ON THE EXTENDED TOTAL GRAPH OF MODULES OVER COMMUTATIVE RINGS

F. Esmaeili Khalil Saraei and E. Navidinia

Received: 23 February 2018; Revised: 7 September 2018; Accepted: 7 September 2018
Communicated by A. Çiğdem Özcan

Abstract. Let $M$ be a module over a commutative ring $R$ and $U$ a nonempty proper subset of $M$. In this paper, the extended total graph, denoted by $ET_U(M)$, is presented, where $U$ is a multiplicative-prime subset of $M$. It is the graph with all elements of $M$ as vertices, and for distinct $m, n \in M$, the vertices $m$ and $n$ are adjacent if and only if $rm + sn \in U$ for some $r, s \in R \setminus (U : M)$. We also study the two (induced) subgraphs $ET_U(U)$ and $ET_U(M \setminus U)$, with vertices $U$ and $M \setminus U$, respectively. Among other things, the diameter and the girth of $ET_U(M)$ are also studied.

Mathematics Subject Classification (2010): 13C13, 05C75, 13A15
Keywords: Total graph, prime submodule, multiplicative-prime subset

1. Introduction

Throughout this paper, $R$ is a commutative ring with nonzero identity and $M$ is a unitary $R$-module. Recently, there has been considerable attention in the literature to associating graphs with algebraic structures (see [1], [2], [3], [5], [6], [7], [8], and [9]). Anderson and Badawi in [4] defined a nonempty proper subset $H$ of $R$ to be a multiplicative-prime subset of $R$ if the following two conditions hold: (i) $rs \in H$ for every $r \in H$ and $s \in R$; (ii) if $rs \in H$ for some $r, s \in R$, then either $r \in H$ or $s \in H$. They introduced the notion of the generalized total graph of a commutative ring $GT_H(R)$ with the vertices all elements of $R$, and two distinct vertices $x, y \in R$ are adjacent if and only if $x + y \in H$, where $H$ is a multiplicative-prime subset of $R$.

Let $R$ be a commutative ring and $U$ be a nonempty subset of an $R$-module $M$. The subset $\{r \in R : rM \subseteq U\}$ will be denoted by $(U :_R M)$ or $(U : M)$. It is clear that if $U$ is a submodule of $M$, then $(U : M)$ is an ideal of $R$. We define a nonempty subset $U$ of $M$ to be a multiplicative-prime subset of $M$ if the following two conditions hold: (i) $rm \in U$ for every $r \in R$ and $m \in U$; (ii) if $sx \in U$ for some $s \in R$ and $x \in M$, then $x \in U$ or $s \in (U : M)$. Note that if $U$ is a...
multiplicative-prime submodule of $M$, then $U$ is necessarily a prime submodule of $M$. One can show that if $U$ is a multiplicative-prime subset of $M$, then $(U : M)$ is a multiplicative-prime subset of $R$.

The total graph of a module $M$ with respect to a multiplicative-prime subset $U$ (denoted by $GT_U(M)$) was introduced in [10]. The set of vertices of $GT_U(M)$ is all elements of $M$, and two distinct vertices $m$ and $n$ are adjacent whenever $m + n \in U$. In this paper, we introduce an extension of the graph $GT_U(M)$, denoted by $ET_U(M)$, such that its vertex set consists of all elements of $M$ and for distinct $m, n \in M$, the vertices $m$ and $n$ are adjacent if and only if $rm + sn \in U$ for some $r, s \in R \setminus (U : M)$, where $U$ is a multiplicative-prime subset of $M$.

Let $ET_U(U)$ be the (induced) subgraph of $ET_U(M)$ with vertex set $U$, and let $ET_U(M \setminus U)$ be the (induced) subgraph $ET_U(M)$ with vertices consisting of $M \setminus U$. Obviously, the total graph $GT_U(M)$ is a subgraph of $ET_U(M)$. It follows that each edge (path) of $GT_U(M)$ is an edge (path) of $ET_U(M)$. The study of $ET_U(M)$ breaks naturally into two cases depending on whether or not $U$ is a submodule of $M$. In the second section, we handle the case when $U$ is a submodule of $M$; in the third section, we do the case when $U$ is not a submodule of $M$. For every case, we characterize the girth and diameter of $ET_U(M)$, $ET_U(U)$ and $ET_U(M \setminus U)$.

We begin with some notation, and definitions. For a graph $\Gamma$, by $E(\Gamma)$ and $V(\Gamma)$, we mean the set of all edges and vertices, respectively. We recall that a graph is connected if there exists a path connecting any two of its distinct vertices. At the other extreme, we say that a graph is totally disconnected if no two vertices of this graph are adjacent. The distance between two distinct vertices $a$ and $b$, denoted by $d(a, b)$, is the length of a shortest path connecting them (if such a path does not exist, then $d(a, b) = \infty$). We also define $d(a, a) = 0$. The diameter of a graph $\Gamma$, denoted by $\text{diam}(\Gamma)$, is equal to $\sup\{d(a, b) : a, b \in V(\Gamma)\}$. A graph is complete if it is connected with diameter less than or equal to one. The girth of a graph $\Gamma$, denoted by $\text{gr}(\Gamma)$, is the length of a shortest cycle in $\Gamma$, provided $\Gamma$ contains a cycle; otherwise, $\text{gr}(\Gamma) = \infty$. We denote the complete graph on $n$ vertices by $K^n$ and the complete bipartite graph on $m$ and $n$ vertices by $K^{m,n}$ (we allow $m$ and $n$ to be infinite cardinals). For a graph $\Gamma$, the degree of a vertex $v$ in $\Gamma$, denoted by $\text{deg}(v)$, is the number of edges of $\Gamma$ incident with $v$. We say that two (induced) subgraphs $\Gamma_1$ and $\Gamma_2$ of $\Gamma$ are disjoint if $\Gamma_1$ and $\Gamma_2$ have no common vertices and no vertex of $\Gamma_1$ is adjacent (in $\Gamma$) to some vertex of $\Gamma_2$. 

---

78

F. ESMAEILI KHALIL SARAEEI AND E. NAVIDINIA
2. The case when $U$ is a submodule of $M$

In this section, we study the case when $U$ is a submodule of $M$. It is clear that if $U$ is a submodule of $M$, then $U$ is a prime submodule of $M$. If $U = M$, then it is clear that $ET_U(M)$ is a complete graph and $ET_U(M)$ is a disconnected graph when $U = 0$ and $|M| \geq 2$. So we may assume that $U \neq 0$ and $U \neq M$.

First, we begin with the following example that shows we may have $ET_U(M) \neq GT_U(M)$.

**Example 2.1.** Let $R = \mathbb{Z}$ and $M = \mathbb{Z}_{10}$. Set $U = \{0, 5\}$. It is clear that $U$ is a submodule of $M$ and $(U : M) = 5\mathbb{Z}$. Since $1 + 3 = 4 \notin U$, so $1 - 3$ is not an edge in $GT_U(M)$. But $2(1) + 1(3) = 5 \in U$ and $2, 1 \in R \setminus (U : M)$. Thus $1 - 3$ is an edge in $ET_U(M)$. Hence $ET_U(M) \neq GT_U(M)$.

The main goal of this section is a general structure theorem (Theorem 2.4) for $ET_U(M \setminus U)$ when $U$ is a submodule of $M$. But first, we record the trivial observation that if $U$ is a submodule of $M$, then $ET_U(U)$ is a complete subgraph of $ET_U(M)$ and is disjoint from $ET_U(M \setminus U)$. Thus we will concentrate on the subgraph $ET_U(M \setminus U)$ throughout this section.

**Theorem 2.2.** Let $M$ be a module over a commutative ring $R$ and $U$ be a prime submodule of $M$. Then $ET_U(U)$ is a complete subgraph of $ET_U(M)$ and is disjoint from $ET_U(M \setminus U)$. In particular, $ET_U(U)$ is connected and $ET_U(M)$ is disconnected.

**Proof.** Let $m, n \in U$. Then it is clear that $m + n \in U$ since $U$ is a submodule of $M$. If $x \in U$ is adjacent to $y \in M \setminus U$, then $rx + sy \in U$ for some $r, s \in R \setminus (U : M)$. This implies that $sy \in U$; so $y \in U$ or $s \in (U : M)$ since $U$ is a prime submodule, which is a contradiction. The “in particular” statement is clear. \qed

**Theorem 2.3.** Let $M$ be a module over a commutative ring $R$ and $U$ be a prime submodule of $M$. Let $G$ be an induced subgraph of $ET_U(M \setminus U)$, and $m$ and $m'$ be distinct nonadjacent vertices of $G$ that are connected by a path in $G$. Then there exists a path in $G$ of length 2 between $m$ and $m'$. In particular, if $ET_U(M \setminus U)$ is connected, then $diam(ET_U(M \setminus U)) \leq 2$.

**Proof.** (1) Let $m_1, m_2, m_3$ and $m_4$ be distinct vertices of $G$. It suffices to show that if there is a path $m_1 - m_2 - m_3 - m_4$ from $m_1$ to $m_4$, then $m_1$ and $m_4$ are adjacent. Now, $r_1m_1 + r_2m_2 + r_3m_3 + r_4m_4 \in U$ for some $r_1, r_2, r_3, r_4 \in R \setminus (U : M)$. Hence $(r_1r_3r_2)m_1 + (r_2r_3r_4)m_4 = r_3r_2(r_1m_1 + r_2m_2) - r_2r_3(r_2m_2 +
\( r_3' m_3 + r_2 r_3' (r_3 m_3 + r_4 m_4) \in U \), and \( r_1 r_3 r_2', r_2 r_3' r_4 \notin (U : M) \) since \((U : M)\) is a prime ideal of \( R \). Thus \( m_1 \) and \( m_4 \) are adjacent. So if \( ET_U(M \setminus U) \) is connected, then \( \text{diam}(ET_U(M \setminus U)) \leq 2 \). \( \square \)

Now, we give the main theorem of this section. Since \( ET_U(U) \) is a complete subgraph of \( ET_U(M) \) by Theorem 2.2, the next theorem gives a complete description of \( ET_U(M \setminus U) \). Let \( |U| = \alpha \). We allow \( \alpha \) to be an infinite cardinal. Compare the next theorem with [10, Theorem 3.5].

**Theorem 2.4.** Let \( M \) be a module over a commutative ring \( R \), \( U \) be a prime submodule of \( M \), and \( |U| = \alpha \).

1. If \( r + s \in (U : M) \) for some \( r, s \in R \setminus (U : M) \), then \( ET_U(M \setminus U) \) is the union of complete subgraphs.

2. If \( r + s \notin (U : M) \) for all \( r, s \in R \setminus (U : M) \), then \( ET_U(M \setminus U) \) is the union of totally disconnected subgraphs and some connected subgraphs.

**Proof.**

1. Suppose that \( r + s \in (U : M) \) for some \( r, s \in R \setminus (U : M) \). For \( m, m' \in M \setminus U \), we write \( m \sim m' \) if and only if \( tm + t'm' \in U \) and \( t + t' \in (U : M) \) for some \( t, t' \in R \setminus (U : M) \). It is straightforward to check that \( \sim \) is an equivalence relation on \( M \setminus U \) since \( U \) is a prime submodule. For \( m \in M \setminus U \), we denote the equivalence class which contains \( m \) by \([m]\). Now let \( m \in M \setminus U \). If \([m] = \{m\}\), then \( r(m + u_1) + s(m + u_2) = (r + s)m + ru_1 + su_2 \in U \) for every \( u_1, u_2 \in U \) since \( r + s \in (U : M) \). Then \( m + U \) is a complete subgraph of \( ET_U(M \setminus U) \) with at most \( \alpha \) vertices. Now let \([|m|] = \nu \) and \( m' \in [m]\). Then \( tm + t'm' \in U \) and \( t + t' \in (U : M) \) for some \( t, t' \in R \setminus (U : M) \). So \( t(m + u_1) + t'(m' + u_2) = tm + t'm' + tu_1 + t'u_2 \in U \) for every \( u_1, u_2 \in U \). Thus \( m + U \) is part of the complete graph \( k^{\nu} \), where \( \nu \leq \alpha \nu \).

2. Assume that \( r + s \notin (U : M) \) for all \( r, s \in R \setminus (U : M) \). Let

\[
A_m = \{ m' \in M \setminus U : rm + sm' \in U \text{ for some } r, s \in R \setminus (U : M) \}
\]

be the set of all vertices adjacent to \( m \). If \( A_m = \emptyset \), then \( pm + qm' \notin U \) for every \( m', m' \in M \setminus U \) and every \( p, q \in R \setminus (U : M) \). In this case, we show that \( m + U \) is a totally disconnected subgraph of \( ET_U(M \setminus U) \). If \( r(m + m_1) + s(m + m_2) \in U \) for some \( r, s \in R \setminus (U : M) \) and \( m_1, m_2 \in U \), then \( (r + s)m \in U \). Since \( U \) is a prime submodule of \( M \) and \( m \notin U \), then \( r + s \in (U : M) \), which is a contradiction. Therefore \( m + U \) is a totally disconnected subgraph of \( ET_U(M \setminus U) \). Now, we may assume that \( A_m \neq \emptyset \). Then \( rm + sm' \in U \) for some \( r, s \in R \setminus (U : M) \) and \( m' \in M \setminus U \). Thus \( r(m + u_1) + s(m' + u_2) = rm + sm' + ru_1 + su_2 \in U \) for every \( u_1, u_2 \in U \); hence each element of \( m + U \) is adjacent to each element of \( m' + U \).
If $|A_m| = \nu$, then we have a connected subgraph of $ET_U(M \setminus U)$ with at most $\alpha \nu$ vertices. So $ET_U(M \setminus U)$ is the union of totally disconnected subgraphs and some connected subgraphs.

Now it is easy to compute the diameter and the girth of $ET_U(M \setminus U)$ using Theorem 2.4.

**Theorem 2.5.** Let $M$ be a module over a commutative ring $R$ such that $U$ is a prime submodule of $M$.

1. $\text{diam}(ET_U(M \setminus U)) = 0$ if and only if $U = \{0\}$ and $|M| = 2$.
2. $\text{diam}(ET_U(M \setminus U)) = 1$ if and only if either $|M\setminus U| = 1$ and $r + s \notin (U : M)$ for some $r, s \in R \setminus (U : M)$ or $|M \setminus U| = 2$, $r + s \in (U : M)$ for every $r, s \in R \setminus (U : M)$ and $x + y \in U$ for some distinct elements $x, y \in M \setminus U$.
3. $\text{diam}(ET_U(M \setminus U)) = 2$ if and only if $|M \setminus U| = 2$, $r + s \notin (U : M)$ for every $r, s \in R \setminus (U : M)$, $x + y \in U$ for some distinct elements $x, y \in M \setminus U$ and $|m + U| \geq 2$ for some $m \in M \setminus U$.
4. Otherwise, $\text{diam}(ET_U(M \setminus U)) = \infty$.

**Proof.** (1) If $\text{diam}(ET_U(M \setminus U)) = 0$, then $ET_U(M \setminus U)$ is a complete graph $K^1$, and so $|U| = |M/U| = 1$ by Theorem 2.4. Hence $U = \{0\}$ and $|M| = 2$. Now, let $U = \{0\}$ and $M = \{0, m\}$. Then $m + U$ is a single graph $K^1$. So $\text{diam}(ET_U(M \setminus U)) = 0$.

(2) It is clear that $ET_U(M \setminus U)$ is a complete graph if and only if $\text{diam}(ET_U(M \setminus U)) = 1$. So the proof is clear by Theorem 2.4.

(3) If $\text{diam}(ET_U(M \setminus U)) = 2$, then $ET_U(M \setminus U)$ is a complete bipartite graph $K^{m,n}$ such that $m \geq 2$ or $n \geq 2$. Thus $r + s \notin (U : M)$ for every $r, s \in R \setminus (U : M)$ by Theorem 2.4. Therefore $|M \setminus U| = 2$ and $x + y \in U$ for some $x, y \in M \setminus U$. Since $m \geq 2$ or $n \geq 2$, we have $|x + U| \geq 2$ or $|y + U| \geq 2$. Conversely, let $r + s \notin (U : M)$ for every $r, s \in R \setminus (U : M)$ and $|M \setminus U| = 2$. Then $M = U \cup (x + U) \cup (y + U)$ and $ET_U(M \setminus U)$ is a complete bipartite graph since $x + y \in U$. Hence $\text{diam}(ET_U(M \setminus U)) = 2$, since $|x + U| \geq 2$ or $|y + U| \geq 2$. □

**Theorem 2.6.** Let $M$ be a module over a commutative ring $R$ such that $U$ is a prime submodule of $M$. Then $\text{gr}(ET_U(M \setminus U)) = 3, 4$, or $\infty$. In particular, $\text{gr}(ET_U(M \setminus U)) \leq 4$ if $ET_U(M \setminus U)$ contains a cycle.

**Proof.** Assume that $ET_U(M \setminus U)$ contains a cycle. Then $ET_U(M \setminus U)$ is not a totally disconnected graph; so by the proof of Theorem 2.4, $ET_U(M \setminus U)$ has either a complete or a complete bipartite subgraph. Therefore it must contain either a 3-cycle or 4-cycle. Thus $\text{gr}(ET_U(M \setminus U)) \leq 4$. □
Theorem 2.7. Let $M$ be a module over a commutative ring $R$ such that $U$ is a prime submodule of $M$.

1. $gr(ET_U(M \setminus U)) = 3$ if and only if $r + s \in (U : M)$ and $|y + U| \geq 3$ for some $r, s \in R \setminus (U : M)$ and $y \in M \setminus U$.

2. $gr(ET_U(M \setminus U)) = 4$ if and only if $r + s \notin (U : M)$ for every $r, s \in R \setminus (U : M)$ and $pm + qm' \in U$ for some $m, m' \in M \setminus U$ and $p, q \in R \setminus (U : M)$.

3. Otherwise, $gr(ET_U(M \setminus U)) = \infty$.

Proof. (1) Assume that $gr(ET_U(M \setminus U)) = 3$. Then by Theorem 2.4, $ET_U(M \setminus U)$ is a complete graph $K^\lambda$, where $\lambda \geq 3$. Then $r + s \in (U : M)$ for some $r, s \in R \setminus (U : M)$ and $|y + U| \geq 3$ for some $y \in M \setminus U$ by Theorem 2.4.

(2) If $gr(ET_U(M)) = 4$, then $ET_U(M \setminus U)$ has a complete bipartite subgraph. So $r + s \notin (U : M)$ for every $r, s \in R \setminus (U : M)$ and $pm + qm' \in U$ for some $m, m' \in M \setminus U$ and $p, q \in R \setminus (U : M)$ by Theorem 2.4.

The other implications of (1) and (2) follows directly from Theorem 2.4.

We end this section with the following theorem.

Theorem 2.8. Let $M$ be a module over a commutative ring $R$ such that $U$ is a prime submodule of $M$.

1. $gr(ET_U(M)) = 3$ if and only if $|U| \geq 3$.

2. $gr(ET_U(M)) = 4$ if and only if $r + s \notin (U : M)$ for every $r, s \in R \setminus (U : M)$, $|U| < 3$, and $pm + qm' \in U$ for some $m, m' \in M \setminus U$ and $p, q \in R \setminus (U : M)$.

3. Otherwise, $gr(ET_U(M)) = \infty$.

Proof. (1) This follows from Theorem 2.2.

(2) Assume that $gr(ET_U(M)) = 4$. Since $gr(ET_U(U)) = 3$ or $\infty$, then $gr(ET_U(M \setminus U)) = 4$. Therefore $r + s \notin (U : M)$ for every $r, s \in R \setminus (U : M)$ and $pm + qm' \in U$ for some $m, m' \in M \setminus U$ and $p, q \in R \setminus (U : M)$ by Theorem 2.7. On the other hand, $gr(ET_U(M)) \neq 3$; so $|U| < 3$. The other implication follows from Theorem 2.4.

3. The case when $U$ is not a submodule of $M$

In this section, we study $ET_U(M)$ when the multiplicative-prime subset $U$ is not a submodule of $M$. Since $U$ is always closed under multiplication by elements of $R$, this just means that $0 \in U$ and there are distinct $x, y \in U$ such that $x + y \in M \setminus U$.

First, we begin with the following example that shows we may have $ET_U(M) \neq GT_U(M)$.
Example 3.1. Let $R = M = \mathbb{Z}$. Set $U = 4\mathbb{Z} \cup 6\mathbb{Z}$. It is clear that $(U : M) = U$ and $U$ is not a submodule of $M$ since $4, 6 \in U$, but $4 + 6 = 10 \notin U$. So $4 - 6$ is not an edge in $GT_U(M)$. But $2(4) + 2(6) = 20 \in U$ and $2 \in R \setminus U$. Thus $4 - 6$ is an edge in $ET_U(M)$. Hence $ET_U(M) \neq GT_U(M)$.

Now, we have the following theorem that shows $ET_U(U)$ is always connected (but never complete), $ET_U(U)$ and $ET_U(M \setminus U)$ are never disjoint subgraphs of $ET_U(M)$, and $ET_U(M)$ is connected when $ET_U(M \setminus U)$ is connected.

Theorem 3.2. Let $M$ be a module over a commutative ring $R$ such that $U$ is a multiplicative-prime subset of $M$ that is not a submodule of $M$. Then the following hold:

1. $ET_U(U)$ is connected with $diam(ET_U(U)) = 2$.
2. Some vertex of $ET_U(U)$ is adjacent to a vertex of $ET_U(M \setminus U)$. In particular, the subgraphs $ET_U(U)$ and $ET_U(M \setminus U)$ are not disjoint.
3. If $ET_U(M \setminus U)$ is connected, then $ET_U(M)$ is connected.

Proof. (1) Let $u \in U^* = U \setminus \{0\}$. Then $u$ is adjacent to $0$. Thus $u - 0 - u'$ is a path in $ET_U(U)$ of length two between any two distinct $u, u' \in U^*$. Moreover, there exist nonadjacent $u, u' \in U^*$ since $U$ is not a submodule of $M$; thus $diam(ET_U(U)) = 2$.

(2) Since $U$ is not a submodule of $M$, there exist distinct $m, n \in U^*$ such that $m + n \notin U$. Then $-m \in U$ and $m + n \notin U$ are adjacent vertices in $ET_U(M)$. Finally, the “in particular” statement is clear.

(3) $ET_U(U)$ and $ET_U(M \setminus U)$ are connected, and there is an edge between $ET_U(U)$ and $ET_U(M \setminus U)$; thus $ET_U(M)$ is connected. \hfill \Box

We determine when $ET_U(M)$ is connected and compute $diam(ET_U(M))$ with the following theorem. Compare the next theorem with [10, Theorem 4.2].

Theorem 3.3. Let $M$ be a module over a commutative ring $R$ such that $U$ is a multiplicative-prime subset of $M$ that is not a submodule of $M$. Then $ET_U(M)$ is connected if and only if for every $m \in M$ there exists $r \in R \setminus (U : M)$ such that $rm \in (U)$.

Proof. Suppose that $ET_U(M)$ is connected and $m \in M$. Then there exists a path $0 - x_1 - x_2 - \cdots - x_n - m$ from $0$ to $m$ in $ET_U(M)$. Thus

$$r_1x_1, r_2x_1 + r_3x_2, \ldots, r_{2n-2}x_{2n-1} + r_{2n-1}x_n, r_{2n}x_n + sm \in U$$

for some $r_1, r_2, \ldots, r_{2n}, s \in R \setminus (U : M)$. Then

$$sr_1r_3r_5 \cdots r_{2n-1}m = (r_1r_3r_5 \cdots r_{2n-1})(sm + r_{2n}x_n) -$$
\[(r_1r_3r_5 \cdots r_{2n-3}r_{2n})(r_{2n-2}\bar{x}_{n-1} + r_{2n-1}\bar{x}_n) + \cdots
\]
\[\quad - (r_1r_3 \cdots r_{2n-2k} - 5r_{2n-2k-3}r_{2n-2k-2} \cdots r_{2n})(r_{2n-2k+1}x_{n-k} + r_{2n-2k-2}x_{n-(k+1)})
\]
\[\quad + (r_1r_3 \cdots r_{2n-2k-5}r_{2n-2k-2}r_{2n-2k-2} \cdots r_{2n})(r_{2n-2k+1}x_{n-(k+1)} + r_{2n-2k-4}x_{n-(k+2)})
\]
\[\quad \cdots - (r_2r_4r_6 \cdots r_{2n})(r_1x_1) \in \langle U \rangle
\]

Since \((U : M)\) is a multiplicative-prime subset of \(R\), we have \(r = sr_1r_3r_5 \cdots r_{2n-1} \in R \setminus \langle U : M \rangle\) and \(rm \in \langle U \rangle\). Conversely, suppose that for every \(m \in M\) there exists \(r \in R \setminus \langle U : M \rangle\) such that \(rm \in \langle U \rangle\). We show that for each \(0 \neq m \in M\), there exists a path in \(ET_U(M)\) from 0 to \(m\). By assumption, there are elements \(u_1, u_2, \ldots, u_n \in U\) such that \(rm = u_1 + u_2 + \cdots + u_n\). Set \(y_0 = 0\) and \(y_k = (-1)^{n+k}(u_1 + u_2 + \cdots + u_k)\) for each integer \(k\) with \(1 \leq k \leq n\). Then \(y_k + y_{k+1} = (-1)^{n+k+1}u_{k+1} \in U\) for each integer \(1 \leq k \leq n - 1\). Also, \(y_{n-1} + rm = y_{n-1} + y_n = u_n \in U\). Thus \(0 - y_1 - y_2 - \cdots - y_{n-1} - m\) is a path from 0 to \(m\) in \(ET_U(M)\). Now, let \(0 \neq x, y \in M\). Then by the preceding argument, there are paths from \(x\) to 0 and 0 to \(y\) in \(ET_U(M)\). Hence there is a path from \(x\) to \(y\) in \(ET_U(M)\). So \(ET_U(M)\) is connected. \(\square\)

**Theorem 3.4.** Let \(M\) be a module over a commutative ring \(R\) such that \(U\) is a multiplicative-prime subset of \(M\) that is not a submodule of \(M\) and for every \(m \in M\) there exists \(r \in R \setminus \langle U : M \rangle\) such that \(rm \in \langle U \rangle\). Let \(n \geq 2\) be the least integer such that \(\langle U \rangle = \langle m_1, m_2, \ldots, m_n \rangle > \langle m_1, m_2, \ldots, m_n \rangle \in U\). Then \(diam(ET_U(M)) \leq n\).

**Proof.** Let \(m\) and \(m'\) be distinct elements in \(M\). We show that there exists a path from \(m\) to \(m'\) in \(ET_U(M)\) with length at most \(n\). By hypothesis, \(rm, r'm' \in \langle U \rangle\) for some \(r, r' \in R \setminus \langle U : M \rangle\); so we can write \(rm = \sum_{i=1}^{n} r_im_i\) and \(r'm' = \sum_{i=1}^{n} s_im_i\) for some \(r_i, s_i \in R\). Define \(x_0 = m\) and \(x_k = (-1)^k(\sum_{i=1}^{k+1} r_im_i + \sum_{i=1}^{k} s_im_i)\); so \(x_k + x_{k+1} = (-1)^k(r_{k+1} + s_{k+1})m_{k+1} \in U\) for each integer \(k\) with \(1 \leq k \leq n - 1\). On the other hand, \(rm + x_1 = (r_1 - s_1)m_1 \in U\) and \(r'm' + (-1)^n x_{n-1} = (s_n - r_n)m_n \in U\). So \(m - x_1 - x_2 - \cdots - x_{n-1} - m'\) is a path from \(m\) to \(m'\) in \(ET_U(M)\) with length at most \(n\) since \(1, (-1)^n \notin \langle U : M \rangle\). \(\square\)

We end the paper with the following theorem.

**Theorem 3.5.** Let \(M\) be a module over a commutative ring \(R\) such that \(U\) is a multiplicative-prime subset of \(M\) that is not a submodule of \(M\). Then the following hold:

1. Either \(gr(ET_U(U)) = 3\) or \(gr(ET_U(U)) = \infty\).
2. If \(gr(ET_U(M)) = 4\), then \(gr(ET_U(U)) = \infty\).
Proof. (1) If $rm + sm' \in U$ for some distinct $m, m' \in U \setminus \{0\}$ and $r, s \in R \setminus (U : M)$, then $0 - m - m' - 0$ is a cycle of length 3 in $ET_U(U)$; so $gr(ET_U(U)) = 3$. Otherwise, $rm + sm' \in M \setminus U$ for all distinct $m, m' \in U \setminus \{0\}$ and all elements $r, s \in R \setminus (U : M)$. Therefore in this case, each nonzero element $m \in U$ is adjacent to 0, and no two distinct $m, m' \in U \setminus \{0\}$ are adjacent. Thus $gr(ET_U(U)) = \infty$.

(2) If $gr(ET_U(M)) = 4$, then it is clear $gr(ET_U(U)) \neq 3$. So $gr(ET_U(U)) = \infty$ by part (1) above. □

Acknowledgement. The authors are deeply grateful to the referee for a careful reading of this manuscript and his valuable suggestions.

References

**F. Esmaeili Khalil Saraei** (Corresponding Author)  
Fouman Faculty of Engineering  
College of Engineering  
University of Tehran  
P.O. Box 43515-1155, Fouman  
43516-66456, Guilan, Iran  
e-mail: f.esmaeili.kh@ut.ac.ir

**E. Navidinia**  
Department of Mathematics  
Faculty of Mathematical Sciences  
University of Guilan  
P.O.Box 1914, Rasht, Iran  
e-mail: elnaz.navidinia@yahoo.com