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ON QUASI COMULTIPLICATION MODULES OVER PULLBACK RINGS

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Abstract. We classify all indecomposable quasi comultiplication modules over pullback of two Dedekind domains. We extend the definitions and the results of comultiplication modules over pullback rings to a more general quasi comultiplication modules case.

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1. Introduction

One of the aims of the modern representation theory is to solve classification problems for subcategories of modules over a unitary ring R. We make the general point that over most rings it is impossible to classify all modules: even algebras of tame representation type typically are "wild" when their infinitely generated representations are considered. The reader is referred to [3], [24], [25, Chapter 1 and 6] and [26] for a detailed discussion of classification problems, their representation types (finite, tame, or wild), and useful computational reduction procedures. Pure-injective modules seem to form one of the classes of modules which arise in practice and where there is hope of some kind of classification. Pure-injective modules play a central role in the model theory of modules. Let R_i be a local Dedekind domain, $\bar{R}$ be a common field and let $v_i : R_i \to \bar{R}$ be a homomorphism of R_i onto $\bar{R}$ for both $i = 1, 2$. Denote the pullback $R = \{(r_1, r_2) \in R_1 \oplus R_2 : v_1(r_1) = v_2(r_2)\}$ by $(R_1 \xrightarrow{v_1} \bar{R} \xleftarrow{v_2} R_2)$, where $\bar{R} = R_1/J(R_1) = R_2/J(R_2)$. Then R is a ring under coordinate-wise multiplication. Denote the kernel of v_i, $i = 1, 2$, by P_i. Then $\mathrm{Ker}(R \to \bar{R}) = P = P_1 \times P_2$, $R/P \cong \bar{R} \cong R_1/P_1 \cong R_2/P_2$, and $P_1P_2 = P_2P_1 = 0$ (so R is not a domain). Furthermore, there is an exact sequence $0 \to P_i \to R \to R_j \to 0$ of R-modules (see [21]), for $i \neq j$. For such a pullback ring R, indecomposable pure-injective modules with finite-dimensional top (for any module M we define its top as $M/rad(M)$) over R have already been classified by the first author [5]. Also, the classification of an arbitrary indecomposable pure-injective module over the $\bar{R}$-algebra $\bar{R}[x,y : xy = 0]_{(x,y)}$ which is the pullback $(\bar{R}[x]_{(x)} \to \bar{R} \leftarrow \bar{R}[y]_{(y)})$ (see [2, Section 6]) appears to be a very difficult problem. Therefore the classification of subclass of pure-injective modules over a pullback of two local Dedekind domains over a common factor field is very important. One point of this paper is to introduce a subclass of pure-injective modules over such rings. Indeed, this article includes the classification of all indecomposable quasi comultiplication modules over $\bar{R}[x,y : xy = 0]_{(x,y)}$.

Modules over pullback rings have been studied by several authors (see for example [4], [8], [9], [11], [12], [14], [15], [18], [19], [23] and [28]). Notably, there is the monumental work of Levy [22], resulting in the classification of all finitely generated indecomposable modules over Dedekind-like rings. Common to all these classification is the reduction to a "matrix problem" over a division ring (see [25, Section 17.9] for background on matrix problems and their applications).

In the present paper we introduce a new class of R-modules, called quasi comultiplication modules (see Definition 2.1), and we study them in detail from the classification point of view. We are mainly interested in the case where R is either a Dedekind domain or a pullback ring of two local Dedekind domains. The classification is divided into two stages: the description of all indecomposable separated quasi comultiplication R-modules and then, using this list of separated quasi comultiplication modules, we show that the only non-zero indecomposable quasi comultiplication non-separated R-module, up to isomorphism, is $E(R/P)$, the R-injective hull of R/P. For the sake of completeness, we state some definitions and notations used throughout. In this paper all rings are commutative with identity and all modules are unitary.

Definition 1.1. An R-module S is defined to be separated if there exist R_i-modules S_i, $i = 1, 2$, such that S is a submodule of $S_1 \oplus S_2$ (the latter is made into an R-module by setting $(r_1, r_2)(s_1, s_2) = (r_1 s_1, r_2 s_2)$).

Equivalently, S is separated if it is a pullback of an R_1-module and an R_2-module and then, using the same notation for pullbacks of modules as for rings, $S = (S/P_2S \to S/PS \leftarrow S/P_1S)$ [21, Corollary 3.3] and $S \subseteq (S/P_2S) \oplus (S/P_1S)$. Also, we show S is separated if and only if $P_1S \cap P_2S = 0$ [21, Lemma 2.9].

If R is a pullback ring, then every R-module is an epimorphic image of a separated R-module, indeed every R-module has a "minimal" such representation: a separated representation of an R-module M is an epimorphism $\varphi = (S \xrightarrow{f} S' \to M)$ of R-modules where S is separated and, if φ admits a factorization $\varphi : S \xrightarrow{f} S' \to M$

with S' separated, then f is one-to-one. The module $K = \mathrm{Ker}(\varphi)$ is an $\bar{R}$-module, since $\bar{R} = R/P$ and $PK = 0$ [21, Proposition 2.3]. An exact sequence $0 \to K \to S \to M \to 0$ of R-modules with S separated and K an $\bar{R}$-module is a separated representation of M if and only if $P_iS \cap K = 0$ for each i and $K \subseteq PS$ [21, Proposition 2.3]. Every module M has a separated representation, which is unique up to isomorphism [21, Theorem 2.8]. Moreover, R-homomorphisms lift to a separated representation, preserving epimorphisms and monomorphisms [20, Theorem 2.6].

Now, in the following definition, we have collected several notions, which we use.

Definition 1.2. (a) If R is a ring and N is a submodule of an R-module M, then the ideal $\{r \in R : rM \subseteq N\}$ is denoted by $(N :_R M)$. The ideal $(0 :_R M)$ is the annihilator of M.

(b) A proper ideal I of R is said to be quasi-prime if for each pair of ideals A and B of R, $A \cap B \subseteq I$ yields either $A \subseteq I$ or $B \subseteq I$ (see [16]). A proper submodule N of an R-module M is called quasi-prime if $(N :_R M)$ is a quasi-prime ideal of R. The set of all quasi-prime submodules of M is denoted by $qSpec_R(M)$ (see [1]). Every maximal submodule of an R-module M is prime and every prime submodule of M is a quasi-prime submodule. Therefore $Max_R(M) \subseteq Spec_R(M) \subseteq qSpec_R(M)$ by [1, Remark 2.3].

(c) An R-module M is defined to be a comultiplication module if for each submodule N of M, $N = (0 :_M I)$, for some ideal I of R. In this case we can take $I = \mathrm{Ann}(N)$.

(d) A submodule N of an R-module M is called pure submodule if any finite system of equations over N which is solvable in M is also solvable in N. A submodule N of an R-module M is called relatively divisible (or an RD-submodule) in M if $rN = N \cap rM$ for all $r \in R$.

(e) A module M is pure-injective if it has the injective property relative to all pure exact sequences.

(f) A ring R is called to be a serial ring, if the set of all ideals of R is linearly ordered.

Remark 1.3.
(1) An R-module M is pure-injective if and only if it is algebraically compact (see [17] and [27]).
(2) Let R be a Dedekind domain, M an R-module and N a submodule of M. Then N is pure in M if and only if $IN = N \cap IM$ for each ideal I of R. Moreover, N is pure in M if and only if N is an RD-submodule of M [27].

2. Some properties of quasi comultiplication modules

In this section, we collect some basic properties concerning quasi comultiplication modules. We begin with the key definition of this paper.

Definition 2.1. Let R be a commutative ring and M be an R-module. Then M is defined to be a quasi comultiplication module if $\mathrm{qSpec}(M) = \emptyset$ or for every quasi-prime submodule N of M, $N = (0 :_M I)$ for some ideal I of R.

One can easily show that if M is a quasi comultiplication module, then $N = (0 :_M \mathrm{ann}(N))$ for every quasi-prime submodule N of M. It is easy to see that the class of quasi comultiplication modules contains the class of comultiplication modules defined in [7].

Lemma 2.2. *Let R be a commutative ring and M be an R-module. If I, J are proper ideals of R and $I \subseteq (0 :_R M)$, then the following hold:*

(1) *If $I \subseteq J$, then J is a quasi-prime ideal of R if and only if J/I is a quasi-prime ideal of R/I.*
(2) *If N is a proper submodule of M, then N is a quasi-prime R-submodule of M if and only if N is a quasi-prime R/I-submodule of M.*
(3) *M is a quasi comultiplication R-module if and only if M is a quasi comultiplication R/I-module.*

Proof. (1) The proof is straightforward.
(2) It is easy to see that $(N :_R M)/I = (N :_{R/I} M)$. So the result follows from Part(1).
(3) One can show that $(0 :_R N)/I = (0 :_{R/I} N)$. So the result follows from Part (2). □

Lemma 2.3. *Let R be a commutative ring and M be an R-module. If $N \subseteq L$ are submodules of M, then the following hold:*

(1) *L is a quasi-prime submodule of M if and only if L/N is a quasi-prime submodule of M/N.*
(2) *If M is a quasi comultiplication R-module and N is a pure submodule of M, then M/N is a quasi comultiplication R-module.*
(3) *If M is a quasi comultiplication R-module, then every direct summand of M is a quasi comultiplication R-module.*

Proof. (1) The proof is straightforward, since $(L :_R M) = (L/N :_R M/N)$.
(2) Let M be a quasi comultiplication module and let L/N be a quasi-prime submodule of M/N. Then by (1), L is a quasi-prime submodule of M, so $L = (0 :_M I)$

for some ideal I of R. Now we show that $L/N = (0 :_{M/N} I)$. Since $L = (0 :_M I)$, so $I(L/N) = (IL + N)/N = 0_{M/N}$. Hence $L/N \subseteq (0 :_{M/N} I)$. Assume that $m+N \in (0 :_{M/N} I)$. Then $a(m+N) = am+N = 0_{M/N}$ for every $a \in I$. Therefore $am \in N$ and then $Im \subseteq N \cap IM = IN = 0$, since $IN \subseteq IL = 0$. Then $m \in L$, so $m + N \in L/N$ and we have the equality.
(3) The result follows from (2), since direct summands are pure submodules. □

Proposition 2.4. *Let R be a local Dedekind domain and M be an R-module. Then the following hold:*

(1) *Every proper submodule of M is quasi-prime.*
(2) *M is a quasi comultiplication R-module if and only if M is a comultiplication R-module.*
(3) *If M is a quasi comultiplication R-module, then M is indecomposable.*

Proof. (1) Since every local Dedekind domain is a serial ring, the proof follows from [1, Lemma 2.4].
(2) Follows from (1).
(3) Let M be a quasi comultiplication R-module such that $M = N \oplus K$ with $N \neq 0$ and $K \neq 0$. By (1), there are positive integers m, n, with $m < n$, such that $M = (0 :_M P^n) + (0 :_M P^m) = (0 :_M P^n)$ and this contradicts $N \cap K = 0$. Thus either $N = 0$ or $K = 0$, as required. □

Theorem 2.5. *Let R be a local Dedekind domain with a unique maximal ideal $P = Rp$. Then the quasi comultiplication modules over R are:*

(1) *R/P^n, $n \geq 1$;*
(2) *$E(R/P)$, the injective hull of R/P.*

Proof. The result follows from Proposition 2.4 and [7, Theorem 2.5]. □

3. The separated quasi comultiplication modules

Throughout this paper we shall assume unless otherwise stated, that

$$R = (R_1 \xrightarrow{v_1} \bar{R} \xleftarrow{v_2} R_2) \tag{1}$$

is the pullback of two local Dedekind domains R_1, R_2 with maximal ideals P_1, P_2 generated respectively by p_1, p_2, P denotes $P_1 \oplus P_2$ and $R_1/P_1 \cong R_2/P_2 \cong R/P \cong \bar{R}$ is a field. In particular, R is a commutative Noetherian local ring with unique maximal ideal P. The other prime ideals of R are easily seen to be P_1 (that is $P_1 \oplus 0$) and P_2 (that is $0 \oplus P_2$).

Theorem 3.1. *Let R be the pullback ring as described in (1), M be an R-module and N be a proper submodule of M. Then N is a quasi-prime submodule of M if and only if either $(P_1 \oplus 0)M \subseteq N$ or $(0 \oplus P_2)M \subseteq N$.*

Proof. Let N be a quasi-prime submodule of M. Since $(P_1 \oplus 0) \cap (0 \oplus P_2) = 0 \subseteq (N :_R M)$, we have that $P_1 \oplus 0 \subseteq (N :_R M)$ or $0 \oplus P_2 \subseteq (N :_R M)$. Therefore $(P_1 \oplus 0)M \subseteq N$ or $(0 \oplus P_2)M \subseteq N$. Conversely, suppose that N is a proper submodule of M and $(P_1 \oplus 0)M \subseteq N$. Since R_2 is a local Dedekind domain, so $N/(P_1 \oplus 0)M$ is a quasi-prime R_2-submodule of $M/(P_1 \oplus 0)M$ by Proposition 2.4. So $N/(P_1 \oplus 0)M$ is a quasi-prime R-submodule of $M/(P_1 \oplus 0)M$ by Lemma 2.2. Then N is a quasi-prime R-submodule of M by Lemma 2.3. So $(N :_R M)/(P_1 \oplus 0)$ is a quasi-prime ideal, then $(N :_R M)$ is a quasi-prime ideal of R by Lemma 2.2 and so N is a quasi-prime submodule of M. □

Remark 3.2. Let R be the pullback ring as described in (1), and let T be an R-submodule of a separated module $S = (S_1 \xrightarrow{f_1} \bar{S} \xleftarrow{f_2} S_2)$, with projection maps $\pi_i : S \to S_i$. Set

$$T_1 = \{t_1 \in S_1 : (t_1, t_2) \in T \text{ for some } t_2 \in S_2\},$$

$$T_2 = \{t_2 \in S_2 : (t_1, t_2) \in T \text{ for some } t_1 \in S_1\}.$$

Then for each i, $i = 1, 2$, T_i is an R_i-submodule of S_i and $T \leq T_1 \oplus T_2$. Moreover, we can define a mapping $\pi'_1 = \pi_1|T : T \to T_1$ by sending (t_1, t_2) to t_1; hence $T_1 \cong T/((0 \oplus Ker(f_2)) \cap T) \cong T/(T \cap P_2S) \cong (T + P_2S)/P_2S \subseteq S/P_2S$. So we may assume that T_1 is a submodule of S_1. Similarly, we may assume that T_2 is a submodule of S_2 (note that $\text{Ker}(f_1) = P_1S_1$ and $\text{Ker}(f_2) = P_2S_2$).

Proposition 3.3. *Let R be the pullback ring as described in (1) and S be a non-zero separated R-module with $\bar{S} = 0$. Then $(0 :_R S) \in \{P_1 \oplus 0, 0 \oplus P_2, 0\}$.*

Proof. It is clear that $S = PS$ since $\bar{S} = 0$. First suppose that $(0 :_R S) = P_1^n \oplus P_2^m$ for some positive integers n and m. Now we consider the various possibilities for m and n.

Case 1. If $n > 1$ and $m > 1$, then $(P_1^{n-1} \oplus P_2^{m-1})S = (P_1^{n-1} \oplus P_2^{m-1})PS = (P_1^n \oplus P_2^m)S = 0$. So $P_1^{n-1} \oplus P_2^{m-1} \subseteq (0 :_R S)$ which is a contradiction.

Case 2. If $n = 1$ and $m > 1$, then $(P_1 \oplus P_2^{m-1})S = (P_1 \oplus P_2^{m-1})PS = (P_1^2 \oplus P_2^m)S = 0$ since $P_1^2 \oplus P_2^m \subseteq P_1 \oplus P_2^m = (0 :_R S)$. So $P_1 \oplus P_2^{m-1} \subseteq (0 :_R S)$ which is a contradiction.

Case 3. If $n > 1$ and $m = 1$, then the proof is similar to Case 2.

Case 4. If $n = 1$ and $m = 1$, then $(0 :_R S) = P$. So $S = PS = 0$, which is a

contradiction. Now suppose that $(0 :_R S) = P_1^n \oplus 0$ for some positive integer $n > 1$. So $(P_1^{n-1} \oplus 0)S = (P_1^{n-1} \oplus 0)PS = (P_1^n \oplus 0)S = 0$. So $P_1^{n-1} \oplus 0 \subseteq (0 :_R S)$ which is a contradiction.
The case $(0 :_R S) = 0 \oplus P_2^m$ for some positive integer $m > 1$ is similar. □

Now, we find the separated quasi comultiplication modules over the pullback ring R. We begin with the following proposition.

Proposition 3.4. *Let S be any separated quasi comultiplication module over the pullback ring as described in (1). Then the following hold:*

(1) *If $(0 :_R S) = 0$, then $\bar{S} = 0$.*
(2) *If $\bar{S} \neq 0$, then $(0 :_R S) \neq P_1^n \oplus 0$ and $(0 :_R S) \neq 0 \oplus P_2^n$ for every positive integer n.*
(3) *If $\bar{S} \neq 0$ and $(0 :_R S) = P_1^n \oplus P_2^m$ for some positive integers n, m, then either $m = 1$ or $n = 1$.*

Proof. (1) Suppose $\bar{S} \neq 0$. Then PS is a quasi-prime submodule of S by Theorem 3.1. Let $(r_1, r_2) \in (0 :_R PS)$. Then $(r_1, r_2)(p_1, p_2)S \subseteq (r_1, r_2)PS = 0$, so $r_1p_1 = 0$ and $r_2p_2 = 0$; hence $r_1 = 0$ and $r_2 = 0$, since R_i is an integral domain for $i = 1, 2$. Therefore, $(0 :_R PS) = 0$. Then S quasi comultiplication gives $PS = (0 :_S (0 :_R PS)) = (0 :_S 0) = S$, which is a contradiction.
(2) Let $(0 :_R S) = P_1^n \oplus 0$. If $(0 \oplus P_2)S = 0$, then $0 \oplus P_2 \subseteq P_1^n \oplus 0$, which is a contradiction. So $(0 \oplus P_2)S \neq 0$ and $(0 :_R (0 \oplus P_2)S) \neq R$. Then $(0 :_R (0 \oplus P_2)S) = P_1 \oplus 0$ by [10, Proposition 3.8]. Moreover, by Theorem 3.1, $(0 \oplus P_2)S$ is a quasi-prime submodule of S, so $(0 \oplus P_2)S = (0 :_S P_1 \oplus 0)$ since, S is quasi comultiplication. We may assume that $n > 1$. Since $(P_1 \oplus 0)(P_1^{n-1} \oplus P_2)S = 0$, we must have $(P_1^{n-1} \oplus P_2)S \subseteq (0 :_S P_1 \oplus 0) = (0 \oplus P_2)S$. Let $s_1 \in S_1$. Then there is an element $s_2 \in S_2$ such that $(s_1, s_2) \in S$. Hence $(p_1^{n-1}, p_2)(s_1, s_2) \in (0 \oplus P_2)S$; hence $p_1^{n-1}s_1 = 0$ and so $P_1^{n-1}S_1 = 0$. Therefore, $P_1^{n-1} \subseteq (0 :_{R_1} S_1) = P_1^n$ by [10, Proposition 3.6], which is a contradiction. Thus $(0 :_R S) \neq P_1^n \oplus 0$ for every positive integer n. Similarly, $(0 :_R S) \neq 0 \oplus P_2^n$ for every positive integer n.
(3) Suppose not. We may assume that $n > 1$ and $m > 1$. Clearly, $0 \neq (P_1 \oplus 0)S \subseteq PS \neq S$, $0 \neq (0 \oplus P_2)S \subseteq PS \neq S$, and they are quasi-prime submodules of S by Theorem 3.1. Since S is a quasi comultiplication R-module, we must have $(P_1 \oplus 0)S = (0 :_S (0 :_R (P_1 \oplus 0)S)) = (0 :_S (P_1^{n-1} \oplus P_2))$ and $(0 \oplus P_2)S = (0 :_S (P_1 \oplus P_2^{m-1}))$ by [10, Lemma 3.4]. Let $s_1 \in S_1$. There exists $s_2 \in S_2$ such that $(s_1, s_2) \in S$. It follows that $p_1^n s_1 = 0$ and $p_2^m s_2 = 0$ by [10, Proposition 3.6]. Therefore, $(p_1, p_2^{m-1})(p_1^{n-1}s_1, p_2s_2) = 0$, so $(p_1^{n-1}s_1, p_2s_2) \in (0 :_S P_1 \oplus P_2^{m-1}) =$

$(0 \oplus P_2)S$; hence $p_1^{n-1}s_1 = 0$. By a similar way, we get $p_1 s_1 = 0$. Therefore, $P_1 S_1 \cong (P_1 \oplus 0)S = 0$, which is a contradiction. □

Theorem 3.5. *Let R be the pullback ring as described in (1), and let $S = (S_1 \to \bar{S} \leftarrow S_2)$ be a separated R-module. Then S is a quasi comultiplication R-module if and only if each S_i is a quasi comultiplication R_i-module, for both $i = 1, 2$.*

Proof. Let S be a quasi comultiplication R-module. If $\mathrm{qSpec}(S) = \emptyset$, then $(P_1 \oplus 0)S = (0 \oplus P_2)S = S$ by Theorem 3.1. Then for each $i = 1, 2$, $S_i = 0$ is a quasi comultiplication R_i-module by [21, Corollary 3.3]. So, we may assume that $\mathrm{qSpec}(S) \neq \emptyset$. If $\bar{S} = 0$. Then by [5, Lemma 2.7], $S = S_1 \oplus S_2$; hence for each $i = 1, 2$; S_i is a quasi comultiplication R-module by Lemma 2.2. Therefore for each $i = 1, 2$; S_i is a quasi comultiplication R_i-module by Lemma 2.2. So, we may assume that $\bar{S} \neq 0$.

Let L (resp. L') be a quasi-prime submodule of S_1 (resp S_2). Then there exists a separated submodule $T = (T/P_2S = T_1 \xrightarrow{g_1} \bar{T} = T/PT \xleftarrow{g_2} T_2 = T/P_1T)$ (resp. $T' = (T'/P_2T' = T'_1 \xrightarrow{g'_1} \bar{T}' = T'/PT' \xleftarrow{g'_2} T'_2 = T'/P_1S))$ of S, where g_i (resp. g'_i) is the restriction of f_i over T_i (resp. T'_i), $i = 1, 2$ such that $L = T_1$ (resp. $L' = T'_2$). Since $(0 \oplus P_2)S \subseteq T$ $((P_1 \oplus 0)S \subseteq T')$; hence T (resp. T') is a proper quasi-prime R-submodule of S by Theorem 3.1. We split the proof into two cases for $(0 :_R S)$ by Proposition 3.4.

Case 1. $(0 :_R S) = P_1 \oplus P_2^m$ for some positive integer m. If $m = 1$, then $(0 :_R S) = P_1 \oplus P_2 = P$. Hence $PT \subseteq PS = 0$ and so $P \subseteq (0 :_R T)$. Then we have $(0 :_R T) = P$. Thus S is quasi comultiplication implies that $T = (0 :_S P) = S$, which is a contradiction. So we may assume that $m > 1$. By [10, Proposition 3.6], $(0 :_{R_1} S_1) = P_1$ and $(0 :_{R_2} S_2) = P_2^m$. Since $(P_1 \oplus 0)S \cong P_1 S_1 = 0$ and $(0 \oplus P_2)S \subseteq T$, we get $PS \subseteq T \subseteq S$, so $(0 :_R S) \subseteq (0 :_R T) \subseteq (0 :_R PS)$; thus $P_1 \oplus P_2^m \subseteq (0 :_R T) \subseteq P_1 \oplus P_2^{m-1}$ by [10, Proposition 3.7]. Therefore, either $(0 :_R T) = P_1 \oplus P_2^m$ or $(0 :_R T) = P_1 \oplus P_2^{m-1}$. Since S is quasi comultiplication, we have either $T = (0 :_S P_1 \oplus P_2^m) = S$ or $T = (0 :_S P_1 \oplus P_2^{m-1}) = PS$; hence $T = PS$ and $T_1 = (PS)/PS = 0$. Then $L = T_1 = (0 :_{S_1} R_1)$ gives S_1 is quasi comultiplication. Now we will prove that S_2 is a quasi comultiplication R_2-module. By hypothesis, $T' = (0 :_S P_1^s \oplus P_2^t)$ for some positive integers s, t. We show that $T'_2 = (0 :_{S_2} P_2^m)$. Since the inclusion $T'_2 \subseteq (0 :_{S_2} P_2^m)$ is clear, we will prove the reverse inclusion. Let $s_2 \in (0 :_{S_2} P_2^m)$. Then $P_2^m s_2 = 0$ and there exists $s_1 \in S_1$ such that $(s_1, s_2) \in S$, so $(P_1^s \oplus P_2^t)(s_1, s_2) = 0$; hence $(s_1, s_2) \in T'$. Therefore, $s_2 \in T'_2$, and so we have the equality.

Case 2. $(0 : S) = P_1^m \oplus P_2$ for some positive integer m. The proof is similar to that in Case 1.

Conversely, assume that S_i is a quasi comultiplication R_i-module for each i, $i = 1, 2$ and let T be a proper quasi-prime submodule of S. We consider the various possibilities for $(0 :_R T)$.

Case 1. If $(0 :_R T) = 0$, then $(0 :_{R_1} T_1) = 0$ and $(0 :_{R_2} T_2) = 0$. So $T_1 = S_1$ and $T_2 = S_2$ implies that $T = S$ which is a contradiction.

Case 2. If $(0 :_R T) = P_1^n \oplus P_2^m$ for some positive integer n and m, then $T_1 = (0 :_{S_1} P_1^n)$ and $T_2 = (0 :_{S_2} P_2^m)$ by Proposition 2.4 and [10, Proposition 3.6]. So $T = (0 :_S P_1^n \oplus P_2^m)$.

Case 3. If $(0 :_R T) = P_1^n \oplus 0$ for some positive integer n, then $T_1 = (0 :_{S_1} P_1^n)$, $T_2 = S_2$ by Proposition 2.4 and [10, Proposition 3.6]. So it is easy to see that $T = (0 :_S P_1^n \oplus 0)$.

The case $(0 :_R T) = 0 \oplus P_2^m$ is similar. So S is a quasi comultiplication R-module. □

Proposition 3.6. *Let R be the pullback ring as described in (1), and let $S = (S_1 \xrightarrow{f_1} \bar{S} \xleftarrow{f_2} S_2)$ be a separated quasi comultiplication R-module with $\bar{S} \neq 0$. Then S is an indecomposable R-module.*

Proof. Let S be a separated quasi comultiplication module. Then, for both i, $i = 1, 2$, S_i is a quasi comultiplication R_i-module by Theorem 3.5. Therefore for both i, $i = 1, 2$, S_i is an indecomposable R_i-module by Proposition 2.4, and so, S is an indecomposable R-module by [5, Lemma 2.7]. □

Theorem 3.7. *Let R be the pullback ring as described in (1). Then the indecomposable separated quasi comultiplication modules over R are:*

(I) $S = (E(R_1/P_1) \to 0 \leftarrow 0)$ *and* $S = (0 \to 0 \leftarrow E(R_2/P_2)$, *where* $E(R_i/P_i)$ *is the R_i-injective hull of R_i/P_i for $i = 1, 2$;*

(II) $S = (R_1/P_1^n \to \bar{R} \leftarrow R_2/P_2^m)$.

Proof. By [5, Lemma 2.8], these modules are indecomposable and by Theorems 2.5 and 3.5 they are quasi comultiplication.

Now, let S be an indecomposable separated quasi comultiplication R-module. First, suppose that $S = PS$. Then by [5, Lemma 2.7 (i)], $S = S_1$ or S_2 and so, S is an indecomposable quasi comultiplication R_i-module for some i and, since $PS = S$, is type (I) in the list. So we may assume that $S \neq PS$. By Theorem 3.5, S_i is a quasi comultiplication R_i-module, for each $i = 1, 2$. Hence, by the structure of quasi comultiplication modules over a local Dedekind domain (see Theorem 2.5), we must have $S_i = E(R_i/P_i)$ or R_i/P_i^n $(n \geq 1)$. Since S is indecomposable and $S/PS \neq 0$, it follows that for each $i = 1, 2$, S_i is torsion and it is not divisible R_i-module. Then there are positive integers m, n and k such that $P_1^m S_1 = 0$, $P_2^k S_2 = 0$ and $P^n S = 0$. For $t \in S$, let $o(t)$ denote the least positive integer l such that $P^l t = 0$. Now choose $t \in S_1 \cup S_2$ with $\bar{t} \neq 0$ and such that $o(t)$ is maximal. There exists a $t = (t_1, t_2)$ such that $o(t) = n$, $o(t_1) = m$ and $o(t_2) = k$. Then $R_i t_i$ is pure in S_i for $i = 1, 2$ (see [5, Theorem 2.9]). Therefore, $R_1 t_1 \cong R_1/P_1^m$ (resp. $R_2 t_2 \cong R_2/P_2^k$) is a direct summand of S_1 (resp. S_2), since for each i, $R_i t_i$ is pure-injective. Hence $S_1 = R_1 t_1 \cong R_1/P_1^m$ since, S_1 is indecomposable. Similarly, $S_2 = R_2 t_2 \cong R_2/P_2^k$. Let $\bar{M}$ be the $\bar{R}$-subspace of $\bar{S}$ generated by $\bar{t}$. Then $\bar{M} \cong \bar{R}$. Let $M = (R_1 t_1 = M_1 \rightarrow \bar{M} \leftarrow M_2 = R_2 t_2)$. Then M is an R-submodule of S which is quasi comultiplication by Theorem 3.5 and is a direct summand of S; this implies that $S = M$, and S is as in (II) in the list (see [5, Theorem 2.9]). □

We refer to modules of type (I) in Theorem 3.7 as P_1-Prüfer and P_2-Prüfer respectively.

Theorem 3.8. *Let R be the pullback ring as described in (1) and let S be a separated quasi comultiplication R-module. Then S has finite-dimensional top.*

Proof. Apply Theorem 3.7 (note that $S = U \oplus X$, where $dim_{\bar{R}}(U/PU) \leq 1$ and $X/PX = 0$). □

4. The non-separated quasi comultiplication modules

We continue to use the notation already established, so R is the pullback ring as described in (1). In this section, we will determine all the indecomposable non-separated quasi comultiplication R-modules over R. It turns out that each can be obtained by amalgamating finitely many indecomposable separated quasi comultiplication modules.

We begin by the following lemma.

Lemma 4.1. *Let R be a pullback ring as described in (1) and M, S be two R-modules. Let $\varphi : S \longrightarrow M$ be an epimorphism.*

(1) *If N is a submodule of M, then $(N :_R M) = (\varphi^{-1}(N) :_R S)$.*
(2) *If T is a proper submodule of S, then $(0 :_R T) = (0 :_R \varphi(T))$.*
(3) *If either $(T :_R S) = P_1 \oplus 0$ or $(T :_R S) = 0 \oplus P_2$, then*

$$(T :_R S) = (\varphi(T) :_R M).$$

(4) *If N is a quasi-prime submodule of M, then $\varphi^{-1}(N)$ is a quasi-prime submodule of S.*

(5) *If T is a quasi-prime submodule of S, then $\varphi(T)$ is a quasi-prime submodule of M.*

Proof. (1) Suppose $(r_1, r_2) \in (N :_R M)$ and $(s_1, s_2) \in S$. Then $\varphi(s_1, s_2) \in M$ and so $\varphi(r_1s_1, r_2s_2) = (r_1, r_2)\varphi(s_1, s_2) \in (r_1, r_2)M \subseteq N$. Thus $(r_1, r_2)(s_1, s_2) = (r_1s_1, r_2s_2) \in \varphi^{-1}(N)$. Hence $(r_1, r_2)S \subseteq \varphi^{-1}(N)$ and we have $(r_1, r_2) \in (\varphi^{-1}(N) :_R S)$. For the reverse inclusion, suppose that $(r_1', r_2') \in (\varphi^{-1}(N) :_R S)$ and let $m \in M$. Then $\varphi(s_1', s_2') = m$ for some $(s_1', s_2') \in S$. Thus $(r_1', r_2')(s_1', s_2') \in \varphi^{-1}(N)$. Hence $(r_1', r_2')m = (r_1', r_2')\varphi(s_1', s_2') = \varphi(r_1's_1', r_2's_2') \in \varphi(\varphi^{-1}(N)) \subseteq N$, and we have the equality.

(2) First suppose that $(0 :_R T) = 0$ and $(r_1, r_2) \in (0 :_R \varphi(T))$. Therefore $\varphi((r_1, r_2)T) = (r_1, r_2)\varphi(T) = 0$, hence $(r_1, r_2)T \subseteq K$. So $(r_1, r_2)(p_1, p_2)T = 0$ since $PK = 0$ by [21, Proposition 2.4]. Thus $(r_1p_1, r_2p_2) \in (0 :_R T)$ implies that $r_1 = 0$ and $r_2 = 0$ since R_i is a domain for each $i = 1, 2$. So we get $(0 :_R \varphi(T)) = 0$ and we have the equality. It is clear that $(0 :_R T) \subseteq (0 :_R \varphi(T))$. Now, we consider the possibilities for $(0 :_R \varphi(T))$:

Case 1. If $(0 :_R \varphi(T)) = P_1^n \oplus 0$ for some positive integer n, then $\varphi((P_1^n \oplus 0)T) = (P_1^n \oplus 0)\varphi(T) = 0$. Hence $(P_1^n \oplus 0)T \subseteq K \cap (P_1 \oplus 0)S = 0$ by [21, Proposition 2.3] and so, $P_1^n \oplus 0 \subseteq (0 :_R T)$ as required.

Case 2. If $(0 :_R \varphi(T)) = 0 \oplus P_2^m$ for some positive integer m, the proof is similar to Case 1.

Case 3. If $(0 :_R \varphi(T)) = P_1^n \oplus P_2^m$ for some positive integer n, m, then by an argument like in Case 1, we get $P_1^n \oplus 0 \subseteq (0 :_R T)$. Similarly, by Case 2, $0 \oplus P_2^m \subseteq (0 :_R \varphi(T))$ implies that $0 \oplus P_2^m \subseteq (0 :_R T)$. Thus $(0 :_R \varphi(T)) = P_1^n \oplus P_2^m \subseteq (0 :_R T)$ and we have the equality.

(3) Let $(T :_R S) = P_1 \oplus 0$. It is clear that $(P_1 \oplus 0)M \subseteq \varphi(T)$. Now, suppose that $(r_1, r_2) \in (\varphi(T) :_R M)$. It suffices to show that $r_2 = 0$. Let $(s_1, s_2) \in S$. Then $(r_1, r_2)\varphi(s_1, s_2) \in (r_1, r_2)M \subseteq \varphi(T)$. So $\varphi(r_1s_1, r_2s_2) = \varphi(t_1, t_2)$ for some $(t_1, t_2) \in T$. Hence $(r_1, r_2)(s_1, s_2) = (t_1, t_2) + (k_1, k_2)$ for some $(k_1, k_2) \in K$. This implies that $(r_1, r_2)S \subseteq T + K$. So $(0, p_2)(r_1, r_2)S \subseteq T$ since $(o, p_2)K \subseteq K \cap (0 \oplus P_2)S = 0$. Hence $(0, p_2r_2) \in (T :_R S) = P_1 \oplus 0$. Then $p_2r_2 = 0$ implies that $r_2 = 0$, since R_2 is a domain. Therefore $(\varphi(T) :_R M) = P_1 \oplus 0$. The case $(T :_R S) = 0 \oplus P_2$ is similar.

(4) Clear by Theorem 3.1 and Case (1).

(5) Let T be a quasi-prime submodule of S. Then by Theorem 3.1, we may assume that $(P_1 \oplus 0)S \subseteq T$. We show that $(P_1 \oplus 0)M \subseteq \varphi(T)$. Assume that $m \in M$, then there exists $s \in S$ where $\varphi(s) = m$. Therefore $(p_1, 0)m = (p_1, 0)\varphi(s) = \varphi((p_1, 0)s) \in \varphi(T)$ for every $m \in M$. Hence $(P_1 \oplus 0)M \subseteq \varphi(T)$. Similarly, if $(0 \oplus P_2)S \subseteq T$, then $(0 \oplus P_2)M \subseteq \varphi(T)$. □

Proposition 4.2. *Let R be the pullback ring as described in (1) and let M be any R-module. Let $0 \to K \xrightarrow{i} S \xrightarrow{\varphi} M \to 0$ be a separated representation of M. Then the following hold:*

(1) $\mathrm{qSpec}_R(S) = \emptyset$ *if and only if* $\mathrm{qSpec}_R(M) = \emptyset$.
(2) $(0 :_R S) = 0$ *if and only if* $(0 :_R M) = 0$.
(3) *If* $(0 :_R S) = 0$, *then* $\mathrm{qSpec}_R(S) \neq \emptyset$.
(4) *If* $(0 :_R M) = 0$, *then* $\mathrm{qSpec}_R(M) \neq \emptyset$.
(5) *If either* $P_1 \oplus 0 \subseteq (0 :_R S)$ *or* $0 \oplus P_2 \subseteq (0 :_R S)$, *then* M *is separated.*

Proof. (1) Follows from Lemma 4.1.
(2) Clear by [13, Proposision 5.2].
(3) Assume that $(0 :_R S) = 0$. If $(P_1 \oplus 0)S = S$, then $(0 \oplus P_2)S = (0 \oplus P_2)(P_1 \oplus 0)S = 0$ and so $0 \oplus P_2 \subseteq (0 :_R S)$ which is a contradiction. So $(P_1 \oplus 0)S$ is a quasi-prime submodule of S by Theorem 3.1. Similarly, $(0 \oplus P_2)S \in \mathrm{qSpec}_R(S)$.
(4) If $(0 :_R M) = 0$, then $(0 :_R S) = 0$ by Part (2). Thus $\mathrm{qSpec}_R(S) \neq \emptyset$ by Part (3). Hence the result follows from Part (1).
(5) Let $P_1 \oplus 0 \subseteq (0 :_R S)$. Then $P_1 \oplus 0 \subseteq (0 :_R M)$ since $(0 :_R S) \subseteq (0 :_R M)$. Then $(P_1 \oplus 0)M = 0$ and M is a separated R-module by [21, Lemma 2.9]. The case $0 \oplus P_2 \subseteq (0 :_R S)$ is similar. □

Theorem 4.3. *Let R be the pullback ring as described in (1) and let M be any non-separated R-module. Let $0 \to K \to S \to M \to 0$ be a separated representation of M. Then S is a quasi comultiplication R-module if and only if M is a quasi comultiplication R-module.*

Proof. Let S be a quasi comultiplication R-module. By Proposition 4.2, we may assume that $\mathrm{qSpec}(S) \neq \emptyset$. First suppose that $\bar{S} \neq 0$. Then either $(0 :_R S) = P_1 \oplus P_2^n$ or $(0 :_R S) = P_1^n \oplus P_2$ for some positive integer n by Proposition 3.4. So either $P_1 \oplus 0 \subseteq (0 :_R S)$ or $0 \oplus P_2 \subseteq (0 :_R S)$; hence M is a separated R-module by Proposition 4.2 which is a contradiction. Now, assume that $\bar{S} = 0$, if either $(0 :_R S) = P_1 \oplus 0$ or $(0 :_R S) = 0 \oplus P_2$, then M is a separated R-module by Proposition 4.2 which is a contradiction. So we may assume that $(0 :_R S) = 0$ by Proposition 3.3. Let N be a quasi-prime submodule of M. Then $\varphi^{-1}(N)$ is a quasi-prime submodule of S by Lemma 4.1. So $\varphi^{-1}(N) = (0 :_S (0 :_R \varphi^{-1}(N)))$. We show that $N = (0 :_M (0 :_R \varphi^{-1}(N)))$. If $n \in N$, then $\varphi(s) = n$ for some $s \in S$. Hence $s = \varphi^{-1}(n) \in \varphi^{-1}(N)$. Let $(r_1, r_2) \in (0 :_R \varphi^{-1}(N))$. Then $(r_1, r_2)s = 0$. Then we have $(r_1, r_2)n = (r_1, r_2)\varphi(s) = \varphi((r_1, r_2)s) = 0$ and $n \in (0 :_M (0 :_R \varphi^{-1}(N)))$. Therefore $N \subseteq (0 :_M (0 :_R \varphi^{-1}(N)))$. Now assume that $m \in (0 :_M (0 :_R \varphi^{-1}(N)))$. By Theorem 3.1, we can assume that $(P_1 \oplus 0)M \subseteq N$

and so by Lemma 4.1, $(P_1 \oplus 0)S \subseteq \varphi^{-1}(N)$. Since $(0 :_R S) = 0$, it is easy to see that $(0 :_R (P_1 \oplus 0)S) = 0 \oplus P_2$. Since $(P_1 \oplus 0)S \subseteq \varphi^{-1}(N)$, we have $(0 :_R \varphi^{-1}(N)) \subseteq (0 :_R (P_1 \oplus 0)S) = 0 \oplus P_2$. Therefore $(0 :_R \varphi^{-1}(N)) = 0 \oplus P_2^k$ for some positive integer k. So $\varphi^{-1}(N) = (0 :_S 0 \oplus P_2^k)$ and $m \in (0 :_M 0 \oplus P_2^k)$. There exists $t = (t_1, t_2) \in S$ such that $\varphi(t) = \varphi(t_1, t_2) = m$. Thus $\varphi(0, p_2^k t_2) = \varphi((0, p_2^k)(t_1, t_2)) = (0, p_2^k)m = 0$. Therefore $(0, p_2^k t_2) \in K \cap (0 \oplus P_2)S = 0$ and so $p_2^k t_2 = 0$ by [21, Proposition 2.3]. Hence $t = (t_1, t_2) \in (0 :_S (0 \oplus P_2^k)) = \varphi^{-1}(N)$ and so $m = \varphi(t) \in N$. Then we have the equality.
Conversely, let M be a quasi comultiplication R-module. By Proposition 4.2, we may assume that $\mathrm{qSpec}(S) \neq \emptyset$. Let T be a non-zero quasi-prime submodule of S. Then $K \subseteq T$ by [7, Proposition 4.3], and so T/K is a quasi-prime submodule of S/K by Lemma 2.2. By an argument like that in [7, Theorem 4.4], S is a quasi comultiplication R-module. □

We are ready to determine all indecomposable non-separated quasi comultiplication R-modules.

Proposition 4.4. *Let R be the pullback ring as described in (1). Then the injective hull $E(R/P)$ of R/P is a non-separated quasi comultiplication R-module.*

Proof. By [7, Proposition 4.2], $E(R/P)$ is a non-separated comultiplication R-module. So it is a non-separated quasi comultiplication R-module. □

Proposition 4.5. *Let R be the pullback ring as described in (1) and let M be a non-zero indecomposable non-separated quasi comultiplication R-module. Let $0 \to K \to S \to M \to 0$ be a separated representation of M. Then $\bar{S} = 0$.*

Proof. Assume to the contrary, $\bar{S} \neq 0$. By Theorem 4.3, S is quasi comultiplication; hence S is type (II) of Theorem 3.7 which are indecomposable. Then $P^m S = 0$ for some m, and so $P^m M = 0$. If $m = 1$, then $(P_1 \oplus 0)M \subseteq PM = 0$; so $(P_1 \oplus 0)M \cap (0 \oplus P_2)M = 0$ that is a contradiction. So suppose that $m \geq 2$. Let k be the least positive integer such that $P^k M = 0$ (so $P^{k-1}M \neq 0$). It follows that $\varphi(P^k x) = 0$ for all $x \in S$; so $\varphi(P_1^k x) = \varphi(P_2^k x) = 0$. Since by [19, Proposition 2.3], φ is one-to-one on $P_i S$ for each i, we get $P_1^k x = P_2^k x = 0$. Thus $P^k x = 0$ and hence $P^k S = 0$. Set $N = P^{k-1}M$. Then $0 \to K \to \varphi^{-1}(N) = P^{k-1}S \to N \to 0$ is a separated representation of N by [6, Lemma 3.1]; hence $K \subseteq PP^{k-1}S = P^k S = 0$ which is a contradiction since M is non-separated. Thus $\bar{S} = 0$. □

Before embarking on the proof of the next result let us explain its idea. Let M be any R-module and let $0 \to K \xrightarrow{i} S \xrightarrow{\varphi} M \to 0$ be a separated representation of M. If M is a quasi comultiplication non-separated R-module, then S is quasi-comultiplication by Theorem 4.3. In this case, $S = S_1 \oplus S_2$, where S_i is of type (I) (in statement of Theorem 3.7). In any separated representation $0 \to K \xrightarrow{i} S \xrightarrow{\varphi} M \to 0$ the kernel of the map φ to M is annihilated by P, hence is contained in the socle of the separated module S. Thus M is obtained by amalgamation in the socles of the various direct summands of S.

Let M be any non-zero indecomposable non-separated quasi comultiplication R-module and let $0 \to K \xrightarrow{i} S \xrightarrow{\varphi} M \to 0$ be a separated representation of M. By Proposition 4.4, the modules finite length do not occur among the direct summands of S and $S = S_1 \oplus S_2$, where S_i is of type (I) (in statement of Theorem 3.7). If there are two modules of type (I), then their generators cannot both be annihilated by the same P_i. This contradicts there being two copies of the P_1-Prüfer or two copies of the P_2-Prüfer. So S_1 is P_1-Prüfer and S_2 is P_2-Prüfer. It is clear that the module obtained this amalgamation is, indeed, $E(R/P)$, the R-injective hull of R/P which is an indecomposable quasi-comultiplication non-separated R-module by Proposition 4.4 (also see [5, p. 4053]). Therefore we have the following theorem:

Theorem 4.6. *Let $R = (R_1 \to \bar{R} \leftarrow R_2)$ be the pullback of two discrete valuation domains R_1, R_2 with common factor field $\bar{R}$. Then the only non-zero indecomposable quasi comultiplication non-separated R-module, up to isomorphism, is $E(R/P)$, the R-injective hull of R/P.*

Corollary 4.7. *Let R be the pullback ring as described in Theorem 4.6. Then every non-zero-indecomposable non-separated quasi comultiplication R-module is pure-injective.*

Proof. Apply [5, Theorem 3.5] and Theorem 4.6. □

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