International Electronic Journal of Algebra
Volume 27 (2020) 218-236
DOI: 10.24330/ieja.663075

INDECOMPOSABLE NON UNISERIAL MODULES OF LENGTH THREE

Gabriella D'Este

Received: 2 May 2019; Revised: 8 October 2019; Accepted: 21 October 2019
Communicated by Christian Lomp

Abstract. We investigate a particular class of indecomposable modules of length three, defined over a K–algebra, with a simple socle and two non isomorphic simple factor modules. These modules may have any projective dimension different from zero. On the other hand their composition factors may have any countable dimension as vector spaces over the underlying field K. Moreover their endomorphism rings are K–vector spaces of dimension ≤ 2.

Mathematics Subject Classification (2010): 16G20, 16D99
Keywords: Pseudo projective module, quasi projective module, quiver and representation

1. Introduction

The aim of this note is to construct left R–modules M, where R is a K–algebra over some field K, with the following properties:

(1) M has exactly three non zero proper submodules N_1, N_2 and $N_1 \cap N_2$.
(2) M/N_1 is not isomorphic to M/N_2.
(3) $\operatorname{End} M/N_i$ is isomorphic to K for $i = 1, 2$.

Given a module M, we say that M is *quasi projective* (resp. *pseudo projective*) if, for any submodule X of M, any morphism (resp. any epimorphism) $f : M \to M/X$ is of the form $\pi \circ g$, where $\pi : M \to M/X$ is the canonical epimorphism and g is an endomorphism of M. By [4, Lemma 2.1] any module M satisfying (1) and (2) is not quasi projective. Moreover, if (1) and (2) holds, then M is pseudo projective if and only if it satisfies (3) with $K = \mathbb{Z}_2$. In the sequel we shall say, for short, that M satisfies (123) if M satisfies conditions (1), (2) and (3). If x is a vertex of a quiver Q, and R is the K–algebra given by Q, then x also denotes the simple module (V_i, f_j), with $V_x = K$, $V_i = 0$ for $i \neq x$ and $f_j = 0$ for any j. Moreover, $P(x)$ and $I(x)$ denote the projective and injective module corresponding to the vertex x. In the examples constructed in [4] R is always a K–algebra given by quivers with at least 2 vertices and at most 4 vertices, and the modules satisfying (123) are three dimensional vector spaces. In this note we consider algebras given by quivers with n vertices for any $n \geq 1$. By making one point extensions or coextensions, or by changing the relations, we obtain modules satisfying (123) of any injective dimension (Proposition 2.1) and of any projective dimension ≥ 1 (Proposition 2.2). We also show that modules satisfying (123) are very far from being selforthogonal (Examples 2.3, 2.4 and 2.5), and that they cannot be faithfully balanced (Proposition 3.8). Moreover their endomorphism ring is a rather small commutative algebra (Theorem 3.6). As we shall see, the free algebra $K < x, y >$ admits modules of dimension three satisfying (123) (Example 2.5, Proposition 2.6, Corollaries 3.10 and 3.11). On the other hand, the free algebra $K < x, y, z >$ admits modules satisfying (123) of any countable dimension ≥ 3 (Theorem 3.2). Finally, suitable matrix algebras admit modules satisfying (123) of any infinite dimension over K (Proposition 3.4 and Corollary 3.5).

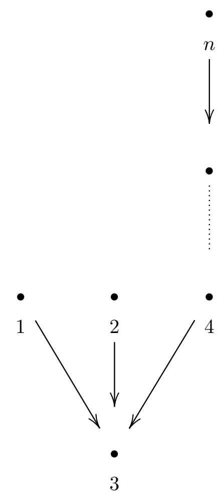
This paper is organized as follows. In Section 1 we recall some definitions, we fix some conventions and we describe the main results. In Section 2 we investigate homological dimensions and extensions of modules satisfying (123). Finally, in Section 3, we describe composition factors and endomorphism rings of these modules. For more background on generalizations of projective modules we refer to [3] and [5]. For more background on quivers and their representations we refer to [1] and [2].

2. Homological dimensions and extensions

We begin with two results on homological dimensions.

Proposition 2.1. *Any $d \in \mathbb{N} \cup \{\infty\}$ is the injective dimension of a module M satisfying (123).*

Proof. If $d = 0, 1, \infty$, then the assertion follows from [4, Example 2.3 and Theorem 2.8]. Assume now $2 \leq d < \infty$. Let $n = d + 3$ and let R be the K-algebra given by the following quiver with relations $ab = 0$ for all arrows a and b.



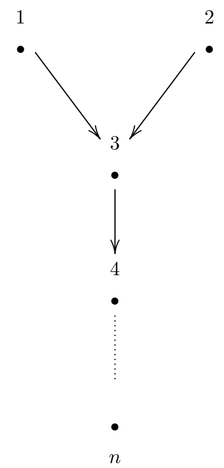
Next let M be the module $\begin{smallmatrix} 1\ 2 \\ 3 \end{smallmatrix}$. Then the injective coresolution of M is

$$0 \longrightarrow M \longrightarrow \begin{smallmatrix} 124 \\ 3 \end{smallmatrix} \longrightarrow \begin{smallmatrix} 5 \\ 4 \end{smallmatrix} \longrightarrow \dots \longrightarrow \begin{smallmatrix} n \\ n-1 \end{smallmatrix} \longrightarrow n \longrightarrow 0 \ .$$

Hence $n - 3 = d$ is the injective dimension of M. □

Proposition 2.2. *Any d with $1 \le d \le \infty$ is the projective dimension of a module M satisfying (123).*

Proof. The modules $\begin{smallmatrix} 1\ 2 \\ 3 \end{smallmatrix}$ and $\begin{smallmatrix} 1\ 2 \\ 2 \end{smallmatrix}$ in [4, Examples 2.3 and 2.4] have projective dimension 1 and ∞, respectively. Assume now $1 < d < \infty$ and let $n = d + 2$. Next let R be the K–algebra given by the following quiver with relations $ab = 0$ for all arrows a and b.



Let M be the module $I(3) = \begin{smallmatrix} 1 & 2 \\ \multicolumn{2}{c}{3} \end{smallmatrix}$. Then the projective resolution of M is

$$0 \longrightarrow n \longrightarrow \begin{smallmatrix} n-1 \\ n \end{smallmatrix} \longrightarrow \dots \longrightarrow \begin{smallmatrix} 3 \\ 4 \end{smallmatrix} \longrightarrow \begin{smallmatrix} 1 \\ 3 \end{smallmatrix} \oplus \begin{smallmatrix} 2 \\ 3 \end{smallmatrix} \longrightarrow M \longrightarrow 0 .$$

Consequently the projective dimension of M is $n - 2 = d$. □

Propositions 2.1 and 2.2 suggest that we investigate the groups of selfextensions of the modules satisfying (123).

Example 2.3. There is a module M satisfying (123) such that $\mathrm{Ext}^n(M, M) \simeq K$ for any $n \geq 1$.

Construction. As in [4, Theorem 2.8], let R be the K–algebra given by the quiver $\underset{1}{\bullet} \xrightarrow{a} \underset{2}{\bullet} \circlearrowright b$ with relations $ba = 0$ and $b^3 = 0$. Next let M be the module $\begin{smallmatrix} 1 & 2 \\ \multicolumn{2}{c}{2} \end{smallmatrix}$.

Then $I(1) = 1$ and $I(2) = \begin{smallmatrix} & 2 \\ 1 & 2 \\ & 2 \end{smallmatrix}$ are the indecomposable injective modules. From the exact sequence $0 \to M \to I(2) \to 2 \to 0$ we obtain the long exact sequence $0 \longrightarrow \mathrm{Hom}(M, M) \xrightarrow{\simeq} \mathrm{Hom}(M, I(2)) \xrightarrow{0} \mathrm{Hom}(M, 2) \xrightarrow{\simeq} \mathrm{Ext}^1(M, M) \longrightarrow 0 \longrightarrow \dots \longrightarrow 0 \longrightarrow \mathrm{Ext}^i(M, 2) \longrightarrow \mathrm{Ext}^{i+1}(M, M) \longrightarrow 0 \dots\dots$

Consequently we have

$$\mathrm{Ext}^1(M, M) \simeq \mathrm{Hom}(M, 2) \simeq K \tag{1}$$

and

$$\mathrm{Ext}^{n+1}(M, M) \simeq \mathrm{Ext}^n(M, 2) \qquad \text{for any} \quad n \geq 1. \tag{2}$$

On the other hand, from the exact sequence $0 \longrightarrow 2 \longrightarrow I(2) \longrightarrow \ 1 \ \oplus \begin{smallmatrix} 2 \\ 2 \end{smallmatrix} \longrightarrow 0$, we obtain the long exact sequence $0 \longrightarrow \operatorname{Hom}(M, 2) \longrightarrow \operatorname{Hom}(M, I(2)) \longrightarrow \operatorname{Hom}\left(M, \ 1 \ \oplus \begin{smallmatrix} 2 \\ 2 \end{smallmatrix}\right) \longrightarrow \operatorname{Ext}^1(M, 2) \longrightarrow 0 \ldots \longrightarrow 0 \longrightarrow \operatorname{Ext}^i\left(M, \ 1 \ \oplus \begin{smallmatrix} 2 \\ 2 \end{smallmatrix}\right) \longrightarrow \operatorname{Ext}^{i+1}(M, 2) \longrightarrow 0 \longrightarrow \ldots.$

Hence, comparing dimensions and using the fact that 1 is an injective module, we have

$$\operatorname{Ext}^1(M, 2) \simeq K \tag{3}$$

and

$$\operatorname{Ext}^{n+1}(M, 2) \simeq \operatorname{Ext}^n\left(M, \begin{smallmatrix} 2 \\ 2 \end{smallmatrix}\right) \qquad \text{for any} \quad n \geq 1. \tag{4}$$

Combining (2) and (3), we get

$$\operatorname{Ext}^2(M, M) \simeq K. \tag{5}$$

Finally, from the exact sequence $0 \longrightarrow \begin{smallmatrix} 2 \\ 2 \end{smallmatrix} \longrightarrow I(2) \longrightarrow \ 1 \ \oplus \ 2 \ \longrightarrow 0$, we obtain the long exact sequence $0 \longrightarrow \operatorname{Hom}\left(M, \begin{smallmatrix} 2 \\ 2 \end{smallmatrix}\right) \longrightarrow \operatorname{Hom}(M, I(2)) \longrightarrow \operatorname{Hom}(M, 1 \oplus 2) \longrightarrow \operatorname{Ext}^1\left(M, \begin{smallmatrix} 2 \\ 2 \end{smallmatrix}\right) \longrightarrow 0 \longrightarrow \ldots \longrightarrow 0 \longrightarrow \operatorname{Ext}^i(M, 1 \oplus 2) \longrightarrow \operatorname{Ext}^{i+1}\left(M, \begin{smallmatrix} 2 \\ 2 \end{smallmatrix}\right) \longrightarrow 0 \to \ldots.$ Therefore we have

$$\operatorname{Ext}^1\left(M, \begin{smallmatrix} 2 \\ 2 \end{smallmatrix}\right) \simeq K \tag{6}$$

and

$$\operatorname{Ext}^{n+1}\left(M, \begin{smallmatrix} 2 \\ 2 \end{smallmatrix}\right) \simeq \operatorname{Ext}^n(M, 2) \qquad \text{for any} \quad n \geq 1. \tag{7}$$

Putting (2), (4) and (6) together, we get

$$\operatorname{Ext}^3(M, M) \simeq K. \tag{8}$$

Combining (4) and (7), we obtain

$$\operatorname{Ext}^{i+2}(M, 2) \simeq \operatorname{Ext}^i(M, 2) \qquad \text{for any} \quad i \geq 1. \tag{9}$$

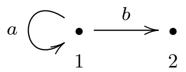
This result and (2) imply that

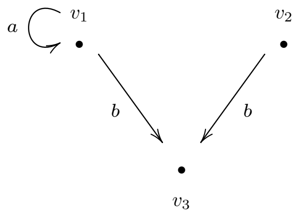
$$\operatorname{Ext}^{i+3}(M, M) \simeq \operatorname{Ext}^{i+1}(M, M) \qquad \text{for any} \quad i \geq 1. \tag{10}$$

Putting (1), (5), (8) and (10) together, we conclude that $\operatorname{Ext}^n(M, M) \simeq K$ for any $n \geq 1$. □

As the next examples show, we may have $\operatorname{Ext}^1(M, M) \neq 0$ also when M is defined over hereditary algebras.

Example 2.4. There is a module M satisfying (123) such that $\dim \mathrm{Ext}^1(M,M) = 2$.

Construction. Let R be the infinite dimensional K–algebra given by the quiver . Let M be the module described by the following picture



where $\{v_1, v_2, v_3\}$ is a basis of M and $av_1 = v_1$, $av_2 = 0$, $bv_1 = v_3 = bv_2$. Then Rv_1, Rv_2 and $Rv_1 \cap Rv_2 = Rv_3$ are the three non zero proper submodules of M. On the other hand the three composition factors of M are one dimensional. Let $U_1 = M/Rv_1$ an let $U_2 = M/Rv_2$. Then $aU_1 = 0$ and $aU_2 \neq 0$. Hence M satisfies (123). From the exact sequence

$$0 \longrightarrow Rv_3 \simeq P(2) \longrightarrow M \longrightarrow U_1 \oplus U_2 \longrightarrow 0 \tag{1}$$

we get the long exact sequence

$$0 = \mathrm{Hom}(U_1 \oplus U_2, M) \longrightarrow \mathrm{Hom}(M,M) \xrightarrow{\simeq} \mathrm{Hom}(Rv_3, M) \xrightarrow{0}$$
$$\longrightarrow \mathrm{Ext}^1(U_1 \oplus U_2, M) \xrightarrow{\simeq} \mathrm{Ext}^1(M,M) \longrightarrow 0. \tag{2}$$

Let e_1 be the path of length zero around the vertex 1. Then the definition of U_1 and U_2 guarantees the existence of epimorphisms $f : P(1) \longrightarrow U_1$ and $g : P(1) \longrightarrow U_2$ such that $\mathrm{Ker}\, f = Ra \oplus Rb$ and $\mathrm{Ker}\, g = R(e_1 - a) \oplus Rb$. Consequently, there are exact sequences of the form

$$0 \longrightarrow P(1) \oplus P(2) \longrightarrow P(1) \longrightarrow U_i \longrightarrow 0 \qquad \text{for} \quad i = 1,2. \tag{3}$$

Hence also the following long sequences are exact

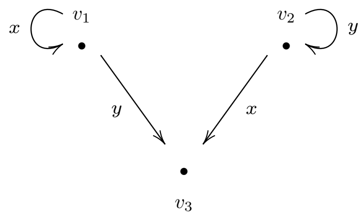
$$0 = \mathrm{Hom}(U_i, M) \longrightarrow \mathrm{Hom}(P(1), M) \longrightarrow \mathrm{Hom}(P(1) \oplus P(2), M) \longrightarrow$$
$$\longrightarrow \mathrm{Ext}^1(U_i, M) \longrightarrow 0 \qquad \text{for} \quad i = 1,2. \tag{4}$$

Since $\mathrm{Hom}(P(1), M) \simeq K^2$ and $\mathrm{Hom}(P(1) \oplus P(2), M) \simeq K^3$, we deduce from (4) that $\mathrm{Ext}^1(U_i, M) \simeq K$ for $i = 1,2$. Consequently we deduce from (2) that $\dim \mathrm{Ext}^1(M,M) = 2$. □

In the next example we use a quiver with one vertex.

Example 2.5. The free algebra $A = K < x, y >$ admits a module M satisfying (123) such that $\dim \operatorname{Ext}^1(M, M) = 10$.

Construction. Let M be the A–module described by the following picture



More precisely, let $\{v_1, v_2, v_3\}$ be a basis of M and assume that $xv_1 = v_1$, $yv_2 = v_2$, $yv_1 = v_3 = xv_2$ and $xv_3 = 0 = yv_3$. Also in this case Av_1, Av_2 and $Av_1 \cap Av_2 = Av_3$ are the three non zero proper submodules of M. Moreover the three composition factors $S_1 = Av_1/Av_3$, $S_2 = Av_2/Av_3$ and $S_3 = Av_3$ are pairwise non isomorphic. Hence M satisfies (123). From the exact sequence

$$0 \longrightarrow A^2 \longrightarrow A \longrightarrow S_i \longrightarrow 0 \qquad \text{for} \quad i = 1, 2, 3, \tag{1}$$

we obtain the long exact sequences

$$0 \longrightarrow \operatorname{Hom}(S_i, M) \longrightarrow \operatorname{Hom}(A, M) \longrightarrow \operatorname{Hom}(A^2, M) \longrightarrow$$
$$\longrightarrow \operatorname{Ext}^1(S_i, M) \longrightarrow 0 \qquad \text{for any} \quad i.$$

Since $\operatorname{Hom}(S_i, M) = 0$ for $i = 1, 2$ and $\operatorname{Hom}(S_3, M) \simeq K$, we obtain

$$\dim \operatorname{Ext}^1(S_i, M) = 3 \qquad \text{for} \quad i = 1, 2 \tag{2}$$

and

$$\dim \operatorname{Ext}^1(S_3, M) = 4. \tag{3}$$

Since the sequence $0 \longrightarrow S_3 \longrightarrow M \longrightarrow S_1 \oplus S_2 \longrightarrow 0$ is exact and A is hereditary, we obtain the long exact sequence

$$0 = \operatorname{Hom}(S_1 \oplus S_2, M) \longrightarrow \operatorname{Hom}(M, M) \xrightarrow{\simeq} \operatorname{Hom}(S_3, M) \longrightarrow$$
$$\longrightarrow \operatorname{Ext}^1(S_1 \oplus S_2, M) \longrightarrow \operatorname{Ext}^1(M, M) \longrightarrow \operatorname{Ext}^1(S_3, M) \longrightarrow 0. \tag{4}$$

Putting (2), (3) and (4) together, we conclude that $\dim \operatorname{Ext}^1(M, M) = 10$. □

In the next statement we compare the dimensions of certain subspaces of small modules satisfying (123).

Proposition 2.6. *Let A be the free K–algebra $K\langle x, y\rangle$, and let M be a module satisfying (123) with* $\dim M = 3$. *Then, among others, the following cases are possible:*

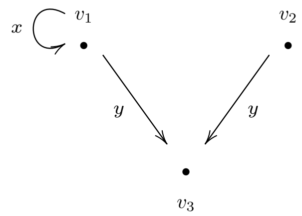
(i) $\dim xM = \dim yM = 2$;

(ii) $\dim xM = \dim yM = 1$;

(iii) $\dim xM = 2$, $\dim yM = 1$.

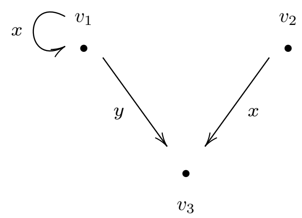
Proof. (i) This follows from Example 2.5, where we have $xM = \langle v_1, v_3\rangle$ and $yM = \langle v_2, v_3\rangle$.

(ii) Let M be the A–module, obtained in an obvious way from Example 2.4, with basis $\{v_1, v_2, v_3\}$ described by the following picture.



Then we have $xM = \langle v_1\rangle$ and $yM = \langle v_3\rangle$.

(iii) Let M be the A–module with basis $\{v_1, v_2, v_3\}$ described by the following picture.



then we have $xM = \langle v_1, v_3\rangle$ and $yM = \langle v_3\rangle$.

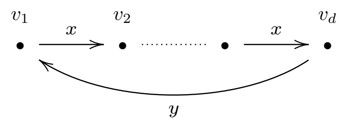
□

3. Composition factors and endomorphism rings

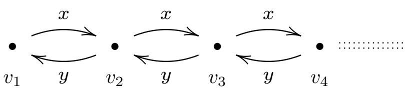
In the sequel we denote by $\mathbb{N}^+$ the set of positive integers. In order to construct modules satisfying (123) of infinite dimension, we need the following lemma.

Lemma 3.1. *Let* $A = K\langle x, y\rangle$. *Then for any* $d \in \mathbb{N}^+ \cup \{\aleph_0\}$ *there is a simple module* S *with* $\dim S = d$ *and* $\operatorname{End} S \simeq K$.

Proof. If $d = 1$, the assertion is obvious. Assume d is finite and $d > 1$. Let S be the module with basis $\{v_1, \ldots, v_d\}$ described in an obvious way by the following picture.



If $0 \neq v = \sum_{i=1}^{d} k_i v_i$, then we have $yx^i v = k_{d-i} v_1$ for $i = 0, \ldots, d-1$. Consequently $Av = Av_1 = S$. Hence S is a simple module of dimension d. Assume now $f \in \operatorname{End} S$ and $f(v_1) = \sum_{i=1}^{d} k_i v_i$. Since $yx^i v_1 = 0$ for $i = 0, \ldots, d-2$, we have $k_{d-i} v_1 = yx^i f(v_1) = f(yx^i v_1) = 0$ for $i = 0, \ldots, d-2$. Hence $k_d = k_{d-1} = \cdots = k_2 = 0$, and so f is the multiplication by k_1. Assume $d = \aleph_0$. Let S be the module with basis $\{v_n | n \geq 1\}$ described in obvious way by the following picture.

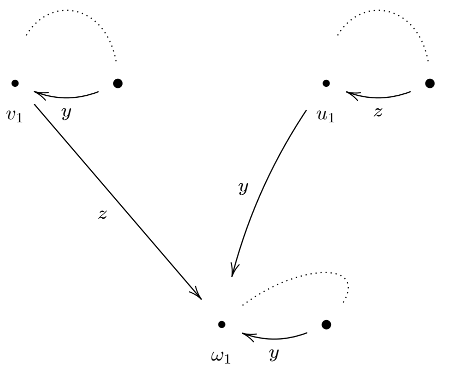


If $0 \neq v = \sum_{i=1}^{n} k_i v_i$ with $k_n \neq 0$, then we have $y^{n-1} v = k_n v_1$. Consequently $Av = Av_1 = S$, and so S is a simple module of dimension $\aleph_0$. Assume now $f \in \operatorname{End} S$ and $f(v_1) = \sum_{i=1}^{n} k_i v_i$ for some $n \geq 2$. Since $yv_1 = 0$, it follows that $\sum_{i=1}^{n-1} k_{i+1} v_i = yf(v_1) = f(yv_1) = 0$. Hence we have $k_2 = \cdots = k_n = 0$, and so f is the multiplication by k_1. The lemma is proved. □

By dealing with reasonably large free K–algebras it is easy to construct big modules satisfying (123).

Theorem 3.2. *If $d_i \in \mathbb{N}^+ \cup \{\aleph_0\}$ for $i = 1, 2, 3$ and $A = K\langle x, y, z\rangle$, then there is an A–module M, satisfying (123), such that the three composition factors of M have dimensions d_1, d_2, d_3 and $d_3 = \dim \operatorname{soc} M$.*

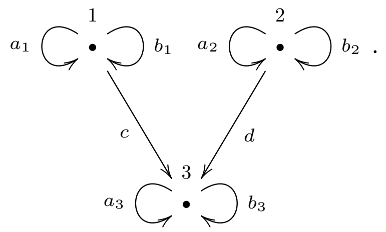
Proof. Let M be the A–module with basis $B_1 \cup B_2 \cup B_3$ with $|B_i| = d_i$ for any i described by the following picture.



More precisely we have $B_1 = \{v_i\}_{i \geq 1}$, $B_2 = \{u_i\}_{i \geq 1}$, $B_3 = \{\omega_i\}_{i \geq 1}$ and the following conditions hold. If $d_1 = 1$, then $yv_1 = v_1$, $xv_1 = 0$ and $zv_1 = \omega_1$. If $d_1 > 1$, then B_1 generates the $K\langle x, y\rangle$–module of dimension d_1 constructed in Lemma 3.1, and we have $zv_1 = \omega_1$ and $zv_i = 0$ for any $i > 1$. If $d_2 = 1$, then $zu_1 = u_1$, $xu_1 = 0$ and $yu_1 = \omega_1$. If $d_2 > 1$, then B_2 generates the $K\langle x, z\rangle$–module of dimension d_2 (with z

instead of y) constructed in Lemma 3.1, and we have $yu_1 = \omega_1$ and $yu_i = 0$ for any $i > 1$. If $d_3 = 1$, then $y\omega_1 = \omega_1$ and $x\omega_1 = 0 = z\omega_1$. If $d_3 > 1$, then B_3 generates the $K\langle x, y\rangle$–module of dimension d_3 constructed in Lemma 3.1, and we have $z\omega_i = 0$ for any i. Let now $N_1 = Av_1$, $N_2 = Au_1$. Then we have $N_1 \cap N_2 = A\omega_1$, and so $\dim N_1 \cap N_2 = d_3$. On the other hand $\dim M/N_1 = d_2$ and $\dim M/N_2 = d_1$. Moreover, by Lemma 3.1, K is the endomorphism ring of M/N_1, M/N_2 and $N_1 \cap N_2$. To end the proof, take any $0 \neq m \in M$. If $m \in N_1 \cap N_2$, then we have $Am = A\omega_1 = N_1 \cap N_2$. If $m \notin N_1$, then there exists $a \in A$ and $k \in K$ such that $zam = u_1 + k\omega_1$. Consequently there is some $b \in A$ such that $zb(u_1 + k\omega_1) = u_1$, and so $N_2 = Au_1 \subseteq Am$. Suppose now $m \notin N_2$. Then there exist $a \in A$ and $s \in N_1 \cap N_2$ such that $yam = v_1 + s$. Consequently $\omega_1 = zyam \in Am$, and so $N_1 \cap N_2 = A\omega_1 \subseteq Am$. It follows that $v_1 = yam - s \in Am$; hence $N_1 = Av_1 \subseteq Am$. Putting things together, we conclude that $Am = N_1$ if $m \in N_1 \setminus N_2$, $Am = N_2$ if $m \in N_2 \setminus N_1$, and $Am = N_1 + N_2 = M$ if $m \notin N_1 \cup N_2$. Hence M satisfies (123) and the proof is complete. □

Corollary 3.3. *Let R be the K–algebra given by the quiver*



If $d_i \in \mathbb{N}^+ \cup \{\aleph_0\}$ for $i = 1, 2, 3$, then there is an R–module $V = (V_i, f_i)$ such that V satisfies (123) and $\dim V_i = d_i$ *for any i.*

Proof. By Lemma 3.1 we can define a simple $K\langle x, y\rangle$–module $(V_i; a_i, b_i)$ with dimension d_i and endomorphism ring K for any $i = 1, 2, 3$. Next we fix a non zero map $c : V_1 \longrightarrow V_3$ and a non zero map $d : V_2 \longrightarrow V_3$. In this way we obtain an R–module $V = (V_1, V_2, V_3; f_j)$ with the following properties:

- The three non zero proper submodules of V are $N_1 = (V_1, 0, V_3; f_j)$, $N_2 = (0, V_2, V_3; f_j)$ and $N_1 \cap N_2 = (0, 0, V_3; f_j)$.
- V/N_1 is not isomorphic to V/N_2.
- $\operatorname{End} V/N_i$ is isomorphic to K for $i = 1, 2$.

Consequently V satisfies (123) and $\dim V_i = d_i$ for any i. □

By hypothesis, two non isomorphic composition factors belonging to the top of a module satisfying (123) have the smallest possible endomorphism ring. However it is easy to see that the socle of a module satisfying (123) may have a large endomorphism ring.

Proposition 3.4. *Let E be an extension field of K with $[E : K] > 1$. Then there are a K–algebra R and a faithful R–module M, satisfying (123), such that* $\mathrm{End}(\mathrm{soc}\, M) \simeq E$ *and M is not injective.*

Proof. Let R be the matrix algebra $\begin{pmatrix} K & 0 & 0 \\ 0 & K & 0 \\ E & E & E \end{pmatrix}$. Next let P be the left ideal $\begin{pmatrix} K & 0 & 0 \\ 0 & K & 0 \\ E & E & 0 \end{pmatrix}$, and let Q be the left ideal $\left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ a & -a & 0 \end{pmatrix} \middle| \, a \in E \right\}$. Finally let $M = P/Q$, $L_1 = \begin{pmatrix} K & 0 & 0 \\ 0 & 0 & 0 \\ E & E & 0 \end{pmatrix}$, $L_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & K & 0 \\ E & E & 0 \end{pmatrix}$, $N_1 = L_1/Q$ and $N_2 = L_2/Q$. Then N_1, N_2 and $N_1 \cap N_2 = \mathrm{soc}\, M$ are the unique proper submodules of M. Moreover M/N_1 and M/N_2 have dimension one over K and we have $M/N_1 \not\simeq M/N_2$. Hence M satisfies (123), and we clearly have $\mathrm{End}(\mathrm{soc}\, M) \simeq E$. Moreover it is easy to check that M is a faithful R–module. Finally, let $\widetilde{P}$ be the left R–module $\begin{pmatrix} E & 0 & 0 \\ 0 & E & 0 \\ E & E & 0 \end{pmatrix}$. It remains to check that $M = P/Q$ is an essential submodule of $\widetilde{P}/Q$. To see this, take any $\widetilde{p} = \begin{pmatrix} x & 0 & 0 \\ 0 & y & 0 \\ z & t & 0 \end{pmatrix} \in \widetilde{P} \setminus P$. Then $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \widetilde{p} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ x & 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \widetilde{p} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & y & 0 \end{pmatrix}$. Consequently $R(\tilde{p}+Q) \cap P/Q \neq 0$, and so P/Q is an essential submodule of $\widetilde{P}/Q$, as claimed. □

Corollary 3.5. *Any infinite cardinal d is the dimension over K of a faithful module satisfying (123).*

Proof. If E is an extension field of K with $[E : K] = d$, then the claim follows from the proof of Proposition 3.4. □

As the next result shows, only two K–algebras occur as the endomorphism ring of a module satisfying (123) and defined over a K–algebra.

Theorem 3.6. *Let M be a module satisfying (123) and defined over a K–algebra. Let N_1 and N_2 be the two maximal submodules of M. The following facts hold:*

(i) *If $N_1 \cap N_2 \not\simeq M/N_i$ for any i, then* $\operatorname{End} M$ *is isomorphic to K.*

(ii) *If $N_1 \cap N_2 \simeq M/N_i$ for some i, then* $\operatorname{End} M$ *is isomorphic to $K[x]/(x^2)$.*

Proof. Let $\pi : M \longrightarrow M/(N_1 \cap N_2)$ be the canonical morphism. Since $N_1 \cap N_2$ is fully invariant in M, for any $f \in \operatorname{End} M$ there is a unique morphism $f^\star \in \operatorname{End}(M/(N_1 \cap N_2))$ such that $\pi \circ f = f^\star \circ \pi$. Let $\rho : \operatorname{End} M \to \operatorname{End}(M/(N_1 \cap N_2)) \simeq K \oplus K$ be the morphism sending f to $f^\star$ for any f. We first note that

$$\operatorname{End}(M/(N_1 \cap N_2)) \quad \text{has non trivial idempotents.} \tag{1}$$

We next observe that

$$f^2 = 0 \quad \text{for every} \quad f \in \operatorname{Ker} \rho. \tag{2}$$

Since M is indecomposable and idempotents lift modulo nil ideals, we conclude that

$$\rho \quad \text{is not surjective.} \tag{3}$$

Suppose first $N_1 \cap N_2 \not\simeq M/N_i$ for any i. In this case ρ is injective. This remark and (3) imply that $\operatorname{End} M$ is isomorphic to K. Hence (i) holds. Assume now $N_1 \cap N_2 \simeq M/N_i$ for some i. In this case there is some $0 \neq f \in \operatorname{Ker} \rho$. Since $\operatorname{End}(M/N_i) \simeq K$, it follows that $\operatorname{Ker} \rho$ is the vector space generated by f. Putting (2) and (3) together, we conclude that $\operatorname{End} M$ is isomorphic to $K[x]/(x^2)$. Thus also (ii) holds. □

Corollary 3.7. *Let R be a K–algebra, let M be a module satisfying (123), and let $E = \operatorname{End} M$. The following facts hold:*

(i) *R is not commutative and E is commutative.*

(ii) *M is a cyclic R–module and M is not a cyclic E–module.*

Proof. (i) We know from [4, Remark 2.2] that R is not commutative, while we know from Theorem 3.6 that E is commutative.

(ii) Let N_1 and N_2 be the maximal submodules of M. Then any $m \in M \setminus (N_1 \cup N_2)$ generates M as an R–module. By Theorem 3.6 we have $\dim E \leq 2$. On the other hand we clearly have $\dim M \geq 3$. Hence M is not a cyclic E–module.

□

As usually, we say that a bimodule ${}_A X_B$ is *faithfully balanced* if the canonical morphisms $A \longrightarrow \operatorname{End} X_B$ and $B \longrightarrow \operatorname{End}_A X$, sending any $a \in A$ and $b \in B$ to the corresponding multiplications, are isomorphisms.

Proposition 3.8. *Let R, M, N_1, N_2 and $E = \operatorname{End} M$ be as in Corollary 3.7. Let T be the algebra of all endomorphisms of M as an E–module. The following facts hold:*

(i) *M is not a faithfully balanced $R-E$ bimodule.*
(ii) *If R is a finite dimensional algebra, then we may have $\dim T = \dim R + 1$.*

Proof. We first note that

$$M \text{ is not a simple } R\text{–module.} \tag{1}$$

Assume first $E \simeq K$. Then $T \simeq \operatorname{End}_K(M)$ and M is a simple T–module. This remark and (1) prove that M is not faithfully balanced. Assume now $E = K[f]$ for some endomorphism f of M such that

$$f(N_1) = 0 \text{ and } f(N_2) = N_1 \cap N_2. \tag{2}$$

Let now B be a basis of M of the form $B_1 \cup B_2 \cup B_3$ such that $B_1 \cup B_3$ and $B_2 \cup B_3$ are bases of N_1 and N_2 respectively, while B_3 is a basis of $N_1 \cap N_2$. Next let $s : M \longrightarrow M$ be a K–linear map such that

$$s(b) = 0 \text{ for any } b \in B_1 \cup B_3, \tag{3}$$

and

$$s(b) \in B_1 \text{ for any } b \in B_2. \tag{4}$$

Consequently we deduce from (2), (3) and (4) that $(s \circ f)(b) = 0 = (f \circ s)(b)$ for any $b \in B$. It follows that $s \in T$. On the other hand we deduce from (4) that $s(N_2) \notin N_2$. Hence s does not act as the multiplication by an element of R, and so (i) holds. To prove (ii), we choose R and M as in [4, Example 2.4]. More precisely, let R be the K –algebra given by the quiver $\underset{1}{\bullet} \xrightarrow{\;a\;} \underset{2}{\bullet} \circlearrowleft b$ with relations $ba = 0$ and $b^2 = 0$, and let M be the module $\begin{smallmatrix}1\ 2\\ 2\end{smallmatrix}$. We first note that

$$R \text{ is isomorphic to the algebra of all matrices over } K \text{ of the form } \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ c & d & b \end{pmatrix}. \tag{5}$$

Consequently E is isomorphic to the algebra of all matrices over K of the form $\begin{pmatrix} x & 0 & 0 \\ 0 & x & 0 \\ 0 & y & x \end{pmatrix}$. Therefore it is easy to check that

$$T \text{ is isomorphic to the algebra of all matrices over } K \text{ of the form } \begin{pmatrix} a & e & 0 \\ 0 & b & 0 \\ c & d & b \end{pmatrix}. \qquad (6)$$

Hence, by (5) and (6), we have $\dim R = 4$ and $\dim T = 5$. Hence also (ii) holds. □

In the next statement we collect some results on dimensions and generators.

Proposition 3.9. *Let R be a K–algebra and let M be an R–module satisfying (123). The following facts hold:*

(i) *R is generated by at least two elements.*
(ii) $\dim R \geq 4$.
(iii) *There is an algebra B, having a B–module satisfying* (123)*, such that B is generated by two elements and* $\dim B = 4$.

Proof. (i) This follows from the fact that R is not commutative (Corollary 3.7).

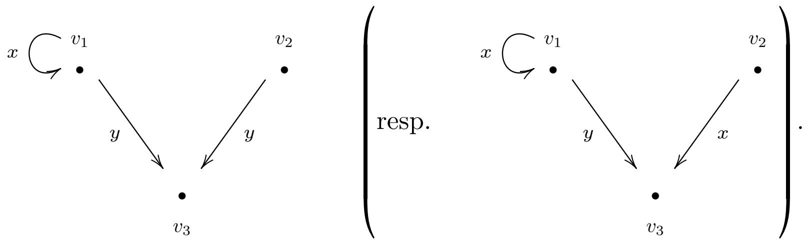
(ii) We first note that M is a cyclic R–module which is not projective (Corollary 3.7 and [4, Lemma 2.1]). Consequently either $\dim R = \infty$ or $3 \leq \dim M < \dim R < \infty$.

(iii) Let B be the K–algebra given by the quiver $\underset{1}{\bullet} \xrightarrow{a} \underset{2}{\bullet} \circlearrowleft b$ with relations $ba = 0$ and $b^2 = 0$, used to prove condition (ii) of Proposition 3.8. Next let e_1 and e_2 be the paths of length zero around 1 and 2. Then $\{e_1, e_2, a, b\}$ is a basis of B, and we clearly have $e_2 = 1 - e_1$, $a = (a+b)e_1$ and $b = (a+b)e_2 = (a+b)(1-e_1)$. Hence B is generated by e_1 and $a+b$. □

The proof of Proposition 2.1 shows that many modules of dimension 3, satisfying (123), are injective modules over suitable algebras. We finally show that also the previous $K\langle x, y\rangle$–modules of dimension 3 have this property.

Corollary 3.10. *Let B be the K–algebra used to prove Proposition 3.9 (iii), and let M be the $K\langle x, y\rangle$–module used to prove Proposition 2.6 (ii) (resp. (iii)). Then M is a faithful injective B–module.*

Proof. By hypothesis M is described by the following picture



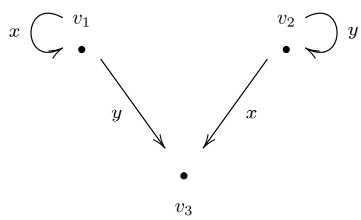
Let f and g be the endomorphisms of M induced by x and y. Then we clearly have $f^2 = f$, $g^2 = 0$, $fg = 0$ and $gf \neq 0$ (resp. $f^3 = f^2$, $g^2 = 0$, $fg = 0$ and $gf = g$). On the other hand, by Proposition 3.9, we have $\dim K\langle f, g\rangle \geq 4$. Consequently $\{1, f, g, gf\}$ (resp. $\{1, f, f^2, g\}$) is a basis of $K\langle f, g\rangle$. Moreover the K–linear map $B \longrightarrow K\langle f, g\rangle$ such that $e_1 \mapsto f$, $e_2 \mapsto 1 - f$, $a \mapsto gf$, $b \mapsto g - gf$ (resp. $e_1 \mapsto f^2$, $e_2 \mapsto 1 - f^2$, $a \mapsto g$, $b \mapsto f - f^2$) is an algebra isomorphism. Since $\begin{smallmatrix}1\\2\end{smallmatrix}$, $\begin{smallmatrix}2\\2\end{smallmatrix}$, $\begin{smallmatrix}1\,2\\2\end{smallmatrix}$, 1 , 2 , are the indecomposable B–modules, we conclude that the B–module M is the injective module $\begin{smallmatrix}1\,2\\2\end{smallmatrix}$. □

Corollary 3.11. *Let C be the K–algebra given by the Dynkin diagram*

$$1 \xrightarrow{a} 3 \xleftarrow{b} 2 \ ,$$

and let M be the $K\langle x, y\rangle$–module constructed in Example 2.5. Then M is a faithful injective C–module.

Proof. By hypothesis M is described by the following picture.

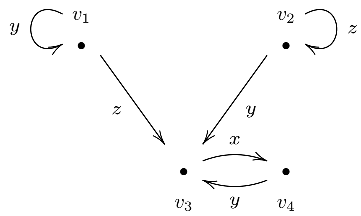


Let f and g be the endomorphisms of M induced by x and y. Then we clearly have $f^3 = f^2$, $g^3 = g^2$, $fg = fg^2 = f - f^2$, $gf = gf^2 = g - g^2$, and $f^2 g = g^2 f = 0$. Moreover $\{1, f, f^2, g, g^2\}$ is a basis of $K\langle f, g\rangle$. As always, let e_i be the path of length zero around the vertex i. Then the K–linear map $C \longrightarrow K\langle f, g\rangle$ such that $e_1 \mapsto f^2$, $e_2 \mapsto g^2$, $e_3 \mapsto 1 - f^2 - g^2$, $a \mapsto g - g^2$, $b \mapsto f - f^2$ is an algebra isomorphism. Hence the C–module M is the injective module $\begin{smallmatrix}1\,2\\3\end{smallmatrix}$. □

In all the previous examples of faithful modules M, satisfying (123) and defined over finite dimensional K–algebras R, we always have $\dim R - \dim M \leq 2$. To see that this is not always true, it suffices to consider the following example.

Example 3.12. There are a K–algebra R and a faithful R–module M, satisfying (123), such that $\dim M = 4$, $\operatorname{End} M \simeq K$ and $\dim R = 10$.

Construction. Let M be the $K\langle x, y, z\rangle$–module described by the following picture.



Then we deduce from Theorems 3.2 and 3.6 that M is a $K\langle x, y, z\rangle$–module satisfying (123) with $\operatorname{End} M \simeq K$ and $\dim M = 4$. Let f, g, h be the endormorphisms of M induced by x, y, z respectively, and let $R = K\langle f, g, h\rangle$. Next let $e = 1-g^2-h^2-gf$ and $\ell = ge$. Then the linear maps $g^2, h^2, hg, gh, gf, \ell, fh, fgh, f, e$ are described by ten different matrices (with entries in K) with one entry equal to 1 and the remaining ones equal to 0, as suggested by the following matrix

$$\begin{pmatrix} g^2 & 0 & 0 & 0 \\ 0 & h^2 & 0 & 0 \\ hg & gh & gf & \ell \\ fh & fgh & f & e \end{pmatrix}.$$

This remark and the shape of M imply that R is isomorphic to the matrix algebra

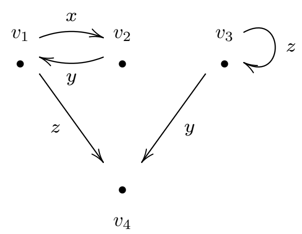
$$\begin{pmatrix} K & 0 & 0 & 0 \\ 0 & K & 0 & 0 \\ K & K & K & K \\ K & K & K & K \end{pmatrix}.$$

Consequently we have $\dim R = 10$, as claimed. □

As the next example shows, the dimension of a faithful module satisfying (123) and that of its endomorphism ring do not determine the dimension of the algebra.

Example 3.13. There are a K–algebra A and a faithful A–module L, satisfying (123), such that $\dim L = 4$, $\operatorname{End} L \simeq K$ and $\dim A = 9$.

Construction. Let L be the $K\langle x,y,z\rangle$–module described by the following picture.



Then we have $\dim L = 4$. Proceeding as in Theorem 3.2 we conclude that the $K\langle x,y,z\rangle$–module L satisfies (123). Hence we deduce from Theorem 3.6 that $\operatorname{End} L \simeq K$. Next let f,g,h be the endomorphisms of L induced by x,y,z respectively, and let $A = K\langle f,g,h\rangle$. Finally let $e = 1 - gf - fg - h^2$. Then the elements $gf, gfg, f, fg, h^2, hgf, hg, gh, e$ are described by nine different matrices (with entries in K) with one entry equal to 1 and the remaining ones equal to 0, as suggested by the following matrix

$$\begin{pmatrix} gf & gfg & 0 & 0 \\ f & fg & 0 & 0 \\ 0 & 0 & h^2 & 0 \\ hgf & hg & gh & e \end{pmatrix}.$$

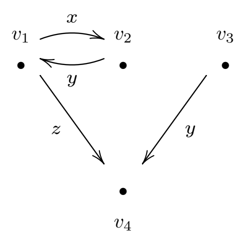
This observation and the shape of M imply that A is isomorphic to the matrix algebra

$$\begin{pmatrix} K & K & 0 & 0 \\ K & K & 0 & 0 \\ 0 & 0 & K & 0 \\ K & K & K & K \end{pmatrix}.$$

Hence we have $\dim A = 9$. □

Example 3.14. There are a K–algebra B and a faithful B–module U, satisfying (123), such that $\dim U = 4$, $\operatorname{End} U \simeq K[x]/(x^2)$ and $\dim B = 8$.

Construction. Let U be the $K\langle x,y,z\rangle$–module described by the following picture.



Let f,g,h be the endomorphisms of U induced by x,y,z respectively, and let $B = K\langle f,g,h\rangle$. Then U is a faithful B–module with $\dim U = 4$ and, proceeding as in

Theorem 3.2, it is easy to see that the B–module U satisfies (123). Finally, by Theorem 3.6, we have $\operatorname{End} U \simeq K[x]/(x^2)$. Let $e = 1 - gf - fg$ and let $\ell = g - gfg$. Then the elements $gf, gfg, f, fg, e, h, hg, \ell$ are described by matrices with one or two entries equal to 1 and the remaining ones equal to 0, as suggested by the following matrix

$$\begin{pmatrix} gf & gfg & 0 & 0 \\ f & fg & 0 & 0 \\ 0 & 0 & e & 0 \\ h & hg & \ell & e \end{pmatrix}.$$

This remark and the shape of U imply that B is the K–algebra of all matrices (with entries in K) of the form

$$\begin{pmatrix} \star & \star & 0 & 0 \\ \star & \star & 0 & 0 \\ 0 & 0 & t & 0 \\ \star & \star & \star & t \end{pmatrix}.$$

Hence we have $\dim B = 8$, as claimed. □

Corollary 3.15. *There exist two finite dimensional factor algebras A and B of $K\langle x, y, z\rangle$ and two faithful modules ${}_AL$ and ${}_BU$, satisfying (123), such that* $\dim L = \dim U$, $\dim \operatorname{soc} L = \dim \operatorname{soc} U$, $\dim A + \dim \operatorname{End} L = \dim B + \dim \operatorname{End} U$ *and* $\dim A \neq \dim B$.

Proof. Let A, B, L and U be as in Examples 3.13 and 3.14. Then we have $\dim L = \dim U = 4$, $\dim \operatorname{soc} L = \dim \operatorname{soc} U = 1$, $\dim A = 9$, $\dim B = 8$, $\dim \operatorname{End} L = 1$ and $\dim \operatorname{End} U = 2$. □

References

[1] I. Assem, D. Simson and A. Skowronski, Elements of the Representation Theory of Associative Algebras, Vol. 1, London Mathematical Society Student Texts, 65, Cambridge University Press, Cambridge, 2006.

[2] M. Auslander, I. Reiten and S. O. Smalø, Representation Theory of Artin Algebras, Cambridge Studies in Advanced Mathematics, 36, Cambridge University Press, Cambridge, 1995.

[3] J. Clark, C. Lomp, N. Vanaja and R. Wisbauer, Lifting Modules, Supplements and Projectivity in Module Theory, Frontiers in Math., Boston, Birkhäuser, 2006.

[4] G. D'Este and D. Keskin Tütüncü, *Pseudo projective modules which are not quasi projective and quivers*, Taiwanese J. Math., 22(5) (2018), 1083-1090.

[5] P. A. Guil Asensio, D. Keskin Tütüncü, B. Kaleboğaz and A. K. Srivastava, *Modules which are coinvariant under automorphisms of their projective covers*, J. Algebra, 466 (2016), 147-152.

Gabriella D'Este
Department of Mathematics
University of Milano
Via Saldini 50
20133 Milano, Italy
e-mail: gabriella.deste@unimi.it