

ULTRA STAR OPERATIONS ON ULTRA PRODUCT OF INTEGRAL DOMAINS

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ABSTRACT. We introduce the notion of ultra star operation on ultraproduct of integral domains as a map from the set of induced ideals into the set of induced ideals satisfying the traditional properties of star operations. A case of special interest is the construction of an ultra star operation on the ultraproduct of integral domains R_i 's from some given star operations \star_i on R_i 's. We provide the ultra b -operation and the ultra v -operation. Given an arbitrary star operation \star on the ultraproduct of some integral domains, we pose the problem of whether the restriction of \star to the set of induced ideals is necessarily an ultra star operation. We show that the ultraproduct of integral domains R_i 's is a \star -Prüfer domain if and only if R_i is a \star_i -Prüfer domain for \mathcal{U} -many i .

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1. Introduction

Let R be an integral domain with quotient field K . Throughout this paper $F(R)$ denotes the set of all nonzero fractional ideals of R and $f(R)$ denotes the set of all nonzero finitely generated fractional ideals of R .

A *star operation* on R is a mapping $A \rightarrow A^*$ of $F(R)$ into $F(R)$ such that for all $A, B \in F(R)$ and for all $a \in K \setminus \{0\}$,

- (i) $(a)^* = (a)$ and $(aA)^* = aA^*$;
- (ii) $A \subseteq A^*$ and $A \subseteq B \Rightarrow A^* \subseteq B^*$, and
- (iii) $A^{**} := (A^*)^* = A^*$.

For an overview of star operations, the reader may refer to [5, Sections 32 and 34]. An $I \in F(R)$ is a \star -ideal if $I^* = I$. An obvious example of a star operation is the identity map traditionally denoted by d and defined as $A^d = A$ for all $A \in F(R)$. Another well known star operation we intensively use in this paper is the v -operation. For $A, B \in F(R)$, $(B : A) = \{x \in K : xA \subseteq B\}$ and $A^{-1} = (R : A)$. The v -operation is defined by $A_v = A^{-1-1}$ for all $A \in F(R)$. The ideal A_v is also called the v -closure of A . If $A_v = A$, then A is called a *divisorial ideal* of R . Another

key example of star operation to be used in the paper is the so called b -operation defined on an integrally closed domain R by $A^b = \bigcap_{\alpha} AV_{\alpha}$ for all $A \in F(R)$, where $\{V_{\alpha}\}$ is the family of all valuation overrings of R .

The notion of star operation in commutative rings is often used to not only provide generalizations of classical domains, but to also produce a common treatment and deeper understanding of those domains. One of these instances is the notion of Prüfer \star -multiplication domain which generalizes the notion of Prüfer domains [3,6]; a notion that will be used in this paper. Under the same spirit, in this paper, we introduce the notion of ultra star operation on an ultraproduct of integral domains and provide some examples. We show that a “true” star operation on a domain does not always restrict to an ultra star operation. For easy reference, in Section 2, we review some definitions and results of ultraproducts of commutative rings that can be found in [10,11] and provide some results to be used in later sections.

In Section 3, we define an ultra star operation on the ultraproduct \mathfrak{R} of integral domains R_i 's and show how to build a “star operation”, called ultra star operation, from star operations on R_i 's. We show that this construction yields the ultra v -operation and the ultra b -operation when we start with the traditional v -operations on the R_i 's and the traditional b -operations on the R_i 's respectively.

In Section 4, we show as an application of the introduction of the notion of ultra star operation that if an ultra star operation \star on \mathfrak{R} is built from star operations \star_i 's on the components R_i 's as in Proposition 3.1, then \mathfrak{R} is a \star -Prüfer domain if and only if R_i is a \star_i -Prüfer domain for \mathcal{U} -many i . We recall here that if \mathcal{U} is an ultrafilter on a nonempty set X , a property \mathcal{P} holds for \mathcal{U} -many i if the set of all $i \in X$ such that R_i satisfy \mathcal{P} is an element of \mathcal{U} (see Section 2 below).

2. Notations and preliminaries

Let X be a set. Let $\mathcal{P}(X)$ be the collection of all subsets of X . A nonempty set $\mathcal{F} \subseteq \mathcal{P}(X)$ is a *filter* on $\mathcal{P}(X)$ if for all $A, B \in \mathcal{F}$ we have:

- (1) if $A, B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$;
- (2) if $A \in \mathcal{F}$ and $A \subseteq B$, then $B \in \mathcal{F}$.

A filter \mathcal{F} on $\mathcal{P}(X)$ is *proper* if it is properly contained in $\mathcal{P}(X)$. A proper filter \mathcal{F} is a *maximal filter* if the only filter that contains it is $\mathcal{P}(X)$. By Zorn's Lemma, every proper filter of $\mathcal{P}(X)$ is contained in a maximal filter of $\mathcal{P}(X)$.

An *ultrafilter* \mathcal{U} on X is a maximal filter on $\mathcal{P}(X)$. Equivalently, a filter \mathcal{U} on X is an ultrafilter if and only if for all $A \subseteq X$, either $A \in \mathcal{U}$ or $X \setminus A \in \mathcal{U}$. If an ultrafilter \mathcal{U} contains a finite subset of X , then it contains a singleton set,

say $\{x\}$, and x is an element of every element of \mathcal{U} . In this case \mathcal{U} is a *principal* ultrafilter. An ultrafilter that is not principal is called a *free* ultrafilter. Let $\{R_i\}_{i \in I}$ be the collection of commutative rings indexed by a set I . The *direct-product* R of the commutative rings $\{R_i\}_{i \in I}$ is a commutative ring consisting of elements $(a_i)_{i \in I} = (a_1, a_2, \dots, a_i, \dots)$, with $a_i \in R_i$.

If \mathcal{U} is an ultrafilter on I , then we define on R a relation \sim by

$$(a_i)_{i \in I} \sim (b_i)_{i \in I} \iff \{i \in I : a_i = b_i\} \in \mathcal{U}.$$

The quotient set R/\sim is denoted $\prod_{\mathcal{U}} R_i =: \mathfrak{R}$ is called the *ultraproduct* of the R_i 's with respect to the ultrafilter \mathcal{U} . We denote simply by (a_i) the element of \mathfrak{R} determined by the equivalence class of (a_i) . In the case that for all $i \in I$, $R_i = R$, then \mathfrak{R} is called the *ultrapower* of R and is denoted $\mathfrak{R}^{\mathcal{U}}$.

2.1. Los' Theorem. A property \mathcal{P} holds for \mathcal{U} -many i if the set of all i such that R_i satisfy \mathcal{P} is an element of \mathcal{U} . Any first order formula over the language of commutative rings utilizes the symbols $+$, \cdot , $=$, 0 , 1 . One of the most fundamental properties of ultraproducts due to Los is that they preserve first order properties.

Theorem 2.1. (Los' Theorem) *Let ϕ be a formula in the first order language of rings. Then R satisfies ϕ if and only if R_i satisfies ϕ for \mathcal{U} -many i .*

It follows from Los' Theorem that \mathfrak{R} is an integral domain (a field resp.) if and only if R_i is an integral domain (a field resp.) for \mathcal{U} -many i . This is because the theory of integral domains and fields consists of finitely many axioms, all of which can be easily be expressed in the first order language of integral domains and fields.

If \mathcal{U} is a principal ultrafilter, say $\{i\} \in \mathcal{U}$, then it is not hard to see that $\mathfrak{R} \cong R_i$. Thus, throughout this paper, we are interested in a collection of integral domains $\{R_i\}_{i \in I}$ indexed by an infinite set I , and \mathcal{U} is a non principal ultrafilter. The quotient field of each R_i is denoted F_i . The quotient field of \mathfrak{R} is the ultraproduct $\prod_{\mathcal{U}} F_i =: \mathfrak{F}$.

An ideal A of a commutative ring R is *definable* in R if there exists a first order formula $\psi(x, y_1, \dots, y_n)$ in the language of commutative rings and $r_1, \dots, r_n \in R$ such that

$$r \in A \iff \psi(r, r_1, \dots, r_n) \text{ is true in } R.$$

A finitely generated ideal A of a domain R with quotient field K and its dual A^{-1} are definable, as well as the maximal ideal of any quasilocal ring (see [10]). For example, if $A := (r_1, r_2, \dots, r_n)$ is a finitely generated ideal of R , then $x \in K$ is in

A^{-1} if there exist some $r, \alpha_1, \alpha_2, \dots, \alpha_n \in R$ such that $r = x\alpha_1r_1 + x\alpha_2r_2 + \dots + x\alpha_nr_n$). Thus A^{-1} is defined in K by the formula

$$\phi(s, t_1, t_2, \dots, t_n) : \exists r, \alpha_1, \alpha_2, \dots, \alpha_n (r = s\alpha_1t_1 + s\alpha_2t_2 + \dots + s\alpha_nt_n),$$

since $x \in A^{-1}$ if and only if $\phi(x, r_1, r_2, \dots, r_n)$ is true in K .

Recall that an ideal A of \mathfrak{R} is *induced* if $A = (A_i)_{\mathcal{U}}$ for some subsets A_i of R_i , $i \in I$. Observe that $A = (A_i)$ is an induced ideal of \mathfrak{R} if and only if A_i is an ideal of R_i for \mathcal{U} -many i . Some facts about the algebra of induced ideals are provided in Proposition 2.2. Recall also that an ideal A of a commutative ring is n -generated if A can be generated by n elements. Note that an ideal A is n -generated if and only if $A = (E_i)_{\mathcal{U}}$ for some n -generated ideals E_i of R_i .

Proposition 2.2. (See also [10, Lemma 2.1]) *Suppose that $A = (A_i)_{\mathcal{U}}$ and $B = (B_i)_{\mathcal{U}}$ are induced ideals of \mathfrak{R} . Then*

- (i) $A \subseteq B$ if and only if $A_i \subseteq B_i$ for \mathcal{U} -many i ;
- (ii) $\mathfrak{R}/A \cong \prod_{\mathcal{U}} R_i/A_i$;
- (iii) A is a prime (maximal) ideal if and only if A_i is a prime (maximal) ideal of R_i for \mathcal{U} -many i ;
- (iv) A is an n -generated ideal if and only if A_i is an n -generated ideal for \mathcal{U} -many i ;
- (v) $A \cap B = (A_i \cap B_i)_{\mathcal{U}}$;
- (vi) $A^{-1} = (A_i^{-1})_{\mathcal{U}}$;
- (vii) A is an invertible ideal of \mathfrak{R} if and only if there exists $n > 0$ such that A_i is an n -generated invertible ideal of R_i for \mathcal{U} -many i ;
- (viii) A is a divisorial ideal of \mathfrak{R} if and only if A_i is a divisorial ideal of R_i for \mathcal{U} -many i .

Proof. All the statements are shown in [10, Lemma 2.1] except (vi). To see (vi), if $A = (A_i)_{\mathcal{U}}$, let $x = (x_i) \in A^{-1}$. Then $x(A_i)_{\mathcal{U}} \subseteq \mathfrak{R}$, that is, by (i), $x_iA_i \subseteq R_i$ for \mathcal{U} -many i . Thus $x_i \in A_i^{-1}$ for \mathcal{U} -many i and so $x \in (A_i^{-1})_{\mathcal{U}}$ by Los' Theorem. Hence $A^{-1} = (A_i^{-1})_{\mathcal{U}}$. □

In general, except in some very special cases, an ultraproduct of Noetherian rings is non-Noetherian (see for example [13, Page 12, 1.4.13]) with a very complicated prime spectrum. However, it is of interest in applications to understand ultraproduct of Noetherian rings. For example H. Shoutens studies in [13, Chapter 8] tight closures in positive characteristic by relying on ultraproduct of Noetherian rings.

We next assume that the R_i 's are Noetherian and are chosen such that the ultraproduct \mathfrak{R} is Noetherian. Then we aim to prove that an induced ideal J of \mathfrak{R}

is integrally closed if and only if each component J_i of J is an integrally closed ideal of R_i for \mathcal{U} -many i . Recall that for an ideal A of an integral domain R , the *integral closure* A^b of A consists of $r \in R$ satisfying the integral dependence equation of r over A : $r^s + a_1 r^{s-1} + a_2 r^{s-2} + \dots + r a_{s-1} + a_s = 0$ for some positive integer s and some $a_i \in A^i$ [7, Definition 1.1.1]. Equivalently $A^b = \bigcap_{\alpha} AV_{\alpha}$, where $\{V_{\alpha}\}$ is the family of all valuation overrings of R [5, page 398]. An ideal A of R is said to be *integrally closed* if $A^b = A$.

Proposition 2.3. *Let us assume that the R_i 's are Noetherian domains such that \mathfrak{R} is Noetherian. Then an induced ideal $A = (A_i)_{\mathcal{U}}$ of \mathfrak{R} is integrally closed if and only if A_i is an integrally closed ideal of R_i for \mathcal{U} -many i .*

Proof. (\Leftarrow) Suppose in one direction that the A_i 's are integrally closed for \mathcal{U} -many i and let us show that $A = (A_i)_{\mathcal{U}}$ is integrally closed. Let $f \in A^b$ satisfying an integral dependence equation witnessing this. Write f as an ultraproduct of elements $f_i \in R_i$, then Los' Theorem gives for \mathcal{U} -many i an integral dependence equation witnessing that f_i lies in the integral closure of A_i , whence in A_i , and hence lies in $A = (A_i)_{\mathcal{U}}$.

(\Rightarrow) Conversely, suppose $A := (A_i)_{\mathcal{U}}$ is integrally closed but not the A_i 's for \mathcal{U} -many i . Let's choose some $f_i \in R_i$ in the integral closure of A_i but not in A_i itself. Let $f_{\mathcal{U}}$ be the ultraproduct of those f_i 's. By Los' Theorem, $f_{\mathcal{U}}$ does not lie in A whence not in its integral closure as $A^b = A$, since \mathfrak{R} is Noetherian, so that by a well-known integrality criterion (see [13, Theorem 8.4.1]), there exists a Discrete Valuation Ring (for short DVR) V containing A such that $f_{\mathcal{U}} \notin AV$. We may assume V to be complete and with algebraic closed residue field, and then by some arguments on cataproducts (see [13, Proof of Theorem 11.1.4]), it follows that there exists for each i a DVR V_i containing R_i such that their cataproduct V_{\sharp} is equal to V (recall that the cataproduct V_{\sharp} is the quotient of the ultraproduct $V_{\mathcal{U}}$ modulo the intersection of the powers of the maximal ideal [13, page 172]). Hence \mathfrak{R} is contained in $V_{\mathcal{U}}$ and we must have $f_{\mathcal{U}} \notin AV_{\mathcal{U}}$ (lest this membership would then also be true in the quotient $V_{\sharp} = V$). By Los' Theorem, $f_i \notin A_i V_i$ for \mathcal{U} -many i , which in turn means by the integrality criterion alluded to above that f_i does not belong to the integral closure of A_i (since the R_i 's are Noetherian), a contradiction. \square

Remark 2.4. 1. Note that even though we assume the R_i 's to be Noetherian, we do not need it to show the direction (\Leftarrow).

2. Observe that an induced ideal $A = (A_i)_{\mathcal{U}}$ is integrally closed if and only if A satisfies the sentence ϕ_n that asserts that if $a, b \in A$ and $f(a, b) = 0$ for

some degree n and for some $f(x, y) := x^n + r_{n-1}x^{n-1}y + \dots + r_0y^n \in A[x, y]$. Note that Schmidt-van den Dries theorem [13, Section 7.2] proves that the necessary bounds exist so that the integral closure of ideals would be a definable condition. And hence Los' Theorem will only prove Proposition 2.3 for ultraproduct of ideals generated by polynomials of degree d (these form a strict subclass of all induced ideals). Hence definability is not always the road to go, especially that the two directions of Proposition 2.3 might need different proofs (notice that the direction (\Leftarrow) of the proof cannot be reversed since we might get integral equations of increasing degree, which become meaningless in the ultraproduct).

3. Ultra star operations on ultra product of domains

In the star operation context, it is customary to replace the v -operation by an arbitrary star operation in results such as Proposition 2.2 (viii) in order to generalize concepts. Proposition 2.2 (viii) is therefore our motivation to introduce the notion of induced star operation on an ultraproduct of integral domains.

From now on assume that \star_i is a star operation on R_i for each $i \in I$ and denote $Ind(\mathfrak{R})$ the collection of induced ideals of \mathfrak{R} . Defined the map:

$$\star : Ind(\mathfrak{R}) \rightarrow Ind(\mathfrak{R}), A = (A_i)_{\mathcal{U}} \mapsto A^\star = (A_i^{\star_i})_{\mathcal{U}}.$$

Proposition 3.1. *The map \star as defined above does satisfy the following properties:*

- (Ind₁) $\mathfrak{R}^\star = \mathfrak{R}$ and $((a_i)A)^\star = (a_i)A^\star$;
- (Ind₂) $A \subseteq A^\star$ and $A \subseteq B \Rightarrow A^\star \subseteq B^\star$, and
- (Ind₃) $A^{\star\star} := (A^\star)^\star = A^\star$;

for all $A, B \in Ind(\mathfrak{R})$ and for all $(a_i) \in \mathfrak{F} \setminus \{0\}$.

Proof. Statements (Ind₂) and (Ind₃) are clear by combining axioms (ii) and (iii) of each \star_i described in the introduction and Proposition 2.2 (i).

For (Ind₁), note that $\mathfrak{R} = (R_i)_{\mathcal{U}}$ and hence $\mathfrak{R}^\star = \mathfrak{R}$, since $R_i^{\star_i} = R_i$. Note also that $((a_i)A)_{\mathcal{U}} = (a_iA_i)_{\mathcal{U}}$ and it will then follow that $((a_i)A)^\star = (a_i)A^\star$, since $(a_iA_i)^{\star_i} = a_iA_i^{\star_i}$. \square

We have then motivated the following definition:

Definition 3.2. (i) An *ultra star operation* \star on \mathfrak{R} is a mapping from $Ind(\mathfrak{R})$ into $Ind(\mathfrak{R})$ satisfying the conditions Ind_1 , Ind_2 and Ind_3 in Proposition 3.1.

- (ii) A *strong ultra star operation* \star on \mathfrak{R} is a star operation \star on \mathfrak{R} in the classical sense preserving induced ideals, i.e., for each induced ideal A of \mathfrak{R} A^\star is an induced ideal of \mathfrak{R} .

Remark 3.3. It is clear that any strong ultra star operation on \mathfrak{R} is an ultra star operation, but the converse is not necessarily true (see Question.)

If we are given some star operations \star_i on R_i for each i , then \star constructed in Proposition 3.1 is an ultra star operation on \mathfrak{R} .

- Example 3.4.**
1. The identity d is clearly an ultra star operation on \mathfrak{R} .
 2. The v -operation on \mathfrak{R} preserves the induced ideals. In fact, by Proposition 2.2 (vi), we have $A_v = A^{-1-1} = (A_i^{-1-1})_{\mathcal{U}} = (A_i^{v_i})_{\mathcal{U}}$. So the v -operation is a strong ultra star operation on \mathfrak{R} .
 3. In Proposition 3.1, if each $\star_i = v_i$ for each i , then the mapping $A = (A_i)_{\mathcal{U}} \mapsto A^\star = (A_i^{v_i})_{\mathcal{U}}$ is an ultra star operation that coincides with the restriction of the v -operation of \mathfrak{R} on the set of induced ideals in view of Proposition 2.2 (vi) and (viii). So the mapping $A = (A_i)_{\mathcal{U}} \mapsto A^\star = (A_i^{v_i})_{\mathcal{U}}$ may be legitimately called the *ultra v -operation* on \mathfrak{R} .
 4. If each $\star_i = b_i$ for each i , where b_i is the integral closure of ideals of R_i , where each R_i is assumed to be integrally closed so that each b_i is a star operation on R_i , then the mapping $A = (A_i)_{\mathcal{U}} \mapsto A^\star = (A_i^{b_i})_{\mathcal{U}}$ is an ultra star operation. Note that by Proposition 2.3, A^\star is an integrally closed ideal of \mathfrak{R} (we do not need the assumption of the R_i 's being Noetherian for this) and therefore can be legitimately labeled as the *ultra b -operation*. Also note that \mathfrak{R} is integrally closed as all the R_i 's are integrally closed. In fact, recall that a domain R is integrally closed if and only if for all $n > 0$, R satisfies the sentence ϕ_n that asserts that if $a, b \in R$ and $f(a, b) = 0$ for some degree n form $f(x, y) := x^n + r_{n-1}x^{n-1}y + \dots + r_0y^n \in R[x, y]$, then b divides $a \in R$. Thus the statement holds by Los' Theorem.

Note however that we do not know whether the ultra b -operation coincide with the restriction of the b -operation of \mathfrak{R} . More generally, we do not expected all star operations on \mathfrak{R} to preserve induced ideals and therefore we may not obtain an ultra star operation just by restricting any star operation of \mathfrak{R} on the set of induced ideals. We can make the following observation for the b -operation in the case we assume that the R_i 's are Noetherian domains such that \mathfrak{R} is Noetherian: if $A = (A_i)_{\mathcal{U}}$ is an induced ideal of \mathfrak{R} , then $A^b = (A_i)_{\mathcal{U}}^b \subseteq (A_i^{b_i})_{\mathcal{U}}^b = (A_i^{b_i})_{\mathcal{U}}$, the later equality holds by combining Proposition 2.2 (i) and Proposition 2.3. If A^b is

induced, then $A^b = (B_i)_\mathcal{U}$. So $(A_i)_\mathcal{U} \subseteq (B_i)_\mathcal{U} \subseteq (A_i^{b_i})_\mathcal{U}$, that is $A_i \subseteq B_i \subseteq A_i^{b_i}$ for \mathcal{U} -many i .

Question: If A is an induced ideal of \mathfrak{R} , must A^b be an induce ideal?

- Remark 3.5.**
1. The \star_i 's in the construction of \star in Proposition 3.1 are arbitrary star operations on R_i 's. If we pick the star operation to be of the same kind (say for example, the \star_i is the identity for all i), then the ultra star operation \star obtained should be denoted $\star_\mathcal{U}$. For instance $d_\mathcal{U}$ means the ultra star operation obtained by considering the identity on each R_i in Proposition 3.1, $v_\mathcal{U}$ is the ultra v -operation on \mathfrak{R} when the v -operation is considered on each R_i , and $b_\mathcal{U}$ is the ultra b -operation on \mathfrak{R} when the b -operation is considered on each R_i .
 2. Though we could not decide, even in the special case where the R_i 's are chosen Noetherian such that \mathfrak{R} is also Noetherian, if the b -operation is a strong ultra star operation, note that we can at least construct an ultra star operation as in Proposition 3.1 from the b_i -operations on R_i 's without the assumption on the R_i 's being Noetherian. So the ultra star operation $b_\mathcal{U}$ may not coincide with the b -operation on \mathfrak{R} .
 3. From Proposition 2.3, we can choose different kinds of star operations on R_i 's (for instance, one could randomly pick the identity, the v -operations, the b -operations at different components of the ultraproduct \mathfrak{R}) to built a complex ultra star operation on \mathfrak{R} .

4. Ultra star operations and Prüfer \star -multiplication domains

In this section we aim to establish that if an ultra star operation \star on \mathfrak{R} is built from star operations on the components R_i 's as in Proposition 3.1, then \mathfrak{R} is a \star -Prüfer domain if and only if R_i is a \star_i -Prüfer domain for \mathcal{U} -many i . We also show that if an ultra star operation \star_f on \mathfrak{R} is built from finite type star operations associated to the $(\star_i)_f$'s, then \mathfrak{R} is a \star_f -Prüfer domain if and only if R_i is a Prüfer \star_i -multiplication domain for \mathcal{U} -many i .

Recall that given a star operation \star on an integral domain R , an ideal A of R is a \star -ideal if $A^\star = A$. We have the following proposition relating the \star -ideals of \mathfrak{R} and the \star_i -ideals of R_i 's.

Proposition 4.1. *Let \star be an ultra star operation on \mathfrak{R} built from the star operations \star_i 's on R_i 's. Then an induced ideal $A = (A_i)_\mathcal{U}$ of \mathfrak{R} is a \star -ideal if and only if A_i is a \star_i -ideal of R_i for \mathcal{U} -many i .*

Proof. Suppose $A = (A_i)_{\mathcal{U}}$ is a \star -ideal of \mathfrak{R} . Then $(A_i)_{\mathcal{U}} = ((A_i)_{\mathcal{U}})^{\star} = (A_i^{\star})_{\mathcal{U}}$. So $A_i^{\star} \subseteq A_i$ for \mathcal{U} -many i , by proposition 2.2 (i). Since $A_i \subseteq A_i^{\star}$ for all i , it follows that $A_i^{\star} = A_i$ for \mathcal{U} -many i .

Conversely, suppose that $A_i^{\star} = A_i$ for \mathcal{U} -many i . Clearly $A = (A_i)_{\mathcal{U}} = (A_i^{\star})_{\mathcal{U}}$ and hence A is a \star -ideal. \square

Given a star operation \star on R , one can construct a new star operation \star_f as follows: for each $A \in F(R)$, $A^{\star_f} = \cup\{B^{\star} | B \subseteq A \text{ and } B \in f(R)\}$. Note that $d_f = d$, where d is the identity star operation and if \star is the v -operation we denote $v_f := t$ and call it the t -operation. A nonzero ideal A of R is a \star -ideal if $A^{\star} = A$. Similarly, we call a \star -ideal of R a \star -prime ideal of R if it is also a prime ideal. We call a maximal element, if such exists, in the set of all proper \star -ideals of R a \star -maximal ideal of R . We denote $\text{Spec}^{\star}(R)$ the set of all \star -prime ideals of R and $\text{Max}^{\star}(R)$ the set of all \star -maximal ideals of R . An $A \in \mathcal{F}(R)$ is said to be \star -invertible if $(AA^{-1})^{\star} = R$, whereas a domain R is a *Prüfer \star -multiplication domain* (in short, $\text{P}\star\text{MD}$) if every finitely generated ideal A of R is \star_f -invertible, i.e., $(AA^{-1})^{\star_f} = R$ for any $A \in f(R)$. Thus a Prüfer domain is a PdMD and PvMD is often called a pseudo Prüfer domain.

For lacking a specific reference, we aim to first prove that a domain R is a $\text{P}\star\text{MD}$ if and only if each ideal of R with two generators is \star_f -invertible. Note that this result is a generalization of the facts that a domain is Prüfer if and only if each ideal with two generators is invertible [12, page 7]. The same result has been shown in the v -domain ([9]) and in the PvMD ([8]) cases. We start with the recollection of some facts about star operations. Let \star be a star operation on R . Recall that \star is *stable* if $(A \cap B)^{\star} = A^{\star} \cap B^{\star}$. Define $\tilde{\star}$ by $A^{\tilde{\star}} := \cap\{AR_M | M \in \text{Max}^{\star_f}(R)\}$, for all $A \in F(R)$. Then it is well known that $\tilde{\star}$ is a stable star operation on R of finite type called the *stable star operation of finite type associated to \star* . It is not hard to see that $\text{Max}^{\tilde{\star}}(R) = \text{Max}^{\star_f}(R)$ [4, Corollary 3.5(2)]. From the latest fact, it then follows that an ideal A is $\tilde{\star}$ -invertible if and only if it is \star_f -invertible (in fact, if a star operation \star is of finite type, then $(AA^{-1})^{\star} = R$ if and only if $AA^{-1} \not\subseteq M$ for all $M \in \text{Max}^{\star}(R)$). From this observation it then follows that $\text{P}\star\text{MD}$, $\text{P}\star_f\text{MD}$, and $\text{P}\tilde{\star}\text{MD}$ coincide.

Lemma 4.2. *Let A be a finitely generated ideal of an integral domain R and \star a star operation on R . If A is \star_f -invertible, then AR_M is principal for every $M \in \text{Max}^{\star_f}(R)$.*

Proof. Suppose that A is \star_f -invertible. From the above observation, it follows that A is $\tilde{\star}$ -invertible, i.e., $(AA^{-1})^{\tilde{\star}} = R$. We have, for each maximal \star_f -ideal M , $R_M = (AA^{-1})^{\tilde{\star}}R_M = \cap\{(AA^{-1})R_M | M \in \text{Max}^{\star_f}(R)\}R_M = (AA^{-1})R_M$. Thus AR_M is invertible and therefore principal. \square

Lemma 4.3. *Let R be an integral domain and let \star be a star operation on R . Then the following statements are equivalent for an integral domain R .*

- (i) R_M is a valuation domain for all $M \in \text{Max}^{\star_f}(R)$.
- (ii) R is a $P\star MD$.
- (iii) Every nonzero fractional ideal with two generators is \star_f -invertible.

Proof. For (i) \Leftrightarrow (ii) [2, Corollary 1.2]. (ii) \Rightarrow (iii) is clear. So it remains to prove that (iii) \Rightarrow (i). Let $x, y \in R$, note that if P is a prime ideal of R , we have $xR_P + yR_P = (a, b)R_P$ for some $a, b \in R$. But if P is a \star_f -maximal ideal of R then, by Lemma 4.2 and by hypothesis, $(a, b)R_P$ is principal, that is, R_P is a valuation domain. \square

Remark 4.4. Note that the above proof of the preceding lemma is a general version of the proof for the PvMD case (see [8, Lemma 1.7]).

Now that we have a bound on finitely generated ideals to characterize $P\star MD$, we return to ultraproduct. Recall that an ideal J of a commutative ring is n -generated if J can be generated by n -elements.

Lemma 4.5. *Let \star be the ultra star operation on \mathfrak{R} built from the \star_i 's on R_i 's. Then an n -generated ideal A is a \star -invertible ideal of \mathfrak{R} if and only if A_i is an n -generated \star_i -invertible ideal of R_i for \mathcal{U} -many i .*

Proof. Suppose that $A = (A_i)_{\mathcal{U}}$ is an n -generated \star -invertible ideal of \mathfrak{R} , that is, $(AA^{-1})^{\star} = \mathfrak{R}$. Note that since A is n -generated, A_i is n -generated for \mathcal{U} -many i . So since $A^{-1} = (A_i^{-1})_{\mathcal{U}}$, we have $(AA^{-1})^{\star} = ((A_i A_i^{-1})^{\star_i})_{\mathcal{U}} = \mathfrak{R}$. Thus $(A_i A_i^{-1})^{\star_i} = R_i$ for \mathcal{U} -many i . So A_i is an n generated \star_i -invertible ideal of R_i for \mathcal{U} -many i . Conversely suppose that A_i is n -generated and $(A_i A_i^{-1})^{\star_i} = R_i$ for \mathcal{U} -many i . Then $((A_i)_{\mathcal{U}}(A_i^{-1})_{\mathcal{U}})^{\star} = ((A_i A_i^{-1})^{\star_i})_{\mathcal{U}} = (R_i)_{\mathcal{U}} = \mathfrak{R}$. \square

Remark 4.6. Let \star_i 's be star operations on R_i 's. Consider $(\star_i)_f$ be the star operation of finite type associated with each \star_i . If we denote \star_f the ultra star operation on \mathfrak{R} built from the $(\star_i)_f$'s (note that \star_f is just a notation, so does not mean \star_f is the finite type associated to \star), then using Lemma 4.5, we have that an n -generated ideal A is a \star_f -invertible ideal of \mathfrak{R} if and only if A_i is an n -generated $(\star_i)_f$ -invertible ideal of R_i for \mathcal{U} -many i .

Let us recall that for a general star operation \star , a \star -Prüfer domain is an integral domain in which every finitely generated nonzero ideal is \star -invertible, equivalently every two generated nonzero ideal is \star -invertible, see [1, Theorem 2.2]. Note then that a P \star MD is a \star_f -Prüfer domain.

Proposition 4.7. *Let \star be the ultra star operation on \mathfrak{R} built from \star_i 's on R_i 's and \star_f be the ultra star operation on \mathfrak{R} built from finite type star operations $(\star_i)_f$ associated with each \star_i . Then*

- (i) \mathfrak{R} is a \star -Prüfer domain if and only if R_i is a \star_i -Prüfer domain for \mathcal{U} -many i .
- (ii) \mathfrak{R} is a \star_f -Prüfer domain if and only if R_i is a P \star_i MD for \mathcal{U} -many i .

Proof. Lemma 4.3 states that a domain is a P \star MD if and only if every nonzero fractional ideal with two generators is \star_f -invertible. By Lemma 4.5 (Resp. Remark 4.6), every 2-generated ideal A of \mathfrak{R} is \star -invertible (Resp. \star_f -invertible) if and only if A_i is a 2-generated \star_i -invertible (Resp. a 2-generated $(\star_i)_f$ -invertible) ideal of R_i for \mathcal{U} -many i . Hence the proposition follows from [1, Theorem 2.2], Lemma 4.3, and Los' Theorem. \square

As a consequence of Proposition 4.7, we obtain [10, Proposition 2.2 (iii)] in the case $\star = d$ and more if $\star = v$ and $\star = t$.

Corollary 4.8. *Let \star be the ultra star operation on \mathfrak{R} obtained from the \star_i 's on R_i 's.*

- (i) *Case $\star_i = d$. \mathfrak{R} is a Prüfer domain if and only if R_i is a Prüfer domain for \mathcal{U} -many i .*
- (ii) *Case $\star_i = v$. \mathfrak{R} is a v -Prüfer domain if and only if R_i is a v -Prüfer domain for \mathcal{U} -many i .*
- (iii) *Case $\star_i = t$. \mathfrak{R} is a v_f -Prüfer domain if and only if R_i is a PvMD for \mathcal{U} -many i .*

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