NOTE ON THE \textit{DW} RINGS

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Received: 19 January 2018; Revised: 26 June 2018; Accepted: 26 June 2018
Communicated by A. Çiğdem Özcan

Abstract. In this paper we are mainly concerned with \textit{DW} rings, i.e., rings in which every ideal is a \textit{w}-ideal. We give some new classes of \textit{DW} rings and we show how the concept of \textit{DW} domains is used to characterize Prüfer domains and Dedekind domains. Namely, we prove that a ring is a Prüfer domain (resp., Dedekind domain) if and only if it a coherent (resp., Noetherian) \textit{DW} domain with finite weak global dimension.

Mathematics Subject Classification (2010): 13D05, 13D07, 13H05
Keywords: \textit{DW} ring and domain, \textit{PvMD}, Krull domain

1. Introduction

Let $R$ be a domain with quotient field $K$, and let $\mathfrak{F}(R)$ denote the set of nonzero fractional ideals of $R$. A map $\star : \mathfrak{F}(R) \to \mathfrak{F}(R)$, $I \mapsto I_\star$, is said to be a star operation on $R$ if the following conditions hold for every nonzero $a \in K$ and $I, J \in \mathfrak{F}(R)$: (1) $(aI)_\star = aI_\star$ and $R_\star = R$; (2) $I \subseteq J$ implies $I_\star \subseteq J_\star$; and (3) $I \subseteq I_\star$ and $(I_\star)_\star = I_\star$. It is common to denote the trivial star operation ($I \mapsto I$) by “$d$”. For any fractional ideal $I$ of $R$, $I$ is called a fractional $\star$-ideal if $I_\star = I$ and $I$ is called a $\star$-ideal of $R$ if $I$ is an ideal of $R$ and $I_\star = I$.

For $I \in \mathfrak{F}(R)$, set $I^{-1} = \{x \in K \mid xI \subseteq R\}$. An ideal $J$ of $R$ is called a $\textit{GV}$-ideal if $J$ is a finitely generated nonzero fractional ideal of $R$ and $J^{-1} = R$. The set of all $\textit{GV}$-ideals of $R$ is denoted by $\textit{GV}(R)$. The $\textit{w}$-operation on $R$ is defined by $I_\textit{w} = \{x \in K \mid$ there exists $J \in \textit{GV}(R)$ such that $xJ \subseteq I\}$. One can see that the notion of a $\textit{w}$-ideal coincides with the notion of a semi-divisorial ideal introduced by Glaz and Vasconcelos in 1977 [4] which may have some far reaching effects on the theory of star operations. As a star operation, the $\textit{w}$-operation was briefly but effectively touched on by Hedstrom and Houston in 1980 under the name of $F_\infty$-operation [5]. Later, this star operation was intensely studied by Wang and McCasland in a more general setting. In particular, Wang and McCasland showed that the $\textit{w}$-envelope notion is a very useful tool in studying strong Mori domains [21,22].
For a domain $R$ and a nonzero fractional ideal $I$ of $R$, the $v$- and $t$-closures of $I$ are defined, respectively, by $I_v := (I^{-1})^{-1}$ and $I_t := \cup J_v$, where $J$ ranges over the set of nonzero finitely generated subideals of $I$. The $t$- and $v$-operations are also examples of star operations. It is well-known that for a domain $R$, $d \leq w \leq t \leq v$ in the sense that for each nonzero fractional ideal $I$ of $R$, $I = I_d \subseteq I_w \subseteq I_t \subseteq I_v$, and the inclusions may be strict [15]. In [6], Heinzer has initiated the study of domains in which each ideal is divisorial (i.e. each ideal is a $v$-ideal, or $d = v$) and called them divisorial domains. Inspired by this work, Houston and Zafrullah studied the so-called $TV$ domains, i.e. domains in which each $t$-ideal is a $v$-ideal (or, $t = v$, see [8]). Mimouni has studied the $TW$ domains, i.e., domains in which each $w$-ideal is a $t$-ideal, or $w = t$ (see [15]) and $DW$ domains, or domains in which each ideal is a $w$-ideal, i.e. the $d = w$ (see [16]).

In [25], the authors extend the notion of the $w$-operation to commutative rings with zero-divisors. Let $R$ be a commutative ring (not necessarily a domain) and $J$ an ideal of $R$. Following [25], $J$ is called a $GV$-ideal if $J$ is finitely generated and the natural homomorphism $\varphi : R \to J^* = \text{Hom}_R(J, R)$ is an isomorphism. Let $M$ be an $R$-module, and define

$$\text{tor}_{GV}(M) = \{ x \in M \mid Jx = 0 \text{ for some } J \in GV(R) \}$$

where $GV(R)$ is the set of $GV$-ideals of $R$. It is clear that $\text{tor}_{GV}(M)$ is a submodule of $M$. Now, $M$ is said to be $GV$-torsion (resp., $GV$-torsion-free) if $\text{tor}_{GV}(M) = M$ (resp., $\text{tor}_{GV}(M) = 0$). A $GV$-torsion-free module $M$ is called a $w$-module if $\text{Ext}_R^1(R/J, M) = 0$ for any $J \in GV(R)$. Projective modules and reflexive modules are $w$-modules. In [27], it was shown that flat modules are $w$-modules.

A commutative ring is called a $DW$ ring if every ideal of $R$ is a $w$-ideal. Over a domain this last definition coincides with the definition of $DW$ domain in [16].

In Section 2, we give some new classes of $DW$ rings. Section 3 gives new characterizations of Krull domains, Dedekind domains and $Pv$MDs.

Throughout, all rings considered are commutative with unity and all modules are unital. Let $R$ be a ring and $M$ be an $R$-module. As usual, we use $\text{pd}_R(M)$, $\text{id}_R(M)$, and $\text{fd}_R(M)$ to denote, respectively, the classical projective dimension, injective dimension, and flat dimension of $M$, and $\text{wdim}(R)$ and $\text{gldim}(R)$ to denote, respectively, the weak and global homological dimensions of $R$.

2. On $DW$ rings

Let $w-\text{Max}(R)$ denote the set of $w$-ideals of $R$ maximal among proper integral $w$-ideals of $R$ (maximal $w$-ideals). By [25, Proposition 3.8], every maximal $w$-ideal is prime. Let $M$ and $N$ be $R$-modules and let $f : M \to N$ be a homomorphism.
Following [18], $f$ is called a $w$-monomorphism if $f_m : M_m \to N_m$ is a monomorphism for all $m \in w\text{-Max}(R)$. An $R$-module $M$ is called a $w$-flat module if the induced map $1 \otimes f : M \otimes A \to M \otimes B$ is a $w$-monomorphism for any $w$-monomorphism $f : A \to B$. Certainly, flat modules are $w$-flat. The notion of $w$-flat modules appeared first in [17] over a domain and was extended to arbitrary commutative rings in [12]. Recently, modules of this type have received attention in several papers in the literature (see for example [12,19,23]).

By [23, Proposition 1.1], it is clear that over a DW ring, $w$-flat modules coincide with flat modules. The next result shows that DW rings are the only rings with this property.

**Proposition 2.1.** Let $R$ be a ring. The following are equivalent:

1. $R$ is a DW ring.
2. Every $w$-flat module is flat.
3. Every finitely presented $w$-flat module is projective.
4. Every GV-torsion module is flat.
5. Every finitely presented GV-torsion module is projective.
6. $\text{fd}_R(F) \leq 1$ for every $w$-flat module $F$.
7. $\text{fd}_R(F) \leq 1$ for every finitely presented $w$-flat module $F$.
8. $\text{fd}_R(F) \leq 1$ for every GV-torsion module $F$.
9. $\text{fd}_R(F) \leq 1$ for every finitely presented GV-torsion module $F$.

**Proof.** It is proved in [24, Theorem 2.7] that an $R$-module $M$ is GV-torsion if and only if $M_m = 0$ for all maximal $w$-ideals $m$ of $R$. Hence, by [23, Proposition 1.1], it is clear that a GV-torsion $R$-module is necessary a $w$-flat $R$-module. Hence, the implications $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (7) \Rightarrow (9)$, $(2) \Rightarrow (4) \Rightarrow (5) \Rightarrow (9)$, and $(2) \Rightarrow (6) \Rightarrow (8) \Rightarrow (9)$ hold. So, we have only to prove the implication $(9) \Rightarrow (1)$. So, let $J \in \text{GV}(R)$. The $R$-module $R/J$ is a finitely presented GV-torsion module, and so $\text{fd}_R(R/J) \leq 1$. Then, $J$ is a flat $R$-module, and so a $w$-ideal. Thus, $J = J_w$. On the other hand, by [25, Proposition 3.5], $J_w = R$. Thus, $\text{GV}(R) = \{R\}$, which means that $R$ is a DW ring (by [18, Theorem 3.8]).

The next proposition gives a new class of DW rings.

**Proposition 2.2.** Let $R$ be a ring such that $\text{fd}_R(I) \leq 1$ for any injective $R$-module $I$. Then $R$ is a DW ring. In particular, if $\text{wdim}(R) \leq 1$, then $R$ is a DW ring.

**Proof.** Let $J$ be a GV ideal of $R$ and let $E(R/J)$ denote the the injective hull of $R/J$. Pick a short exact sequence $0 \to K \to F \to R/J \to 0$ where $F$ is a flat $R$-module. By hypothesis, $K$ is a flat $R$-module. Then, by [20, Theorem 6.1.17],

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$E(R/J)$ is a $GV$-torsion free $R$-module. Hence, $R/J$ is a $GV$-torsion free $R$-module (as a submodule of $E(R/J)$). Then, $R/J = \{0\}$ (since $R/J$ is also a $GV$-torsion $R$-module). Thus, $GV(R) = \{R\}$, which means that $R$ is a $DW$ ring (by [18, Theorem 3.8]). □

Remark 2.3. Let $R$ be a ring. An $R$-module $M$ is called Gorenstein flat, if there exists an exact sequence of flat $R$-modules $F : \ldots \rightarrow F_1 \rightarrow F_0 \rightarrow F^0 \rightarrow F^1 \rightarrow \ldots$ such that $M \cong \text{Im}(F_0 \rightarrow F^0)$ and such that the functor $- \otimes_R I$ leaves $F$ exact whenever $I$ is an injective $R$-module. The Gorenstein flat dimension is defined in terms of Gorenstein flat resolutions and denoted by $\text{Gfd}(-)$ (see [7]). The weak Gorenstein global dimension of $R$ is defined by

$$\text{wGgd}(R) = \sup\{\text{Gfd}(M) \mid M \text{ is an } R\text{-module}\}.$$  

The class of rings indicated in Proposition 2.2 is exactly the class of rings with $\text{wGgd}(R) \leq 1$ (by [13, Theorem 2.12]).

Example 2.4. Let $n$ be a positive integer and set $R := \mathbb{Z}/n\mathbb{Z}$. It is well known that $R$ is a quasi-Frobenius ring. Hence, every injective module is projective (and so flat). Thus, $R$ is a $DW$ ring. Moreover, by [13, Theorem 3.2], $\text{wGgd}(R[X]) = 1$ since $\text{wGgd}(R) = 0$ (by [13, Theorem 2.12]). Thus, $R[X]$ is also a $DW$ ring. Moreover, $R$ (and so $R[X]$) has an infinite weak global dimension when $n$ is not square-free.

Proposition 2.5. Let $R_1$ and $R_2$ be two rings. Then $R_1 \times R_2$ is a $DW$ ring if and only if $R_1$ and $R_2$ are $DW$ rings.

Proof. Follows immediately from [25, Proposition 1.2(5)] and [18, Theorem 3.8]. □

The next example shows that, for a positive integer $n > 1$, we can always find an example of a ring $R$ with $\sup\{\text{fd}_R(E) \mid E$ is an injective $R$-module\} = $n$ (that is $\text{wGgd}(R) = n$ by [13, Theorem 2.12]) and $R$ is not a $DW$ ring.

Example 2.6. (1) Let $(R, \mathfrak{m})$ be a regular local ring with $\text{gldim}(R) = n \geq 2$. By [11, 13, Exercise 2, p. 102], $\mathfrak{m}^{-1} = R$ (since $\text{grad}(\mathfrak{m}) = n \geq 2$ where $\text{grad}(\mathfrak{m})$ is the grade of $\mathfrak{m}$). Thus, $\mathfrak{m} \in GV(R)$, and so $R$ is not $DW$ (by [18, Theorem 3.8]). By [1, Corollary 3.3] and [13, Theorems 2.12], we have $\text{wGgd}(R) = n$.

(2) Let $T := \mathbb{Z}/4\mathbb{Z}$ which is a quasi-Frobenius ring with infinite weak global dimension. Then, $\text{wdim}(R \times T) = \infty$, $\text{wGgd}(R \times T) = n$ (by [13, Theorem 3.1] since $\text{wGgd}(T) = 0$), and $R \times T$ is not a $DW$ ring by Proposition 2.5.
In [16, Proposition 2.12], Mimouni proved that, for an integral domain $R$, the polynomial ring $R[X]$ is a DW domain if and only if $R$ is a field. However, even outside the context of integral domains, the ring $R[X]$ may be a DW ring. For example, the ring $\mathbb{Z}/4\mathbb{Z}[X]$ is a DW ring which is not a domain (by Example 2.4).

**Proposition 2.7.** Let $R$ be a ring and let $X$ be an indeterminate over $R$. If the ring $R[X]$ is DW then:

1. every non-zero-divisor element of $R$ is unit (that is $R = T(R)$ where $T(R)$ denotes the total ring of fractions of $R$).
2. ([20, Corollary 6.3.15]) $R$ is a DW ring.

**Proof.** (1) Let $a$ be a non-zero-divisor element of $R$ and set $J = (a, X)$. Then $J$ is a finitely generated regular ideal of $R[X]$. Thus, by [20, Corollary 6.6.9], $J \in GV(R[X])$ if and only if $J^{-1} = R[X]$ (with $J^{-1} := \{u \in T(R[X]) \mid uJ \subseteq R[X]\}$). Let $u \in J^{-1}$. Then $au \in R[X]$, and so $u \in T(R[X])$. Moreover, $uX \in R[X]$ implies that $u \in R[X]$. Hence, $J^{-1} = R[X]$, and since $R[X]$ is a DW ring, $J = R[X]$. Thus, $a$ is a unit.

(2) Let $J \in GV(R)$, then $J[X] \in GV(R[X]) = \{R[X]\}$. Thus, $J = R$, and so $R$ is DW.

Recall that a ring $R$ is called Gorenstein Von Neumann regular [14] if $wGgldim(R) = 0$ (that is every $R$-module is Gorenstein flat).

**Corollary 2.8.** If $R$ is a Gorenstein Von Neumann regular ring, then $T(R) = R$.

**Proof.** By [13, Theorem 3.2], we have $wGgldim(R[X]) = 1$. Hence, by Proposition 2.2 and Remark 2.3, $R[X]$ is a DW ring. Accordingly, by Proposition 2.7, $T(R) = R$.

3. On DW domains

Let $\ast$ be a star operation on a domain $R$. A fractional ideals $I$ of $R$ is said to be $\ast$-invertible if $(II^{-1})_\ast = R$. A domain $R$ is called a Krull domain if it satisfies the following three conditions:

1. for every prime ideal $p$ of $R$ of height one, $R_p$ is a discrete valuation ring;
2. $R = \cap R_p$, where $p$ ranges over all prime ideals of $R$ of height one;
3. any nonzero element of $R$ lies in only a finite number of prime ideals of height one.

It is proved that a ring $R$ is a Krull domain if and only if $R$ is a domain over which every nonzero $w$-ideal is $w$-invertible (see [20]).
Let $I$ be a nonzero fractional ideal of $R$. Recall that $I$ is a $t$-finite (or $v$-finite) ideal if there exists a finitely generated fractional ideal $J$ of $R$ such that $I = J_t = J_v$; and $R$ is called a Prüfer $v$-multiplication domain ($PvMD$) if the set of its $t$-finite $t$-ideals forms a group under ideal $t$-multiplication ($((I, J) \mapsto (IJ)_t)$). A useful characterizations is that $R$ is a $PvMD$ if and only if each localization at a maximal $t$-ideal is a valuation domain if and only if every nonzero finitely generated ideal of $R$ is $t$-invertible if and only if every nonzero finitely generated ideal of $R$ is $w$-invertible. The class of $PvMD$’s includes Krull domains. A domain $R$ is a $v$-domain if each nonzero finitely generated ideal of $R$ is $v$-invertible. An integrally closed domain $R$ is an integral domain whose integral closure in its field of fractions is $R$ itself. We have that

Prüfer domain $\rightarrow$ $PvMD$ $\rightarrow$ $v$-domain $\rightarrow$ integrally closed domain,

and all arrows are irreversible (see [9]). Clearly, Prüfer domains are $DW$ domains. However, this is not the case for the $PvMD$’s. Moreover, a $DW$ domain needs not to be integrally closed.

Recall that a ring $R$ is called a regular ring if every finitely generated ideal of $R$ has finite projective dimension [3]. This notion, extending Noetherian regularity, was extensively studied for coherent rings. Coherent rings of finite weak global dimensions are regular rings. In particular, Von Neumann regular rings and semi-hereditary rings are regular rings. But, there are coherent rings, even local, with infinite weak global dimension which are regular.

**Proposition 3.1.** Let $R$ be a ring. The following are equivalent:

1. $R$ is a coherent $DW$ domain with finite weak dimension.
2. $R$ is a coherent regular and $DW$ domain.
3. $R$ is $PvMD$ and $DW$ domain.
4. $R$ is coherent $DW$ and $v$-domain.
5. $R$ is coherent integrally closed and $DW$ domain.
6. $R$ is a Prüfer domain.

**Proof.** The implications (1) $\Rightarrow$ (2), (6) $\Rightarrow$ (4), (6) $\Rightarrow$ (1), and (4) $\Rightarrow$ (5) are clear. (2) $\Rightarrow$ (3) Follows from [20, Theorem 9.1.13]. (3) $\Rightarrow$ (6) Follows from [20, Corollary 7.5.10]. (5) $\Rightarrow$ (3) Follows from the fact that every coherent integrally closed ring is a $PvMD$. □

The condition “domain” in the previous result is necessary. Indeed, a Prüfer ring (every finitely generated regular ideal is invertible) may have an infinite weak global
dimension even if it is coherent and $DW$. As an easy example, we can consider $R = \mathbb{Z}/4\mathbb{Z}$ which is a Noetherian Prüfer ring with infinite (weak) global dimension. However, $R$ is quasi-Frobenius (and so every injective module is projective). Then, by Proposition 2.2, $R$ is a $DW$ ring.

Recall that, according to Zafrullah [26], a domain $R$ is said to be an $fgv$ domain if each finitely generated ideal is divisorial. Clearly $R$ is an $fgv$ domain if and only if the $t$-operation on $R$ is trivial, that is $t = d$. Trivially, every $fgv$ domain is a $DW$ domain, while the converse is not true ([16, Example 2.1]). Kang [10] showed that a domain $R$ is a $PvMD$ if and only if $R$ is integrally closed with $t = w$ (see also [20, Theorem 7.5.12]). As a consequence of Proposition 3.1, we obtain the following well-known result:

**Corollary 3.2.** Let $R$ be a ring. Then, $R$ is a Prüfer domain if and only if $R$ is an $fgv$ integrally closed domain.

**Corollary 3.3.** Let $R$ be a ring. Then, $R$ is a valuation domain if and only if $R$ is a coherent local $DW$ ring with finite weak dimension.

**Proof.** It is known that local coherent rings with finite weak dimension are domains ([3, Corollary 4.2.4]). Hence, the desired result follows from Proposition 3.1. □

A domain $R$ is called a Bézout domain if every finitely generated ideal of $R$ is principal, and $R$ is called a $GCD$ domain if for any two nonzero $a, b \in R$, $(a) \cap (b)$ is principal. It is known that an integral domain is a Prüfer $GCD$-domain if and only if it is a Bézout domain, and that a Prüfer domain need not be a $GCD$ domain. Hence, clearly Bézout domains are $DW$ domains. Now, seen [20, Theorem 7.6.3], we obtain easily the following result.

**Proposition 3.4.** Let $R$ be a ring. The following are equivalent:

1. $R$ is a $GCD$ and $DW$.
2. $R$ is a Bézout domain.

Let $R$ be a ring and let $f \in R[X]$ be a polynomial in one variable over $R$. The content of $f$, denoted by $c(f)$, is the ideal of $R$ generated by the coefficients of $f$. Let $\star$ be a star operation on a domain $R$ and set $S_\star = \{f \in R[X] \mid (c(f))_\star = R\}$. It is easy to see that $S_\star$ is a multiplicatively closed set of $R[x]$. In [10], the author introduced and studied the ring $R[X]_{S_\star}$. He proved that a ring $R$ is $PvMD$ if and only if $R[X]_{S_\star}$ is a $PvMD$ if and only if $R[X]_{S_\star}$ is a Prüfer domain if and only if $R[X]_{S_\star}$ is a Bézout domain ([10, Theorem 3.7]). In [23], the authors defined the $w$-Nagata rings (not necessary a domain) to be $R\{X\} := R[X]_{S_w}$. It is proved that $R\{X\}$ is a $DW$ ring ([23, Proposition 4.5]). Note that the notation $R\{X\}$
was used by many authors to denote the ring $R[X]_{S_v}$. However, even if $R$ is a $PvMD$ (which is a subject of our next result), we can have $v \neq w$. For example, [16, Example 2.1(2)] gave an example of a $PvMD$ (and so $w = t$) with $t \neq v$. In [23], the authors introduced and investigated also the $w$-flat dimensions of modules and rings. Let $R$ be a ring and $n$ be a positive integer. We say that an $R$-module $M$ has a $w$-flat dimension less than or equal to $n$, $w-\text{fd}_R(M) \leq n$, if $\text{Tor}^{n+1}_R(M, N)$ is a $GV$-torsion $R$-module for all $R$-modules $N$. Hence, the $w$-weak global dimension of $R$ is defined to be

$$w-\text{wdim}(R) = \sup\{w-\text{fd}_R(M) \mid M \text{ is an } R-\text{module}\}.$$ 

Let $R$ be a ring. An $R$-module $M$ is said to be of finitely presented type (with respect to the $w$-operation) if there exists a $w$-isomorphism $f : M \to N$ (that is $f_m : M_m \to N_m$ is an isomorphism for all $m \in w-\text{Max}(R)$) where $N$ is a finitely presented $R$-module. An $R$-module is called $w$-coherent if every finitely generated ideal of $R$ is of finitely presented type. Clearly, every coherent ring is $w$-coherent with equivalence if $R$ is a $DW$ ring. In what follows, we characterize the ring $R\{X\}$ to be a Prüfer domain.

**Proposition 3.5.** Let $R$ be a domain. The following are equivalent:

1. $R$ is a $PvMD$.
2. $R\{X\}$ is a Prüfer domain.
3. $w-\text{dim}(R\{X\}) < \infty$ and $R\{X\}$ is coherent.
4. $w-\text{dim}(R) < \infty$ and $R\{X\}$ is $w$-coherent.
5. $R\{X\}$ is a GCD.
6. $R\{X\}$ is a Bézout domain.

**Proof.** (1) $\Leftrightarrow$ (2) Follows from [20, Theorem 7.5.14].

(2) $\Rightarrow$ (3) Since $R\{X\}$ is a coherent $DW$ domain with finite weak global dimension, it is a Prüfer domain (by Proposition 3.1).

(3) $\Rightarrow$ (2) Since $R\{X\}$ is a coherent $DW$ domain with finite weak global dimension, it is a Prüfer domain (by Proposition 3.1).

(3) $\Leftrightarrow$ (4) Note that if $R\{X\}$ is $w$-coherent then $R\{X\}$ is coherent (by [23, Corollary 4.6]), and under this last condition, $w-\text{wdim}(R) = \text{wdim}(R\{X\})$ (by [23, Proposition 4.2]). So, we have the desired equivalence.

(5) $\Leftrightarrow$ (6) Clear since $R\{X\}$ is a $DW$ domain.

(6) $\Rightarrow$ (2) Clear.

(2) $\Rightarrow$ (6) Let $I$ be a non-zero finitely generated ideal of $R\{X\}$. Since $R\{X\}$ is a Prüfer domain, $I$ is invertible. Using [10, Theorem 2.14], $I$ is principal, and so $R\{X\}$ is a Bézout domain. \qed
Recall that a ring $R$ is called regular if $R$ is Noetherian such that $\text{gldim}(R_m) < \infty$ for every maximal ideal $m$ of $R$. Note that if $R$ is a Noetherian ring with $\text{gldim}(R) < \infty$, then $R$ is regular with equivalence when $R$ is local. However, there is an example of a Noetherian domain with infinite weak global dimension which is regular (see [2]).

**Proposition 3.6.** Let $R$ be a ring. The following are equivalent:

1. $R$ is a Noetherian DW domain with finite global dimension.
2. $R$ is a regular DW domain.
3. $R$ is a Krull DW domain.
4. $R$ is Dedekind domain.

**Proof.** The equivalence (1) ⇔ (4) follows from Proposition 3.7. Note that Dedekind domains are clearly DW domains. Also, recall that a domain $R$ is Krull domain if and only if every nonzero $w$-ideal is $w$-invertible (by [20, Theorem 7.9.3]). Hence, if $R$ is a DW domain, $R$ is Krull if and only if every nonzero ideal is invertible if and only if $R$ is a Dedekind domain. Thus, the equivalence (3) ⇔ (4) holds.

(1) ⇒ (2). Clear.

(2) ⇒ (4). If $R$ is a field then the result is trivial. Otherwise, let $m$ be a maximal ideal of $R$. Then, $R_m$ is a local regular ring. Hence, by Proposition 3.1, $R_m$ is a Noetherian local Prüfer domain, and so a discrete valuation domain. Hence, $R$ is a Dedekind domain. $\square$

An element $p$ in a ring $R$ is called prime if the principal ideal $(p)$ generated by $p$ is a nonzero prime ideal of $R$. A unique factorization domain (UFD) is an integral domain $R$ in which every non-zero element can be written as a finite product of prime elements of $R$. By [20, Theorem 7.9.5], we have the following result.

**Proposition 3.7.** Let $R$ be a ring. The following are equivalent:

1. $R$ is a UFD and DW.
2. $R$ is a PID.

A Strong Mori domain (SM domain, called also $w$-Noetherian domain) is domain for which the ascending chain condition on $w$-ideals holds. Clearly, Noetherian domains are Strong Mori domains with equivalence when the domain is DW.

**Proposition 3.8.** Let $R$ be a domain. The following are equivalent:

1. $R$ is a Krull domain.
2. $R\{X\}$ is a Dedekind domain.
3. $\text{gldim}(R\{X\}) < \infty$ and $R\{X\}$ is Noetherian.
4. $w - \text{wdim}(R) < \infty$ and $R$ is SM domain.
(5) $R\{X\}$ is an UFD.
(6) $R\{X\}$ is a PID.

**Proof.** (1) $\Rightarrow$ (2) Since $R$ is a Krull domain, then so is $R\{X\}$ as a localization of $R[X]$. Moreover, $R\{X\}$ is a DW domain (by [23, Proposition 4.5]). Then, $R\{X\}$ is a Dedekind domain (by Proposition 3.6).

(2) $\Rightarrow$ (3) Clear.

(3) $\Rightarrow$ (1) Since $R\{X\}$ is Noetherian DW domain with finite global dimension, it is a Dedekind domain (by Proposition 3.6). Hence, by [23, Proposition 4.2], we have that $w - \text{wdim}(R) = \text{wdim}(R\{X\}) = \text{gldim}(R\{X\}) \leq 1$. Then, by [23, Theorem 3.5] and [20, Theorem 6.8.8], $R$ is a PeMD and SM domain. Thus, $R$ is a Krull domain (by [20, Theorem 7.9.3]).

(3) $\Leftrightarrow$ (4) Note that $R$ is an SM domain if and only if $R\{X\}$ is a Noetherian domain (by [20, Theorem 6.8.8]), and under this condition, $w - \text{wdim}(R) = \text{wdim}(R\{X\}) = \text{gldim}(R\{X\})$ (by [23, Proposition 4.2]). So, we have the desired equivalence.

(5) $\Leftrightarrow$ (6) Clear since $R\{X\}$ is a DW domain.

(6) $\Rightarrow$ (2) Clear.

(2) $\Rightarrow$ (6) Let $I$ be a nonzero ideal of $R\{X\}$. Since $R\{X\}$ is a Dedekind domain, $I$ is invertible. Using [10, Theorem 2.14], $I$ is principal, and so $R\{X\}$ is a PID. □

**Proposition 3.9.** Let $R$ be a domain and suppose that there is a prime ideal $p$ of $R$ such that $pR_p = p$. Then, $R$ is a DW domain if and only if $R/p$ is a DW domain.

**Proof.** Consider the following pullback of rings:

$$
\begin{array}{ccc}
R & \xrightarrow{\pi_2} & R/p \\
\downarrow{\iota_2} & & \downarrow{\iota_1} \\
R_p & \xrightarrow{\pi_1} & R_p/p
\end{array}
$$

Since $R_p$ is local, by applying [16, Theorem 3.1(2)] to the above pullback, we get that $R$ is a DW domain if and only if so is $R/p$. □

**Remark 3.10.**
(1) Examples of rings $R$ with a prime ideal $p$ such that $pR_p = p$ are the non-Noetherian coherent local rings with $\text{wdim}(R) = \text{gldim}(R) = 2$ (by [3, Theorem 6.3.3]).

(2) The requirement $pR_p = p$ in the above result can’t be dropped. For example, consider the ring $R = k[x, y]$ where $k$ is a field. The ideal $(x)$ of $R$ is prime and $R/(x) \cong k[y]$ which is a DW ring, while $R$ is not.
Acknowledgement. The authors would like to thank the anonymous referees for their comments and suggestions, which have substantially improved the paper.

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