International Electronic Journal of Algebra
Volume 27 (2020) 169-177
DOI: 10.24330/ieja.663001

A NOTE ON FLAG-TRANSITIVE 5-$(v, k, 4)$ DESIGNS

Shaojun Dai and Shangzhao Li

Received: 22 March 2019; Revised: 30 May 2019; Accepted: 29 June 2019
Communicated by Abdullah Harmancı

Abstract. This article is a contribution to the study of the automorphism groups of 5-$(v, k, 4)$ designs. Let $\mathcal{S}$=$(\mathcal{P}, \mathcal{B})$ be a non-trivial 5-$(q+1, k, 4)$ design. If G acts flag-transitively on $\mathcal{S}$, then G is not two-dimensional projective linear group $PSL(2, q)$.

Mathematics Subject Classification (2010): 05B05, 20B25
Keywords: Flag-transitive, 5-designs, $PSL(2, q)$ groups

1. Introduction

For positive integers $t \leq k \leq v$ and λ, we define a t-(v, k, λ) design to be a finite incidence structure $\mathcal{S}$=$(\mathcal{P}, \mathcal{B})$, where $\mathcal{P}$ denotes a set of points, $|\mathcal{P}| = v$, and $\mathcal{B}$ a set of blocks, $|\mathcal{B}| = b$, with the properties that each block is incident with k points, and each t-subset of $\mathcal{P}$ is incident with λ blocks. A flag of $\mathcal{S}$ is an incident point-block pair (x, B) with x is incident with B, where $x \in \mathcal{P}$ and $B \in \mathcal{B}$. We consider automorphisms of $\mathcal{S}$ as pairs of permutations on $\mathcal{P}$ and $\mathcal{B}$ which preserve incidence structure. The full automorphism group of an incidence structure $\mathcal{S}$ will be denotes by $Aut(\mathcal{S})$. We call a group $G \leq Aut(\mathcal{S})$ of automorphisms of $\mathcal{S}$ flag-transitive (respectively block-transitive, point t-transitive, point t-homogeneous) if G acts transitively on the flags (respectively transitively on the blocks, t-transitively on the points, t-homogeneously on the points) of $\mathcal{S}$. For short, $\mathcal{S}$ is said to be, e.g., flag-transitive if $\mathcal{S}$ admits a flag-transitive automorphism group.

For historical reasons, a t-(v, k, λ) design with $\lambda = 1$ is called a Steiner t-design (sometimes this is also known as a Steiner system). If $t < k < v$ holds, then we speak of a non-trivial Steiner t-design.

Investigating t-designs for arbitrary λ, but large t, Cameron and Praeger proved the following result:

Supported by the National Natural Science Foundation of China (11301377, 11701046, 11671402), the Natural Science Foundation of Jiangsu Province (BK20170433) and the Universities Natural Science Foundation of Jiangsu Province (16KJB110001).

Theorem 1.1. ([2]) *Let $\mathcal{S}$=$(\mathcal{P}, \mathcal{B})$ be a t-(v, k, λ) design. If $G \leq Aut(\mathcal{S})$ acts block-transitively on $\mathcal{S}$, then $t \leq 7$, while if $G \leq Aut(\mathcal{S})$ acts flag-transitively on $\mathcal{S}$, then $t \leq 6$.*

Among the properties of homogeneity of incidence structures, flag transitivity obviously is a particularly important and natural one. Originally, F. Buekenhout et al. ([1]) reached a classification of flag-transitive Steiner 2-designs. Recently, Huber ([5]) completely classified all flag-transitive Steiner t-designs using the classification of the finite 2-transitive permutation groups. Hence the determination of all flag-transitive t-designs with $\lambda \geq 2$ has remained of particular interest and has been known as a long-standing and still open problem.

In 2010, Xu ([9]) completely classified flag-transitive 6-(v, k, λ) designs with $\lambda \leq 5$. In 2010, Liu ([7]) completely classified flag-transitive 5-$(v, k, 2)$ design and $PSL(2, q)$ groups. In 2017, Dai ([3]) completely classified flag-transitive 4-$(v, k, 4)$ design and $PSL(2, q)$ groups. The present paper continues the work of classifying flag-transitive t-designs. We discuss the flag-transitive 5-$(v, k, 4)$ designs and $PSL(2, q)$ groups and get the following:

Theorem 1.2. *Let $\mathcal{S}$=$(\mathcal{P}, \mathcal{B})$ be a non-trivial 5-$(q + 1, k, 4)$ design. If G acts flag-transitively on $\mathcal{S}$, then G is not two-dimensional projective linear group $PSL(2, q)$.*

The second section describes the definitions and contains several preliminary results about flag-transitivity t-designs. In the third section we give the proof of Theorem 1.2.

2. Preliminary results

Here we gather notation which are used throughout this paper. For a t-design $\mathcal{S}$=$(\mathcal{P}, \mathcal{B})$ with $G \leq Aut(\mathcal{S})$, let r denotes the number of blocks through a given point, G_x denotes the stabilizer of a point $x \in \mathcal{P}$ and G_B the setwise stabilizer of a block $B \in \mathcal{B}$. We define $G_{xB} = G_x \cap G_B$.

For integers m and n, let (m, n) denotes the greatest common divisor of m and n, and $m \mid n$ if m divides n. All other notation is standard.

Lemma 2.1. ([5]) *Let G act flag-transitively on t-(v, k, λ) design $\mathcal{S}$=$(\mathcal{P}, \mathcal{B})$. If $t \geq 3$, then G is 2-transitive and the following cases hold:*

(1) $|G| = |G_x||x^G| = |G_x|v$, *where $x \in \mathcal{P}$;*
(2) $|G| = |G_B||B^G| = |G_B|b$, *where $B \in \mathcal{B}$;*
(3) $|G| = |G_{xB}||(x, B)^G| = |G_{xB}|bk$, *where $x \in B$.*

Lemma 2.2. ([8]) *Let* $\mathcal{S}$=$(\mathcal{P}, \mathcal{B})$ *be a non-trivial* t-(v, k, λ) *design. Then* $v > k + t$ *and*

$$\lambda(v - t + 1) \geq (k - t + 2)(k - t + 1).$$

Lemma 2.3. ([8]) *Let* $\mathcal{S}$=$(\mathcal{P}, \mathcal{B})$ *be a non-trivial* 5-(v, k, λ) *design. Then*

(1) $bk = vr$;

(2) $b = \frac{\lambda v(v-1)(v-2)(v-3)(v-4)}{k(k-1)(k-2)(k-3)(k-4)}$.

Lemma 2.4. ([8]) *Let* $1 \leq i < t$, $\mathcal{S}$=$(\mathcal{P}, \mathcal{B})$ *is a* t-(v, k, λ) *design. Then* $\mathcal{S}$ *is also an* i-(v, k, λ_i) *design, where*

$$\lambda_i = \lambda \frac{\begin{pmatrix} v-i \\ t-i \end{pmatrix}}{\begin{pmatrix} k-i \\ t-i \end{pmatrix}}.$$

Let q be a prime power p^f, and U a subgroup of $PSL(2, q)$. Furthermore, let N_l denotes the number of orbits of length l and let $(2, q-1) = n$. For the list of subgroups of $PSL(2, q)$, we refer to [4, 6].

Lemma 2.5. *Let* U *be the cyclic group of order* c *with* $c \mid \frac{q\pm1}{n}$. *Then*

(1) *if* $c \mid \frac{q+1}{n}$, *then* $N_c = (q + 1)/c$;

(2) *if* $c \mid \frac{q-1}{n}$, *then* $N_1 = 2$, $N_c = (q - 1)/c$.

Lemma 2.6. *Let* U *be the dihedral group of order* $2c$ *with* $c \mid \frac{q\pm1}{n}$. *Then*

(1) *for* $q \equiv 1\,(mod\,4)$, *we have*

(a) *if* $c \mid \frac{q+1}{2}$, *then* $N_c = 2$ *and* $N_{2c} = (q + 1 - 2c)/(2c)$;

(b) *if* $c \mid \frac{q-1}{2}$, *then* $N_2 = 1$, $N_c = 2$, *and* $N_{2c} = (q - 1 - 2c)/(2c)$, *unless* $c = 2$, *in which case* $N_2 = 3$ *and* $N_4 = (q - 5)/4$.

(2) *for* $q \equiv 3\,(mod\,4)$, *we have*

(a) *if* $c \mid \frac{q+1}{2}$, *then* $N_{2c} = (q + 1)/(2c)$;

(b) *if* $c \mid \frac{q-1}{2}$, *then* $N_2 = 1$ *and* $N_{2c} = (q - 1)/(2c)$.

(3) *for* $q \equiv 0\,(mod\,2)$, *we have*

(a) *if* $c \mid (q + 1)$, *then* $N_c = 1$ *and* $N_{2c} = (q + 1 - c)/(2c)$;

(b) *if* $c \mid (q - 1)$, *then* $N_2 = 1$, $N_c = 2$, *and* $N_{2c} = (q - 1 - c)/(2c)$.

Lemma 2.7. *Let* U *be the elementary Abelian group of order* $\bar{q} \mid q$. *Then* $N_1 = 1$, $N_{\bar{q}} = q/\bar{q}$.

Lemma 2.8. *Let* U *be a semi-direct product of an elementary Abelian subgroup of order* $\bar{q} \mid q$ *and the cyclic subgroup of order* c, *where* c *divides* $\bar{q} - 1$ *and* $q - 1$. *Then* $N_1 = 1$, $N_{\bar{q}} = 1$, $N_{c\bar{q}} = (q - \bar{q})/(c\bar{q})$.

Lemma 2.9. *Let U be $PSL(2,\bar{q})$ and $\bar{q}^m = q$, $m \geq 1$. Then $N_{\bar{q}+1} = 1$, $N_{\bar{q}(\bar{q}-1)} = 1$ if m is even, and all other orbits are regular.*

Lemma 2.10. *Let U be $PGL(2,\bar{q})$ and $\bar{q}^m = q$, $m > 1$ even. Then $N_{\bar{q}+1} = 1$, $N_{\bar{q}(\bar{q}-1)} = 1$, and all other orbits are regular.*

Lemma 2.11. *Let U be isomorphic to A_4. Then*

(1) *for $q \equiv 1\,(mod\,4)$, we have*

(a) *if $3 \mid \frac{q+1}{2}$, then $N_6 = 1$ and $N_{12} = (q-5)/12$;*

(b) *if $3 \mid \frac{q-1}{2}$, then $N_4 = 2$, $N_6 = 1$, and $N_{12} = (q-13)/12$;*

(c) *if $3 \mid q$, then $N_4 = 1$, $N_6 = 1$ and $N_{12} = (q-9)/12$.*

(2) *for $q \equiv 3\,(mod\,4)$, we have*

(a) *if $3 \mid \frac{q+1}{2}$, then $N_{12} = (q+1)/12$;*

(b) *if $3 \mid \frac{q-1}{2}$, then $N_4 = 2$ and $N_{12} = (q-7)/12$;*

(c) *if $3 \mid q$, then $N_4 = 1$ and $N_{12} = (q-3)/12$.*

(3) *for $q = 2^f$, $f \equiv 0\,(mod\,2)$, then $N_1 = 1$, $N_4 = 1$, and $N_{12} = (q-4)/12$.*

Lemma 2.12. *Let U be isomorphic to S_4. Then*

(1) *for $q \equiv 1\,(mod\,8)$, we have*

(a) *if $3 \mid \frac{q+1}{2}$, then $N_6 = 1$, $N_{12} = 1$, and $N_{24} = (q-17)/24$;*

(b) *if $3 \mid \frac{q-1}{2}$, then $N_6 = 1$, $N_8 = 1$, $N_{12} = 1$, and $N_{24} = (q-25)/24$;*

(c) *if $3 \mid q$, then $N_4 = 1$, $N_6 = 1$, and $N_{24} = (q-9)/24$.*

(2) *for $q \equiv -1\,(mod\,8)$, we have*

(a) *if $3 \mid \frac{q+1}{2}$, then $N_{24} = (q+1)/24$;*

(b) *if $3 \mid \frac{q-1}{2}$, then $N_8 = 1$ and $N_{24} = (q-7)/12$.*

Lemma 2.13. *Let U be isomorphic to A_5. Then*

(1) *for $q \equiv 1\,(mod\,4)$, we have*

(a) *if $q = 5^f$, $f \equiv 1\,(mod\,2)$, then $N_6 = 1$ and $N_{60} = (q-5)/60$;*

(b) *if $q = 5^f$, $f \equiv 0\,(mod\,2)$, then $N_6 = 1$, $N_{20} = 1$, and $N_{60} = (q-25)/60$;*

(c) *if $15 \mid \frac{q+1}{2}$, then $N_{30} = 1$ and $N_{60} = (q-29)/60$;*

(d) *if $3 \mid \frac{q+1}{2}$ and $5 \mid \frac{q-1}{2}$, then $N_{12} = 1$, $N_{30} = 1$, and $N_{60} = (q-41)/60$;*

(e) *if $3 \mid \frac{q-1}{2}$ and $5 \mid \frac{q+1}{2}$, then $N_{20} = 1$, $N_{30} = 1$, and $N_{60} = (q-49)/60$;*

(f) *if $15 \mid \frac{q-1}{2}$, then $N_{12} = 1$, $N_{20} = 1$, $N_{30} = 1$, and $N_{60} = (q-61)/60$;*

(g) *if $3 \mid q$ and $5 \mid \frac{q+1}{2}$, then $N_{10} = 1$ and $N_{60} = (q-9)/60$;*

(h) *if* $3 \mid q$ *and* $5 \mid \frac{q-1}{2}$, *then* $N_{10} = 1$, $N_{12} = 1$, *and* $N_{60} = (q-21)/60$.

(2) *for* $q \equiv 3\,(mod\,4)$, *we have*

(a) *if* $15 \mid \frac{q+1}{2}$, *then* $N_{60} = (q+1)/60$;

(b) *if* $3 \mid \frac{q+1}{2}$ *and* $5 \mid \frac{q-1}{2}$, *then* $N_{12} = 1$ *and* $N_{60} = (q-11)/60$;

(c) *if* $3 \mid \frac{q-1}{2}$ *and* $5 \mid \frac{q+1}{2}$, *then* $N_{20} = 1$ *and* $N_{60} = (q-19)/60$;

(d) *if* $15 \mid \frac{q-1}{2}$, *then* $N_{12} = 1$, $N_{20} = 1$, *and* $N_{60} = (q-31)/60$.

3. Proof of Theorem 1.2

Suppose that $G{=}PSL(2,q)$ acts flag-transitively on 5-$(q+1,k,4)$ designs. Then G is point-transitive and $|G| = q(q^2-1)/n$, where $q = p^f > 3$, $n = (2, q-1)$.

By Lemma 2.1(1), we have

$$|G_x| = \frac{|G|}{v} = \frac{q(q^2-1)/n}{q+1} = q(q-1)/n.$$

Again by Lemma 2.3(2) and Lemma 2.1(3),

$$b = \frac{4v(v-1)(v-2)(v-3)(v-4)}{k(k-1)(k-2)(k-3)(k-4)} = \frac{v|G_x|}{k|G_{xB}|}.$$

Thus

$$4|G_{xB}|(q-2)(q-3)n = (k-1)(k-2)(k-3)(k-4), \tag{1}$$

which is equivalent to

$$4|G_{xB}|(q-2)(q-3)n - 24 = k(k^3 - 10k^2 + 35k - 50). \tag{2}$$

By Lemma 2.2,

$$4(q-3) \geq (k-3)(k-4). \tag{3}$$

Thus

$$|G_{xB}|(q-2)n \leq (k-1)(k-2). \tag{4}$$

If $k < 9$, then

$$|G_{xB}|(q-2)(q-3)n = 280, 120, 40 \tag{5}$$

by Eq.(1). By Lemma 2.2, we get $k > 5$ and $q > 10$. Thus q is not exist by Eq.(5).

If $k \geq 9$, then $(k-1)(k-2) < 2(k-3)(k-4)$ and $q \geq 14$. We have

$$|G_{xB}|(q-2)n < 8(q-3). \tag{6}$$

In particular,

$$|G_{xB}|n \leq 7. \tag{7}$$

Since G_B acts transitively on the points of B, we have

$$k = |x^{G_B}| = |G_B : G_{xB}|. \tag{8}$$

We assume that $k \geq 9$ and distinguish three cases:

Case 1. $|G_{xB}| = 1$.

If q is even, then $n = 1$ and $k \mid (4q^2 - 20q)$ by Eq.(2). By Lemmas 2.5-2.13, we have to consider G_B is conjugate to a cyclic group of order c with $c \mid (q+1)$ and $c = k$. Thus

$$k \mid (4q^2 - 20q, q+1) = (q+1, 24) = (q+1, 3).$$

Obviously, $k = 3$ which is clearly impossible.

If q is odd, then $n = 2$ and $k \mid (8q^2 - 40q + 24)$ by Eq.(2). Examining the list of subgroups of $PSL(2, q)$ with their orbits on the projective line by Lemmas 2.5-2.13, we have to consider the following subcase:

Subcase 1.1. G_B is conjugate to a cyclic group of order c with $c \mid \frac{q+1}{2}$ and $c = k$. Thus

$$k \mid (8q^2 - 40q + 24, \frac{q+1}{2}) = (\frac{q+1}{2}, 72).$$

We have $k = 9, 12, 18, 24, 36, 72$. If $k = 9$, then $q = 17$ by Eq.(1). Obviously, $\mathcal{S}$ is a 5-$(18, 9, 4)$ design. By Lemma 2.4, $\mathcal{S}$ is also a 4-design which is impossible since λ_4 is not integer. If $k = 18$, then $q = 87$ by Eq.(1) which is impossible since q is prime power. If $k = 12, 24, 36, 72$, then q is not exist by Eq.(1).

Subcase 1.2. G_B is conjugate to a dihedral group of order $2c$ with $c \mid \frac{q+1}{2}$, $q \equiv 3\,(mod\,4)$ and $2c = k$. Thus

$$k \mid (8q^2 - 40q + 24, q+1) = (q+1, 72).$$

We have $k = 12, 18, 24, 36, 72$. This is easily ruled out as Subcase 1.1.

Subcase 1.3. G_B is conjugate to A_4 with $k = 12$, S_4 with $k = 24$ or A_5 with $k = 60$. We get that q is not exist by Eq.(1).

Case 2. $|G_{xB}| = 2$.

If q is even, then $n = 1$ and $k \mid (8q^2 - 40q + 24)$ by Eq.(2). By Lemmas 2.5-2.13, we have to consider the following subcase:

Subcase 2.1. G_B is conjugate to a cyclic group of order c with $c \mid (q-1)$, which is impossible as $c = 2k$ is even.

Subcase 2.2. G_B is conjugate to a dihedral group of order $2c$ with $c \mid (q+1)$ and $c = k$. Thus

$$k \mid (8q^2 - 40q + 24, q+1) = (q+1, 72) = (q+1, 9).$$

Obviously, $k = 9$ with $q = 17$, which is a contradiction.

Subcase 2.3. G_B is conjugate to a elementary Abelian group of order $\bar{q} \mid q$ and $2k = \bar{q}$. Thus

$$k \mid (8q^2 - 40q + 24, \frac{q}{2}) = (\frac{q}{2}, 24) = (\frac{q}{2}, 8).$$

Obviously, $k \leq 8$, which is a contradiction.

If q is odd, then $n=2$ and $k \mid (16q^2-80q+72)$ by Eq.(2). By Lemmas 2.5-2.13, we have to consider the following subcase:

Subcase 2.4. G_B is conjugate to a cyclic group of order c with $c \mid \frac{(q-1)}{2}$ and $c=2k$. Thus

$$k \mid (16q^2-80q+72, \frac{q-1}{4}) = (\frac{q-1}{4}, 8).$$

We have $k \leq 8$, which is a contradiction.

Subcase 2.5. G_B is conjugate to a dihedral group of order $2c$ with $c=k$. If $c \mid \frac{q+1}{2}$ and $q \equiv 1\,(mod\,4)$, then

$$k \mid (16q^2-80q+72, \frac{q+1}{2}) = (\frac{q+1}{2}, 168) = (\frac{q+1}{2}, 8).$$

If $c \mid \frac{q-1}{2}$ and $q \equiv 3\,(mod\,4)$, then

$$(16q^2-80q+72, \frac{q-1}{2}) = (\frac{q-1}{2}, 8).$$

We have $k \leq 8$, which is a contradiction.

Subcase 2.6. G_B is conjugate to a elementary Abelian group of order $\bar{q} \mid q$ and $2k=\bar{q}$, which is impossible as q is odd.

Subcase 2.7. G_B is conjugate to A_4 with $k=6$, S_4 with $k=12$ or A_5 with $k=30$. We get that q is not exist by Eq.(1).

Case 3. $|G_{xB}| \geq 3$.

If q is even, then $n=1$ and $|G_{xB}|=3,4,5,6,7$. Thus $k \mid (4|G_{xB}|(q-2)(q-3)-24)$ by Eq.(2). By Lemmas 2.5-2.13, we have to consider the following subcase:

Subcase 3.1. G_B is conjugate to a dihedral group of order $2c$ with $c \mid (q-1)$ and $2c=3k$. Thus let $k=4m+1$. We have $q \mid b$ by Lemma 2.3(2). Again by Lemma 2.1(2), $b=\frac{q(q^2-1)}{|G_B|}$. Then $|G_B|$ is odd, which is impossible as $|G_B|=2c$ is even.

Subcase 3.2. G_B is conjugate to a semi-direct product of an elementary Abelian subgroup of order $\bar{q} \mid q$ and the cyclic subgroup of order c, where c divides $\bar{q}-1$ and $q-1$, and $k \mid \bar{q}$. If $|G_{xB}|=3$, then $k \mid (12q^2-60q+48, q)=(q,48)=(q,16)$. If $|G_{xB}|=4$, then $k \mid (16q^2-80q+72, q)=(q,72)=(q,8)$. If $|G_{xB}|=5$, then $k \mid (20q^2-100q+96, q)=(q,96)=(q,32)$. If $|G_{xB}|=6$, then $k \mid (24q^2-120q+120, q)=(q,120)=(q,8)$. If $|G_{xB}|=7$, then $k \mid (28q^2-140q+144, q)=(q,144)=(q,16)$. We have $k=16,32$. But q is not exist.

Subcase 3.3. G_B is conjugate to $PSL(2,\bar{q})$ with $\bar{q}^m=q$, $m \geq 1$ and $k=\bar{q}+1$ or $\bar{q}(\bar{q}-1)$ if m is even. If $k=\bar{q}+1$, then $|G_{xB}|=\bar{q}(\bar{q}-1)$. Thus, by Eq.(1),

$$4(q-2)(q-3)=(\bar{q}-2)(\bar{q}-3).$$

This is impossible as $\bar{q}^m = q$. If $k = \bar{q}(\bar{q}-1)$, then $|G_{xB}| = \bar{q}+1$ and

$$\bar{q}(\bar{q}-1) \mid (4(q-2)(q-3)(\bar{q}+1)-24) = 4\bar{q}^{2m+1} + 4\bar{q}^{2m} - 20\bar{q}^{m+2} - 20\bar{q}^{m+1} + 24\bar{q}.$$

Since $(4\bar{q}^{2m+1} + 4\bar{q}^{2m} - 20\bar{q}^{m+2} - 20\bar{q}^{m+1} + 24\bar{q}, \bar{q}-1) = (\bar{q}-1, 8) = 1$, which is a contradiction.

Subcase 3.4. G_B is conjugate to $PGL(2,\bar{q})$ with $\bar{q}^m = q$, $m > 1$ even and $k = \bar{q}+1$ or $\bar{q}(\bar{q}-1)$. By q even, $PGL(2,\bar{q}) \cong PSL(2,\bar{q})$. We get that q is not exist by subcase 3.3.

Subcase 3.5. G_B is conjugate to A_4 and $k = 4$, which is impossible since $k \geq 9$.

If q is odd, then $n = 2$ and $|G_{xB}| = 3$. Thus $k \mid (24q^2 - 120q + 120)$ by Eqs.(2) and (7).

Subcase 3.6. G_B is conjugate to a dihedral group of order $2c$ with $c \mid \frac{(q-1)}{2}$ with $q \equiv 1\,(mod\,4)$ and $2c = 3k$. Thus

$$k \mid (24q^2 - 120q + 120, \frac{q-1}{3}) = (\frac{q-1}{3}, 24).$$

We have $k = 12, 24$ which is impossible since q is not exist by Eq.(1).

Subcase 3.7. G_B is conjugate to a semi-direct product of an elementary Abelian subgroup of order $\bar{q} \mid q$ and the cyclic subgroup of order c, where c divides $\bar{q}-1$ and $q-1$, and $k \mid \bar{q}$. Thus

$$k \mid (24q^2 - 120q + 120, q) = (q, 120) = (q, 8).$$

We have $k = 8$, which is impossible since $k \geq 9$.

Subcase 3.8. G_B is conjugate to $PSL(2,\bar{q})$ with $\bar{q}^m = q$, $m \geq 1$ and $k = \bar{q}+1$ or $\bar{q}(\bar{q}-1)$ if m is even. If $k = \bar{q}+1$, then $|G_{xB}| = \frac{\bar{q}(\bar{q}-1)}{2} = 3$. If $k = \bar{q}(\bar{q}-1)$, then $|G_{xB}| = \frac{\bar{q}+1}{2} = 3$. We have $k = 4$ or 20, which is impossible since q is not exist by Eq.(1).

Subcase 3.9. G_B is conjugate to $PGL(2,\bar{q})$ with $\bar{q}^m = q$, $m > 1$ even and $k = \bar{q}+1$ or $\bar{q}(\bar{q}-1)$. If $k = \bar{q}+1$, then $|G_{xB}| = \bar{q}(\bar{q}-1) = 3$. If $k = \bar{q}(\bar{q}-1)$, then $|G_{xB}| = \bar{q}+1 = 3$. We have $k = 2$ which impossible since since $k \geq 9$.

Subcase 3.10. G_B is conjugate to S_4 with $k = 8$ or A_5 with $k = 20$, which is impossible by Eq.(1).

This completes the proof of Theorem 1.2.

Acknowledgement. The authors would like to thank the referee for the valuable suggestions and comments.

References

[1] F. Buekenhout, A. Delandtsheer, J. Doyen, P. B. Kleidman, M. W. Liebeck and J. Saxl, *Linear spaces with flag-transitive automorphism groups*, Geom. Dedicata, 36(1) (1990), 89-94.

[2] P. J. Cameron and C. E. Praeger, *Block-transitive t-designs, II: large t*, In Finite geometry and combinatorics, Deinze, 1992, London Math. Soc. Lecture Note Ser., Cambridge Univ. Press, 191 (1993), 103-119.

[3] S. Dai and S. Li, *Flag-transitive* 4-$(v, k, 4)$ *designs and* $PSL(2, q)$ *groups*, Util. Math., 105 (2017), 3-11.

[4] M. Huber, *A census of highly symmetric combinatorial designs*, J. Algebraic Combin., 26(4) (2007), 453-476.

[5] M. Huber, Flag-Transitive Steiner Designs, Frontiers in Mathematics, Birkhäuser Verlag, Basel, 2009.

[6] B. Huppert, Endliche Gruppen I, Die Grundlehren der Mathematischen Wissenschaften, Band 134, Springer-Verlag, Berlin-New York, 1967.

[7] W. J. Liu, Q. H. Tan and L. Z. Gong, *Flag-transitive* 5-$(v, k, 2)$ *designs*, J. Jiangsu Univ. (Natural Science Edition), 31(5) (2010), 612-615.

[8] H. Shen, The Theory of Combinatorial Design, Shanghai Jiao Tong Univ. Press, Shanghai, 2008.

[9] X. Xu and W. Liu, *On flag-transitive* 6-(v, k, λ) *designs with* $\lambda \leq 5$, Ars Combin., 97 (2010), 507-510.

Shaojun Dai (Corresponding Author)
School of Mathematical Sciences
Tianjin Polytechnic University
No.399 Binshuixi Road, Xiqing District
300387 Tianjin, P. R. China
e-mail: daishaojun@tjpu.edu.cn

Shangzhao Li
School of Mathematics and Statistics
Changshu Institute of Technology
215500 Changshu, Jiangsu, P. R. China
e-mail: lszfd2004@163.com