideal of R is v-invertible. An integrally closed domain R is an integral domain whose integral closure in its field of fractions is R itself. We have that

$$\text{Prüfer domain } \rightarrow \; PvMD \; \rightarrow v-\text{domain} \rightarrow \text{ integrally closed domain,}$$

and all arrows are irreversible (see [9]). Clearly, Prüfer domains are DW domains. However, this is not the case for the $PvMD$'s. Moreover, a DW domain needs not to be integrally closed.

Recall that a ring R is called a regular ring if every finitely generated ideal of R has finite projective dimension [3]. This notion, extending Noetherian regularity, was extensively studied for coherent rings. Coherent rings of finite weak global dimensions are regular rings. In particular, Von Neumann regular rings and semi-hereditary rings are regular rings. But, there are coherent rings, even local, with infinite weak global dimension which are regular.

Proposition 3.1. *Let R be a ring. The following are equivalent:*

(1) *R is a coherent DW domain with finite weak dimension.*
(2) *R is a coherent regular and DW domain.*
(3) *R is $PvMD$ and DW domain.*
(4) *R is coherent DW and v-domain.*
(5) *R is coherent integrally closed and DW domain.*
(6) *R is a Prüfer domain.*

Proof. The implications (1) ⇒ (2), (6) ⇒ (4), (6) ⇒ (1), and (4) ⇒ (5) are clear.
(2) ⇒ (3) Follows from [20, Theorem 9.1.13].
(3) ⇒ (6) Follows from [20, Corollary 7.5.10].
(5) ⇒ (3) Follows from the fact that every coherent integrally closed ring is a $PvMD$. □

The condition "domain" in the previous result is necessary. Indeed, a Prüfer ring (every finitely generated regular ideal is invertible) may have an infinite weak global dimension even if it is coherent and DW. As an easy example, we can consider $R = \mathbb{Z}/4\mathbb{Z}$ which is a Noetherian Prüfer ring with infinite (weak) global dimension. However, R is quasi-Frobenius (and so every injective module is projective). Then, by Proposition 2.2, R is a DW ring.

Recall that, according to Zafrullah [26], a domain R is said to be an fgv domain if each finitely generated ideal is divisorial. Clearly R is an fgv domain if and only if the t-operation on R is trivial, that is $t = d$. Trivially, every fgv domain is a DW domain, while the converse is not true ([16, Example 2.1]). Kang [10] showed that a domain R is a $PvMD$ if and only if R is integrally closed with $t = w$ (see also [20, Theorem 7.5.12]). As a consequence of Proposition 3.1, we obtain the following well-known result:

Corollary 3.2. *Let R be a ring. Then, R is a Prüfer domain if and only if R is an fgv integrally closed domain.*

Corollary 3.3. *Let R be a ring. Then, R is a valuation domain if and only if R is a coherent local DW ring with finite weak dimension.*

Proof. It is known that local coherent rings with finite weak dimension are domains ([3, Corollary 4.2.4]). Hence, the desired result follows from Proposition 3.1. □

A domain R is called a Bézout domain if every finitely generated ideal of R is principal, and R is called a GCD domain if for any two nonzero $a, b \in R$, $(a) \cap (b)$ is principal. It is known that an integral domain is a Prüfer GCD-domain if and only if it is a Bézout domain, and that a Prüfer domain need not be a GCD domain. Hence, clearly Bézout domains are DW domains. Now, seen [20, Theorem 7.6.3], we obtain easily the following result.

Proposition 3.4. *Let R be a ring. The following are equivalent:*

(1) *R is a GCD and DW.*
(2) *R is a Bézout domain.*

Let R be a ring and let $f \in R[X]$ be a polynomial in one variable over R. The content of f, denoted by $c(f)$, is the ideal of R generated by the coefficients of f. Let $\star$ be a star operation on a domain R and set $S_\star = \{f \in R[X] \mid (c(f))_\star = R)\}$. It is easy to see that $S_\star$ is a multiplicatively closed set of $R[x]$. In [10], the author introduced and studied the ring $R[X]_{S_\star}$. He proved that a ring R is $PvMD$ if and only if $R[X]_{S_v}$ is a $PvMD$ if and only if $R[X]_{S_v}$ is a Prüer domain if and only if $R[X]_{S_v}$ is a Bézout domain ([10, Theorem 3.7]). In [23], the authors defined the w-Nagata rings (not necessary a domain) to be $R\{X\} := R[X]_{S_w}$. It is proved that $R\{X\}$ is a DW ring ([23, Proposition 4.5]). Note that the notation $R\{X\}$

was used by many authors to denote the ring $R[X]_{S_v}$. However, even if R is a $PvMD$ (which is a subject of our next result), we can have $v \neq w$. For example, [16, Example 2.1(2)] gave an example of a $PvMD$ (and so $w = t$) with $t \neq v$. In [23], the authors introduced and investigated also the w-flat dimensions of modules and rings. Let R be a ring and n be a positive integer. We say that an R-module has a w-flat dimension less than or equal to n, $w-\mathrm{fd}_R(M) \leq n$, if $\mathrm{Tor}_R^{n+1}(M, N)$ is a GV-torsion R-module for all R-modules N. Hence, the w-weak global dimension of R is defined to be

$$w - \mathrm{wdim}(R) = \sup\{w - \mathrm{fd}_R(M) \mid M \text{ is an } R - \text{module}\}.$$

Let R be a ring. An R-module M is said to be of finitely presented type (with respect to the w-operation) if there exists a w-isomorphism $f : M \to N$ (that is $f_{\mathfrak{m}} : M_{\mathfrak{m}} \to N_{\mathfrak{m}}$ is an isomorphism for all $\mathfrak{m} \in w - Max(R)$) where N is a finitely presented R-module. An R-module is called w-coherent if every finitely generated ideal of R is of finitely presented type. Clearly, every coherent ring is w-coherent with equivalence if R is a DW ring. In what follows, we characterize the ring $R\{X\}$ to be a Prüfer domain.

Proposition 3.5. *Let R be a domain. The following are equivalent:*

1. (1) *R is a $PvMD$.*
2. (2) *$R\{X\}$ is a Prüfer domain.*
3. (3) $\mathrm{wdim}(R\{X\}) < \infty$ *and $R\{X\}$ is coherent.*
4. (4) $w - \mathrm{wdim}(R) < \infty$ *and $R[X]$ is w-coherent.*
5. (5) *$R\{X\}$ is a GCD.*
6. (6) *$R\{X\}$ is a Bézout domain.*

Proof. (1) $\Leftrightarrow$ (2) Follows from [20, Theorem 7.5.14].
(2) $\Rightarrow$ (3) Clear.
(3) $\Rightarrow$ (2) Since $R\{X\}$ is a coherent DW domain with finite weak global dimension, it is a Prüfer domain (by Proposition 3.1).
(3) $\Leftrightarrow$ (4) Note that if $R[X]$ is w-coherent then $R\{X\}$ is coherent (by [23, Corollary 4.6]), and under this last condition, $w - \mathrm{wdim}(R) = \mathrm{wdim}(R\{X\})$ (by [23, Proposition 4.2]). So, we have the desired equivalence.
(5) $\Leftrightarrow$ (6) Clear since $R\{X\}$ is a DW domain.
(6) $\Rightarrow$ (2) Clear.
(2) $\Rightarrow$ (6) Let I be a non-zero finitely generated ideal of $R\{X\}$. Since $R\{X\}$ is a Prüfer domain, I is invertible. Using [10, Theorem 2.14], I is principal, and so $R\{X\}$ is a Bézout domain. □

Recall that a ring R is called regular if R is Noetherian such that $\mathrm{gldim}(R_{\mathfrak{m}}) < \infty$ for every maximal ideal $\mathfrak{m}$ of R. Note that if R is a Noetherian ring with $\mathrm{gldim}(R) < \infty$, then R is regular with equivalence when R is local. However, there is an example of a Noetherian domain with infinite weak global dimension which is regular (see [2]).

Proposition 3.6. *Let R be a ring. The following are equivalent:*

(1) *R is a Noetherian DW domain with finite global dimension.*
(2) *R is a regular DW domain.*
(3) *R is a Krull DW domain.*
(4) *R is Dedekind domain.*

Proof. The equivalence (1) $\Leftrightarrow$ (4) follows from Proposition 3.7. Note that Dedekind domains are clearly DW domains. Also, recall that a domain R is Krull domain if and only if every nonzero w-ideal is w-invertible (by [20, Theorem 7.9.3]). Hence, if R is a DW domain, R is Krull if and only if every nonzero ideal is invertible if and only if R is a Dedekind domain. Thus, the equivalence (3) $\Leftrightarrow$ (4) holds.
(1) $\Rightarrow$ (2) Clear.
(2) $\Rightarrow$ (4). If R is a field then the result is trivial. Otherwise, let $\mathfrak{m}$ be a maximal ideal of R. Then, $R_{\mathfrak{m}}$ is a local regular ring. Hence, by Proposition 3.1, $R_{\mathfrak{m}}$ is a Noetherian local Prüfer domain, and so a discrete valuation domain. Hence, R is a Dedekind domain. □

An element p in a ring R is called prime if the principal ideal (p) generated by p is a nonzero prime ideal of R. A unique factorization domain (UFD) is an integral domain R in which every non-zero element can be written as a finite product of prime elements of R. By [20, Theorem 7.9.5], we have the following result.

Proposition 3.7. *Let R be a ring. The following are equivalent:*

(1) *R is a UFD and DW.*
(2) *R is a PID.*

A Strong Mori domain (SM domain, called also w-Noetherian domain) is domain for which the ascending chain condition on w-ideals holds. Clearly, Noetherian domains are Strong Mori domains with equivalence when the domain is DW.

Proposition 3.8. *Let R be a domain. The following are equivalent:*

(1) *R is a Krull domain.*
(2) *$R\{X\}$ is a Dedekind domain.*
(3) $\mathrm{gldim}(R\{X\}) < \infty$ *and $R\{X\}$ is Noetherian.*
(4) $w-\mathrm{wdim}(R) < \infty$ *and R is SM domain.*

(5) *$R\{X\}$ is an UFD.*
(6) *$R\{X\}$ is a PID.*

Proof. (1) ⇒ (2) Since R is a Krull domain, then so is $R\{X\}$ as a localization of $R[X]$. Moreover, $R\{X\}$ is a DW domain (by [23, Proposition 4.5]). Then, $R\{X\}$ is a Dedekind domain (by Proposition 3.6).
(2) ⇒ (3) Clear.
(3) ⇒ (1) Since $R\{X\}$ is Noetherian DW domain with finite global dimension, it is a Dedekind domain (by Proposition 3.6). Hence, by [23, Proposition 4.2], we have that $w-\mathrm{wdim}(R) = \mathrm{wdim}(R\{X\}) = \mathrm{gldim}(R\{X\}) \leq 1$. Then, by [23, Theorem 3.5] and [20, Theorem 6.8.8], R is a $PvMD$ and SM domain. Thus, R is a Krull domain (by [20, Theorem 7.9.3]).
(3) ⇔ (4) Note that R is an SM domain if and only if $R\{X\}$ is a Noetherian domain (by [20, Theorem 6.8.8]), and under this condition, $w-\mathrm{wdim}(R) = \mathrm{wdim}(R\{X\}) = \mathrm{gldim}(R\{X\})$ (by [23, Proposition 4.2]). So, we have the desired equivalence.
(5) ⇔ (6) Clear since $R\{X\}$ is a DW domain.
(6) ⇒ (2) Clear.
(2) ⇒ (6) Let I be a nonzero ideal of $R\{X\}$. Since $R\{X\}$ is a Dedekind domain, I is invertible. Using [10, Theorem 2.14], I is principal, and so $R\{X\}$ is a PID. □

Proposition 3.9. *Let R be a domain and suppose that there is a prime ideal $\mathfrak{p}$ of R such that $\mathfrak{p}R_{\mathfrak{p}} = \mathfrak{p}$. Then, R is a DW domain if and only if $R/\mathfrak{p}$ is a DW domain.*

Proof. Consider the following pullback of rings:

$$\begin{array}{ccc} R & \xrightarrow{\pi_2} & R/\mathfrak{p} \\ \downarrow{\scriptstyle \iota_2} & & \downarrow{\scriptstyle \iota_1} \\ R_{\mathfrak{p}} & \xrightarrow{\pi_1} & R_{\mathfrak{p}}/\mathfrak{p} \end{array}$$

Since $R_{\mathfrak{p}}$ is local, by applying [16, Theorem 3.1(2)] to the above pullback, we get that R is a DW domain if and only if so is $R/\mathfrak{p}$. □

Remark 3.10. (1) *Examples of rings R with a prime ideal $\mathfrak{p}$ such that $\mathfrak{p}R_{\mathfrak{p}} = \mathfrak{p}$ are the non-Noetherian coherent local rings with* $\mathrm{wdim}(R) = \mathrm{gldim}(R) = 2$ *(by* [3, Theorem 6.3.3]*).*
(2) *The requirement $\mathfrak{p}R_{\mathfrak{p}} = \mathfrak{p}$ in the above result can't be dropped. For example, consider the ring $R = k[x, y]$ where k is a field. The ideal (x) of R is prime and $R/(x) \cong k[y]$ which is a DW ring, while R is not.*

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