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A GENERALIZATION OF THE ESSENTIAL GRAPH FOR MODULES OVER COMMUTATIVE RINGS

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Abstract. Let R be a commutative ring with nonzero identity and let M be a unitary R-module. The essential graph of M, denoted by $EG(M)$ is a simple undirected graph whose vertex set is $Z(M)\backslash \mathrm{Ann}_R(M)$ and two distinct vertices x and y are adjacent if and only if $\mathrm{Ann}_M(xy)$ is an essential submodule of M. Let $r(\mathrm{Ann}_R(M)) \neq \mathrm{Ann}_R(M)$. It is shown that $EG(M)$ is a connected graph with $\mathrm{diam}(EG(M)) \leq 2$. Whenever M is Noetherian, it is shown that $EG(M)$ is a complete graph if and only if either $Z(M) = r(\mathrm{Ann}_R(M))$ or $EG(M) = K_2$ and $\mathrm{diam}(EG(M)) = 2$ if and only if there are $x, y \in Z(M)\backslash \mathrm{Ann}_R(M)$ and $\mathfrak{p} \in \mathrm{Ass}_R(M)$ such that $xy \notin \mathfrak{p}$. Moreover, it is proved that $\mathrm{gr}(EG(M)) \in \{3, \infty\}$. Furthermore, for a Noetherian module M with $r(\mathrm{Ann}_R(M)) = \mathrm{Ann}_R(M)$ it is proved that $|\mathrm{Ass}_R(M)| = 2$ if and only if $EG(M)$ is a complete bipartite graph that is not a star.

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1. Introduction

The concept of the zero-divisor graph of a commutative ring was introduced and studied by I. Beck in [6]. Subsequently, D. F. Anderson and P. S. Livingston in [2] studied and investigated the concept of zero-divisor graph on nonzero zero-divisors of a commutative ring. Let R be a commutative ring and let $Z(R)$ be its set of zero-divisors. The zero-divisor graph of R, which is the graph with vertex set $Z^*(R) = Z(R) \setminus \{0\}$ and two distinct vertices x and y are adjacent if and only if $xy = 0$, has been studied by many authors (see [1,3,4]). Variations of the zero-divisor graph are created by changing the vertex set, the edge condition, or both. The essential graph of R is a variation of the zero-divisor graph that changes the edge condition, and is introduced and studied in [10]. The essential graph of R is a simple undirected graph, denoted by $EG(R)$, with vertex set $Z^*(R)$ and two distinct vertices x and y are adjacent if and only if $\mathrm{Ann}_R(xy)$ is an essential ideal of R. Recently, a lot of research (e.g., [5,7,8,11,12]) has been devoted to the zero-divisor graph of a module (Definition 4.1). Let M be an R-module and let $Z(M)$ be its set of zero-divisors. In this paper, we associate a graph to the module M, denoted by $EG(M)$, with vertex set $Z(M) \setminus \mathrm{Ann}_R(M)$ and two distinct vertices $x, y \in Z(M) \setminus \mathrm{Ann}_R(M)$ are adjacent if and only if $\mathrm{Ann}_M(xy)$ is an essential submodule of M. Before we state some results, let us introduce some graphical notations.

Let $G = (V(G), E(G))$ be a simple undirected graph, $V(G)$ and $E(G)$ are called vertex set and edge set of G, respectively. Let $x, y \in V(G)$. Whenever x and y are joint by an edge, it is denoted by $x - y$. The vertex x is said to be a universal vertex if it is adjacent to every other vertex of G. The graph G is connected if there is a path between any two distinct vertices. For vertices x and y of G, we define $\mathrm{d}(x, y)$ to be the length of a shortest path between x and y (if there is no path, then $\mathrm{d}(x, y) = \infty$). The open neighborhood of a vertex x is defined to be the set $N(x) = \{y \in V(G) \ : \ \mathrm{d}(x, y) = 1\}$. The diameter of G is $\mathrm{diam}(G) = \sup\{\mathrm{d}(x, y) | x, y \in V(G)\}$. A graph G is complete if any two distinct vertices are adjacent and a complete graph with n vertices is denoted by K_n. A bipartite graph is one whose vertex set can be partitioned into two subsets so that an edge has both ends in no subset. A complete bipartite graph is a bipartite graph in which each vertex is adjacent to every vertex that is not in the same subset. The complete bipartite graph with part sizes m and n is denoted by $K_{m,n}$. If $m = 1$, then the bipartite graph is called star graph. The girth of G, denoted by $\mathrm{gr}(G)$ is the length of a shortest cycle contained in the graph (if there is no cycle, then $\mathrm{gr}(G) = \infty$).

Throughout this paper, R is a commutative ring with nonzero identity and M is a unitary R-module. Recall that $Z(M) = \{r \in R : rm = 0 \text{ for some } 0 \neq m \in M\}$, $\mathrm{Ass}_R(M) = \{\mathfrak{p} \in \mathrm{Spec}(R) : \mathfrak{p} = \mathrm{Ann}_R(m) \text{ for some } 0 \neq m \in M\}$, $\mathrm{Ann}_R(M) = \{r \in R : rM = 0\}$ and $r(\mathrm{Ann}_R(M)) = \{x \in R : x^t \in \mathrm{Ann}_R(M) \text{ for some } t \in \mathbb{N}\}$. For $x \in R$, $\mathrm{Ann}_M(x) = \{m \in M : xm = 0\}$. Let $\mathrm{Spec}_R(M)$ denote the set of prime submodules of M. Then m-$\mathrm{Ass}_R(M) = \{P \in \mathrm{Spec}_R(M) : P = \mathrm{Ann}_M(x) \text{ for some } 0 \neq x \in R\}$. For notations and terminologies not given in this paper, the reader is referred to [13].

Here is a brief summary of the paper. In the second section, for a Noetherian R-module M with $r(\mathrm{Ann}_R(M)) \neq \mathrm{Ann}_R(M)$, we show that $EG(M)$ is a connected graph with $\mathrm{diam}(EG(M)) \leq 2$ and $\mathrm{gr}(EG(M)) \in \{3, \infty\}$ (Theorem 2.6). We show that $EG(M)$ is a complete graph if and only if either $Z(M) = r(\mathrm{Ann}_R(M))$ or $EG(M) = K_2$ (Theorem 2.10). Whenever $r(\mathrm{Ann}_R(M)) = \mathrm{Ann}_R(M)$, among other things, we prove that $|\mathrm{Ass}_R(M)| = 2$ if and only if $EG(M)$ is a complete bipartite graph that is not a star (Theorem 3.7). In the fourth section, for a Noetherian R-module with $r(\mathrm{Ann}_R(M)) = \mathrm{Ann}_R(M)$, we show that $\Gamma(M) = EG(M)$ (Theorem 4.6), where $\Gamma(M)$ denotes the zero divisor graph of M.

2. Properties of the essential graph for modules

Let R be a commutative ring and M be an R-module. A submodule of M is called essential if it has a non-trivial intersection with every non-trivial submodule of M.

Definition 2.1. Let M be an R-module. The essential graph of M, denoted by $EG(M)$ is a simple undirected graph associated to M with vertex set $Z(M) \setminus \mathrm{Ann}_R(M)$, and a pair of distinct vertices x and y are adjacent if and only if $\mathrm{Ann}_M(xy)$ is an essential submodule of M.

Suppose that $x, y \in Z(M) \setminus \mathrm{Ann}_R(M)$. It is easy to see that x and y are adjacent in $EG(M)$ if and only if $\mathrm{Ann}_M(x) + \mathrm{Ann}_M(y)$ is an essential submodule of M.

Lemma 2.2. *Let M be an R-module. If $c \in r(\mathrm{Ann}_R(M)) \setminus \mathrm{Ann}_R(M)$, then c is a universal vertex of $EG(M)$.*

Proof. Let $c \in r(\mathrm{Ann}_R(M)) \setminus \mathrm{Ann}_R(M)$. Then $\mathrm{Ann}_M(c)$ is an essential submodule of M, by [5, Theorem 5(i)]. Hence, for each $a \in Z(M) \setminus \mathrm{Ann}_R(M)$, $\mathrm{Ann}_M(ac)$ is an essential submodule of M. This means that c is a universal vertex of $EG(M)$. □

Lemma 2.3. *Let M be an R-module and let $c \in Z(M) \setminus \mathrm{Ann}_R(M)$ be a universal vertex of $EG(M)$. Then either $\mathrm{Ann}_M(c)$ is an essential submodule of M or $R = R_1 \oplus R_2$ and $M = M_1 \oplus M_2$, where R_1 and R_2 are subrings of R, M_1 and M_2 are R-submodules of M and $(a, 0)$ is a universal vertex of $EG(M)$, for all $a \in Z_{R_1}(M_1)$.*

Proof. Suppose that $c \in Z(M) \setminus \mathrm{Ann}_R(M)$ is a universal vertex of $EG(M)$. If $c^2M = 0$, then the result follows by [5, Theorem 5(i)]. Suppose that $c^2M \neq 0$ and $c \neq c^2$. Thus $\mathrm{Ann}_M(c^3)$ is an essential submodule of M so $\mathrm{Ann}_M(c)$ is an essential submodule of M.

Now, assume that $c^2 = c$. Thus $R = cR \oplus (1-c)R$ and $M = cM \oplus (1-c)M$. Assume that $R_1 = cR$ and $R_2 = (1-c)R$. Then R_1 and R_2 are subrings of R. In addition, $M_1 = cM$ and $M_2 = (1-c)M$ are R-submodules of M. Moreover, if $r = (r_1, r_2)$ and $m = (m_1, m_2)$, then $rm = (r_1m_1, r_2m_2)$. It is easy to see that $c = (1, 0)$. Then $(1, 0)$ is a universal vertex of $EG(M)$. Assume that $0 \neq b \in Z_{R_2}(M_2)$. Thus there exists $0 \neq m_2 \in M_2$ such that $(1, b)(0, m_2) = (0, 0)$ but

$(1, b)(M_1 \oplus M_2) = M_1 \oplus bM_2 \neq 0$. This means that $(1, b) \in Z(M) \setminus \mathrm{Ann}_R(M)$. Since $(1, 0)$ is a universal vertex $\mathrm{Ann}_M((1, 0)(1, b)) = \mathrm{Ann}_M((1, 0)) = 0 \oplus M_2$ is an essential submodule of M that is impossible. Therefore, $Z_{R_2}(M_2) = 0$. Moreover, if $a \in Z_{R_1}(M_1)$, then there exists $0 \neq m_1 \in M_1$ such that $(a, 1)(m_1, 0) = (0, 0)$ so $(a, 1) \in Z(M) \setminus \mathrm{Ann}_R(M)$. Thus $\mathrm{Ann}_M((1, 0)(a, 1)) = \mathrm{Ann}_M((a, 0))$ is an essential submodule of M, as required. □

Remark 2.4. Let the situation be as Lemma 2.3. Since $\mathrm{Ann}_M((a, 0))$ is an essential submodule of M so $\mathrm{Ann}_{M_1}(a)$ is an essential submodule of M_1. Moreover R_1 has characteristic 2; because, if $(1, 0) \neq (-1, 0)$, then $\mathrm{Ann}_M((1, 0)(-1, 0)) = \mathrm{Ann}_M((-1, 0)) = 0 \oplus M_2$ is an essential submodule of M, that is impossible. Thus $1 = -1 \in R_1$ and R_1 has characteristic 2.

Theorem 2.5. *Let M be a Noetherian R-module with $r(\mathrm{Ann}_R(M)) \neq \mathrm{Ann}_R(M)$. Then $x, y \in Z(M) \setminus \mathrm{Ann}_R(M)$ are adjacent in $EG(M)$ if and only if $xy \in \mathfrak{p}$, for all $\mathfrak{p} \in \mathrm{MinAss}_R(M)$.*

Proof. Suppose that M is a Noetherian R-module and $\mathrm{MinAss}_R(M) = \{\mathfrak{p}_1, \cdots, \mathfrak{p}_k\}$. Thus there exists $m_i \in M$ such that $\mathfrak{p}_i = \mathrm{Ann}_R(m_i)$, for all $i = 1, \cdots, k$. Assume that $x, y \in Z(M) \setminus \mathrm{Ann}_R(M)$ are adjacent in $EG(M)$ so $\mathrm{Ann}_M(xy)$ is an essential submodule of M. Hence, $\mathrm{Ann}_M(xy) \cap Rm_i \neq 0$, for all $i = 1, \cdots, k$. Therefore, $xyr_im_i = 0$ for some $r_i \in R$ with $0 \neq r_im_i$ so $xy \in \mathfrak{p}_i$.

Conversely, suppose that $xy \in \mathfrak{p}$, for all $\mathfrak{p} \in \mathrm{MinAss}_R(M)$. We may assume that $x \in \cap_{j=1}^{t}\mathfrak{p}_j$ and $y \in \cap_{j=t+1}^{k}\mathfrak{p}_j$, for some t with $1 \leq t \leq k$. So $xy \in r(\mathrm{Ann}_R(M)) = \cap_{j=1}^{k}\mathfrak{p}_j$. Hence, $\mathrm{Ann}_M(xy)$ is an essential submodule of M by Lemma 2.2. Therefore, x and y are adjacent in $EG(M)$, as needed. □

Theorem 2.6. *Let M be an R-module such that $r(\mathrm{Ann}_R(M)) \neq \mathrm{Ann}_R(M)$. Then the following statements hold:*

(i) *$EG(M)$ is a connected graph with $\mathrm{diam}(EG(M)) \leq 2$.*

(ii) *If M is Noetherian, then $\mathrm{gr}(EG(M)) \in \{3, \infty\}$.*

Proof. (i) It is clear by Lemma 2.2.

(ii) If $|r(\mathrm{Ann}_R(M)) \setminus \mathrm{Ann}_R(M)| \geq 2$, then either $EG(M) = K_{1,1}$ or $EG(M)$ has a cycle with length three, by Lemma 2.2, so $\mathrm{gr}(EG(M)) = 3$. Now, assume that $r(\mathrm{Ann}_R(M)) \setminus \mathrm{Ann}_R(M) = \{c\}$. Two following cases may occur:

Case 1. Let $\mathrm{MinAss}_R(M) = \{\mathfrak{p}_1, \cdots, \mathfrak{p}_k\}$ $(k \geq 2)$. Then for $x \in \mathfrak{p}_i \setminus \cup_{j=1, j\neq i}^{k}\mathfrak{p}_j$ and $y \in \cap_{j=1, j\neq i}^{k}\mathfrak{p}_j \setminus \mathfrak{p}_i$ we have $x \neq y$, $x, y \notin r(\mathrm{Ann}_R(M))$ and $xy \in r(\mathrm{Ann}_R(M)) = \cap_{j=1}^{k}\mathfrak{p}_j$. Hence, x, y are adjacent in $EG(M)$ so $c - x - y - c$ is a cycle in $EG(M)$.

Case 2. Let $\mathrm{MinAss}_R(M) = \{\mathfrak{p}\}$. If $x, y \in Z(M) \setminus r(\mathrm{Ann}_R(M))$ are adjacent in $EG(M)$, then $\mathrm{Ann}_M(xy)$ is an essential submodule of M. So $\mathrm{Ann}_M(xy) \cap Rm \neq 0$, where $\mathfrak{p} = \mathrm{Ann}_R(m)$ and $0 \neq m \in M$. Thus it is easy to see that either $x \in \mathfrak{p}$ or $y \in \mathfrak{p}$ which is a contradiction. Therefore, either $Z(M) \setminus r(\mathrm{Ann}_R(M)) = \{x\}$ and $EG(M) = K_{1,1}$ or $|Z(M) \setminus r(\mathrm{Ann}_R(M))| \geq 2$ and $EG(M) = K_{1,|Z(M)\setminus r(\mathrm{Ann}_R(M))|}$ that has no any cycle. $\square$

Corollary 2.7. *Let M be a Noetherian R-module with $r(\mathrm{Ann}_R(M)) \neq \mathrm{Ann}_R(M)$. If $\mathfrak{p} = r(\mathrm{Ann}_R(M))$ is a prime ideal of R, then $EG(M) = K_{|\mathfrak{p}\setminus \mathrm{Ann}_R(M)|} \vee \bar{K}_{|Z(M)\setminus \mathfrak{p}|}$. In particular,* $\mathrm{diam}(EG(M)) = 2$.

Proof. It is immediate by the proof of Theorem 2.6. $\square$

Corollary 2.8. *Let M be a Noetherian R-module with $r(\mathrm{Ann}_R(M)) \neq \mathrm{Ann}_R(M)$. Then* $\mathrm{diam}(EG(M)) = 2$ *if and only if there are $x, y \in Z(M) \setminus \mathrm{Ann}_R(M)$ and $\mathfrak{p} \in \mathrm{Ass}(M)$ such that $xy \notin \mathfrak{p}$.*

Proof. It is immediate by Theorems 2.5 and 2.6. $\square$

Lemma 2.9. *Let M be an R-module. Then $EG(M)$ is a complete graph if and only if one of the following statements holds:*

(i) $\mathrm{Ann}_M(x)$ *is an essential submodule of M, for all $x \in Z(M) \setminus \mathrm{Ann}_R(M)$.*
(ii) $R = \mathbb{Z}_2 \oplus \mathbb{Z}_2$, $M = (\oplus \mathbb{Z}_2) \oplus (\oplus \mathbb{Z}_2)$ *and* $Z(M) = \{(0,0), (1,0), (0,1)\}$.

Proof. Suppose that $EG(M)$ is complete and $x \in Z(M) \setminus \mathrm{Ann}_R(M)$. If either $x \in r(\mathrm{Ann}_R(M))$ or $x \notin r(\mathrm{Ann}_R(M))$ and $x \neq x^2$, then $\mathrm{Ann}_M(x)$ is an essential submodule of M, by Lemmas 2.2 and 2.3. Now, assume that $x \notin r(\mathrm{Ann}_R(M))$ and $x = x^2$. Then, $R = R_1 \oplus R_2$ and $M = M_1 \oplus M_2$, where R_1 and R_2 are subrings of R, M_1 and M_2 are R-submodules of M and $Z(M) = (Z_{R_1}(M_1) \oplus R_2) \cup (R_1 \oplus 0)$, which follows from the proof of Lemma 2.3. If $0 \neq a \in Z_{R_1}(M_1)$, then there exists $0 \neq m_1 \in M_1$ such that $(a,1)(m_1,0) = (0,0)$ so $(a,1) \in Z(M) \setminus \mathrm{Ann}_R(M)$. Thus $\mathrm{Ann}_M((0,1)(a,1)) = \mathrm{Ann}_M((0,1))$ is an essential submodule of M, that is impossible. Hence, $Z(M) = (R_1 \oplus 0) \cup (0 \oplus R_2)$. Let $(a,0) \in (R_1 \oplus 0) \setminus \{(0,0),(1,0)\}$. Then $(a,0) \in Z(M) \setminus \mathrm{Ann}_R(M)$ and $\mathrm{Ann}_M((1,0)(a,0)) = \mathrm{Ann}_M((a,0)) = 0 \oplus M_2$ is an essential submodule of M, that is impossible. So $R_1 = \{0,1\}$ and $Z(M) = \{(0,0),(1,0)\} \cup (0 \oplus R_2)$. By a similar argument one can show that $R_2 = \{0,1\}$ and $Z(M) = \{(0,0),(1,0)\} \cup \{(0,0),(0,1)\}$ as desired. The converse is obvious. $\square$

Theorem 2.10. *Let M be a Noetherian R-module with $r(\mathrm{Ann}_R(M)) \neq \mathrm{Ann}_R(M)$. Then $EG(M)$ is a complete graph if and only if one of the following statements holds:*

(i) $Z(M) = r(\mathrm{Ann}_R(M))$.
(ii) $EG(M) = K_2$.

Proof. Of course, (i) and (ii) imply that $EG(M)$ is a complete graph, see Lemma 2.2. Hence, it is enough to prove that if $EG(M)$ is complete, then either (i) or (ii) holds. Suppose that $EG(M) \neq K_2$ is a complete graph and $x \in Z(M) \setminus \mathrm{Ann}_R(M)$. Then the increasing chain of submodules $\mathrm{Ann}_M(x) \subseteq \mathrm{Ann}_M(x^2) \subseteq \ldots \subseteq \mathrm{Ann}_M(x^n) \subseteq \ldots$ does stabilize. Suppose that $n \in \mathbb{N}$ and $\mathrm{Ann}_M(x^n) = \mathrm{Ann}_M(x^{n+i})$, for all $i \geq 0$. If $m \in \mathrm{Ann}_M(x) \cap x^n M$, then $m = x^n m'$, for some $m' \in M$. Hence, $x^{n+1}m' = xm = 0$ which implies that $m' \in \mathrm{Ann}_M(x^{n+1}) = \mathrm{Ann}_M(x^n)$. So $m = 0$ and then $x^n M = 0$ since $\mathrm{Ann}_M(x)$ is an essential submodule of M. Therefore, $x \in r(\mathrm{Ann}_R(M))$ and $Z(M) = r(\mathrm{Ann}_R(M))$. □

The following example has been presented to show that the property of being Noetherian is a necessary condition in Theorem 2.10.

Example 2.11. Let p be a prime number and consider $\mathbb{Z}_{p^\infty}$ as a $\mathbb{Z}$-module. It is easy to see that $\mathrm{Ann}_{\mathbb{Z}_{p^\infty}}(p^i)$ is an essential submodule of $\mathbb{Z}_{p^\infty}$, for all $i \geq 1$. Thus $EG(\mathbb{Z}_{p^\infty})$ is a complete graph, but neither $Z_{\mathbb{Z}}(\mathbb{Z}_{p^\infty}) = r(\mathrm{Ann}(\mathbb{Z}_{p^\infty}))$ nor $EG(\mathbb{Z}_{p^\infty}) = K_2$. Therefore, the Noetherian condition in Theorem 2.10 is necessary.

3. Results when $r(\mathrm{Ann}_R(M)) = \mathrm{Ann}_R(M)$

In this section, we investigate more results about the essential graph of M whenever $r(\mathrm{Ann}_R(M)) = \mathrm{Ann}_R(M)$.

Lemma 3.1. *Let M be a Noetherian R-module such that $r(\mathrm{Ann}_R(M)) = \mathrm{Ann}_R(M)$ and let $0 = \cap_{j=1}^n Q_j$ be a minimal primary decomposition of the zero submodule of M with $r(\mathrm{Ann}_R(M/Q_j)) = \mathfrak{p}_j$, for each $j = 1, \cdots, n$. Then the following statements hold:*

(i) *If $\mathfrak{p}_i$ is a minimal element of $\mathrm{Ass}_R(M)$, for some i with $1 \leq i \leq n$, then there exists $a_i \in R$ such that $Q_i = \mathrm{Ann}_M(a_i)$.*
(ii) *If $\mathfrak{p}_i$ is a minimal element of $\mathrm{Ass}_R(M)$, for some i with $1 \leq i \leq n$, then Q_i is a prime submodule of M.*
(iii) *If $P = \mathrm{Ann}_M(a)$ is a prime submodule of M and $\mathfrak{p} = \mathrm{Ann}_R(M/\mathrm{Ann}_M(a))$, then $\mathfrak{p}$ is a minimal element of $\mathrm{Ass}_R(M)$.*
(iv) *If $P = \mathrm{Ann}_M(a)$ is a prime submodule of M, $\mathfrak{p} = \mathrm{Ann}_R(M/\mathrm{Ann}_M(a))$ and $\mathfrak{p} = \mathfrak{p}_i$ for some i with $1 \leq i \leq n$, then $P = Q_i$.*

Proof. (i) Suppose that $0 = \cap_{j=1}^n Q_i$ is a minimal primary decomposition of the zero submodule of M with $r(\mathrm{Ann}_R(M/Q_j)) = \mathfrak{p}_j$, for each $j = 1, \cdots, n$. Assume that $\mathfrak{p}_i = r(\mathrm{Ann}_R(M/Q_i))$ is a minimal element of $\mathrm{Ass}_R(M)$ for some i with $1 \leq i \leq n$. Then $\cap_{j=1,j\neq i}^n \mathrm{Ann}_R(M/Q_j) \nsubseteq \mathfrak{p}_i$. Let $a_i \in \cap_{j=1,j\neq i}^n \mathrm{Ann}_R(M/Q_j) \setminus \mathfrak{p}_i$. We show that $\mathrm{Ann}_M(a_i) = Q_i$. Of course, we have $\mathrm{Ann}_M(a_i) = (0 :_M a_i) = (\cap_{j=1}^n Q_j :_M a_i) = \cap_{j=1}^n (Q_j :_M a_i) = (Q_i :_M a_i)$ and $Q_i \subseteq (Q_i :_M a_i)$. If $m \in (Q_i :_M a_i) \setminus Q_i$, then there exists $t \in \mathbb{N}$ such that $a_i^t M \subseteq Q_i$ so $a_i \in \mathfrak{p}_i$ which is a contradiction. Hence, $Q_i = \mathrm{Ann}_M(a_i)$.

(ii) From (i) it follows that $Q_i = \mathrm{Ann}_M(a_i)$, for some $a_i \in \cap_{j=1,j\neq i}^n \mathfrak{p}_j \setminus \mathfrak{p}_i$. We show that Q_i is a prime submodule. Suppose that $b \in R, m \in M$ are such that $bm \in Q_i$ but $m \notin Q_i$. Thus there is $t \in \mathbb{N}$ such that $b^t M \subseteq Q_i$. So $(a_i b)^t M \subseteq Q_i$. On the other hand, $a_i b \in \cap_{j=1,j\neq i}^n \mathfrak{p}_i$ thus $(a_i b)^t M \subseteq \cap_{j=1,j\neq i}^n Q_i$. Hence, $(a_i b)^t M \subseteq \cap_{j=1}^n Q_j = 0$. Therefore, by the hypothesis $a_i bM = 0$ so $bM \subseteq Q_i$ and Q_i is prime.

(iii) Let $P = \mathrm{Ann}_M(a)$ be a prime submodule of M and $\mathfrak{p} = \mathrm{Ann}_R(M/\mathrm{Ann}_M(a))$. It is easy to see that $\mathfrak{p} = \mathrm{Ann}_R(aM)$. Let $m \in M$ and $am \neq 0$. We show that $\mathfrak{p} = \mathrm{Ann}_R(am)$. It is obvious that $\mathfrak{p} \subseteq \mathrm{Ann}_R(am)$. Assume that $r \in R$ and $ram = 0$. Thus $rm \in P = \mathrm{Ann}_M(a)$ and $m \notin P = \mathrm{Ann}_M(a)$ so $raM = 0$ and $r \in \mathrm{Ann}_R(aM) = \mathfrak{p}$. Hence, $\mathfrak{p} = \mathrm{Ann}_R(am) \in \mathrm{Ass}_R(M)$. If $a \in \cap_{j=1}^n \mathfrak{p}_j$, then there is $t \in \mathbb{N}$ such that $a^t \in \cap_{j=1}^n (Q_j :_R M)$ so $a^t M \subseteq \cap_{j=1}^n Q_j = 0$. Therefore, $a^t M = 0$ and $aM = 0$ which is a contradiction. Thus there are $1 \leq i \leq n$ and $\mathfrak{p}_i \in \mathrm{MinAss}_R(M)$ such that $a \notin \mathfrak{p}_i$. Assume that $r \in \mathfrak{p}$. Thus $raM = 0$ and so $raM \subseteq \cap_{j=1}^n Q_j$. Hence, $ra \in (\cap_{j=1}^n Q_j :_R M) \subseteq \cap_{j=1}^n (Q_j :_R M) \subseteq \cap_{j=1}^n \mathfrak{p}_j$. Now, from $ra \in \mathfrak{p}_i$ and $a \notin \mathfrak{p}_i$ it follows that $\mathfrak{p} \subseteq \mathfrak{p}_i$ so $\mathfrak{p} = \mathfrak{p}_i$.

(iv) Suppose that $1 \leq i \leq n$ and $\mathfrak{p} = \mathfrak{p}_i$. We show that $P = Q_i$. Assume that $m \in Q_i$. Thus $a_i m = 0 \in P$. If $m \notin P$, then $a_i \in \mathfrak{p} = \mathfrak{p}_i$, which is a contradiction so $Q_i \subseteq P$. Assume that $m \in P$ so $am = 0 \in Q_i$. If $m \notin Q_i$, then there is $s \in \mathbb{N}$ such that $a^s M \subseteq Q_i = \mathrm{Ann}_M(a_i)$. Hence, $a_i a^s M = 0$ and so $a_i a^{s-1}(aM) = 0$. Therefore, $a_i a^{s-1} \in \mathfrak{p} = \mathfrak{p}_i$ which implies that $a^{s-1} \in \mathfrak{p} = \mathfrak{p}_i$ since $a_i \notin \mathfrak{p}_i$. Hence, $a \in \mathfrak{p}$. This means that $a^2 M = 0$ and $a \in r(\mathrm{Ann}_R(M)) = \mathrm{Ann}_R(M)$ which is a contradiction. Therefore, $m \in Q_i$ and so $P \subseteq Q_i$. □

Theorem 3.2. *Let M be a Noetherian R-module with $r(\mathrm{Ann}_R(M)) = \mathrm{Ann}_R(M)$. Then $EG(M)$ is a null graph if and only if $|\mathrm{MinAss}_R(M)| = 1$.*

Proof. ($\Leftarrow$) Suppose that $a \in Z(M) \setminus \mathrm{Ann}_R(M)$ and $\mathrm{Ann}_M(a)$ is a prime submodule of M. Thus $\mathfrak{p} = \mathrm{Ann}_R(aM)$ is a minimal element of $\mathrm{Ass}_R(M)$, by Lemma 3.1(iii). In view of [9, Lemma 3.2], $\mathrm{Ann}_M(a)$ is a unique maximal element of $X = \{\mathrm{Ann}_M(x) : x \in Z(M) \setminus \mathrm{Ann}_R(M)\}$. So the zero submodule of M has only one minimal primary component. Thus $r(\mathrm{Ann}_R(M)) = \mathrm{Ann}_R(M) = \mathfrak{p}$. Assume that x and y are adjacent vertices of $EG(M)$ so $\mathrm{Ann}_M(xy)$ is an essential submodule of M. If $\mathrm{Ann}_M(xy)$ is a proper submodule of M, then $\mathrm{Ann}_M(xy) \subseteq \mathrm{Ann}_M(a)$ implies that $\mathrm{Ann}_M(a)$ is an essential submodule of M, which contradicts [5, Theorem 5(iii)]. Hence, $\mathrm{Ann}_M(xy) = M$. So $xy \in \mathrm{Ann}_R(M) = \mathfrak{p}$ which implies that either $x \in \mathrm{Ann}_R(M)$ or $y \in \mathrm{Ann}_R(M)$ that is impossible. Therefore, $EG(M)$ is a null graph.

($\Rightarrow$) Since M is a Noetherian R-module we have $|\mathrm{MinAss}_R(M)| \geq 1$. Moreover, $|\mathrm{MinAss}_R(M)| = |m\text{-}\mathrm{Ass}(M)| \leq 1$, by Lemma 3.1 and [5, Theorem 6(i)]. □

Theorem 3.3. [5, Theorem 7] *Let M be a Noetherian R-module and* $\mathrm{Ann}_R(M) = r(\mathrm{Ann}_R(M))$. *Then $EG(M)$ is a disconnected graph if and only if there exists $b \in Z(M) \setminus \mathrm{Ann}_R(M)$ such that* $\mathrm{Ann}_M(b) \subseteq \bigcap_{P \in m-\mathrm{Ass}(M)} P$.

Corollary 3.4. *Let M be a Noetherian R-module with $r(\mathrm{Ann}_R(M)) = \mathrm{Ann}_R(M)$. Then $EG(M)$ is a connected graph if and only if for all $b \in Z(M) \setminus \mathrm{Ann}_R(M)$ there exists $P \in m$-$\mathrm{Ass}(M)$ such that $\mathrm{Ann}_M(b) \nsubseteq P$.*

Corollary 3.5. *Let M be a Noetherian R-module. If $EG(M)$ is a connected graph, then* $\mathrm{diam}(EG(M)) \leq 3$.

Proof. If $r(\mathrm{Ann}_R(M)) \neq \mathrm{Ann}_R(M)$, then the result follows by Theorem 2.6. Otherwise, it follows by Corollary 3.4, [5, Corollary 2] and [5, Remark 1(iii)], note that by Theorem 3.2, $|m\text{-}\mathrm{Ass}(M)| \geq 2$. □

Corollary 3.6. *Let M be a Noetherian R-module. If the connected graph $EG(M)$ has a cycle, then* $\mathrm{gr}(EG(M)) \leq 4$.

Proof. If $r(\mathrm{Ann}_R(M)) \neq \mathrm{Ann}_R(M)$, then the result follows by Theorem 2.6. Now, assume that $r(\mathrm{Ann}_R(M)) = \mathrm{Ann}_R(M)$. For $|m\text{-}\mathrm{Ass}(M)| \geq 3$ there is nothing to prove, see [5, Remark 1(iii)]. So we may assume that $|m\text{-}\mathrm{Ass}(M)| \leq 2$. On the other hand, $|m\text{-}\mathrm{Ass}(M)| > 1$ since $EG(M)$ is a connected graph, see Corollary 3.4. Hence, $|m\text{-}\mathrm{Ass}(M)| = 2$. Now, by a similar argument to that of [11, Theorem 3.3] one can show that $\mathrm{gr}(EG(M)) \leq 4$. □

Theorem 3.7. *Let M be a Noetherian R-module with $r(\mathrm{Ann}_R(M)) = \mathrm{Ann}_R(M)$ and assume that $EG(M)$ is not a star graph. Then $EG(M)$ is a complete bipartite graph if and only if $|\mathrm{Ass}_R(M)| = 2$.*

Proof. Let $I = \mathrm{Ann}_R(M)$. Note that

$$r(\mathrm{Ann}_R(M)) = \mathrm{Ann}_R(M) \text{ and } r(\mathrm{Ann}_R(M))/I = r(\mathrm{Ann}_{R/I}(M)).$$

Thus $r(\mathrm{Ann}_{R/I}(M)) = 0$. Moreover, for each $a \in R$, we have $a \in Z(M) \setminus \mathrm{Ann}_R(M)$ if and only if $a + I \in Z_{R/I}(M) \setminus \{0\}$ and $\mathfrak{p} \in \mathrm{Ass}_R(M)$ if and only if $\mathfrak{p}/I \in \mathrm{Ass}_{R/I}(M)$. It is therefore enough for us to prove this result under the additional hypothesis that $r(\mathrm{Ann}_R(M)) = 0$.

Let $EG(M)$ be a complete bipartite graph and $\{V_1, V_2\}$ be a partition of the vertex set of $EG(M)$. We prove that $V_i \cup \{0\}$ for $i = 1, 2$ is a prime ideal of R. Let $a, b \in \bar{V_1} = V_1 \cup \{0\}$. If $a = 0$ or $b = 0$ or $a + b = 0$, then $a + b \in \bar{V_1}$ and we are done. Suppose that $a, b \in V_1$. Thus there exist $x, y \in V_2$ such that $\mathrm{Ann}_M(ax)$ and $\mathrm{Ann}_M(by)$ are essential submodules of M. If $\mathrm{Ann}_M(ax) \cap \mathrm{Ann}_M(by) \subseteq \mathrm{Ann}_M(xy)$, then $\mathrm{Ann}_M(xy)$ is an essential submodules of M which is a contradiction. Thus assume that $m \in \mathrm{Ann}_M(ax) \cap \mathrm{Ann}_M(by) \setminus \mathrm{Ann}_M(xy)$. Hence, $(a + b)xym = 0$ and $xym \neq 0$ this means $a + b \in Z(M)$. If $a + b \in V_1$ we are done; otherwise $a + b \in V_2$ and it follows that $\mathrm{Ann}_M(a(a + b))$ and $\mathrm{Ann}_M(b(a + b))$ are essential submodules of M. Then $\mathrm{Ann}_M((a + b)^2)$ and so $\mathrm{Ann}_M(a + b)$ is an essential submodule of M which is a contradiction. So $a + b \in V_1$. Let $a, b \in R$ and $ab \in \bar{V_1}$. We show that either $a \in \bar{V_1}$ or $b \in \bar{V_1}$. If $a = 0$ or $b = 0$, then there is nothing to prove. If $ab = 0$, then $\mathrm{Ann}_M(ab)$ is an essential submodule of M contrary to the assumption. So assume that $0 \neq a, b, 0 \neq ab$ and $a, b \notin V_1$. Thus either $\mathrm{Ann}_M(a^2b)$ or $\mathrm{Ann}_M(ab^2)$ is an essential submodule of M and so $\mathrm{Ann}_M((ab)^2)$ is an essential submodule of M which implies that $\mathrm{Ann}_M(ab)$ is an essential submodule of M, this is a contradiction. Hence, either $a \in V_1$ or $b \in V_1$.

Conversely, assume that $\mathrm{Ass}_R(M) = \{\mathfrak{p}_1, \mathfrak{p}_2\}$. Thus $\mathfrak{p}_1 \cap \mathfrak{p}_2 = r(\mathrm{Ann}_R(M)) = 0$. Suppose that $a, b \in \mathfrak{p}_1 \setminus \{0\}$ and $\mathrm{Ann}_M(ab)$ is an essential submodule of M. Moreover, suppose that $\mathfrak{p}_2 = \mathrm{Ann}_R(m)$, for some $m \in M$. Thus $\mathrm{Ann}_M(ab) \cap Rm \neq 0$. If $0 \neq rm \in \mathrm{Ann}_M(ab)$, then $abr \in \mathfrak{p}_2$ which implies that $ab \in \mathfrak{p}_2$ and so either $a \in \mathfrak{p}_2$ or $b \in \mathfrak{p}_2$. Hence, either $a = 0$ or $b = 0$ which is a contradiction. Therefore, the elements of $\mathfrak{p}_1 \setminus \{0\}$ are not adjacent with each other. By a similar argument, one can show that any two distinct elements of $\mathfrak{p}_2 \setminus \{0\}$ are not adjacent. Let $a \in \mathfrak{p}_1 \setminus \{0\}$ and $b \in \mathfrak{p}_2 \setminus \{0\}$. Then $ab \in \mathfrak{p}_1\mathfrak{p}_2 \subseteq \mathfrak{p}_1 \cap \mathfrak{p}_2 = 0$ so $abM = 0$ which means that an element of $\mathfrak{p}_1 \setminus \{0\}$ is adjacent to all elements of $\mathfrak{p}_2 \setminus \{0\}$. Therefore, $EG(M)$ is a complete bipartite graph. □

Corollary 3.8. *Let M be a Noetherian R-module with $r(\mathrm{Ann}_R(M)) = \mathrm{Ann}_R(M)$. Then $EG(M)$ is a star graph if and only if $R = \mathbb{Z}_2 \oplus R'$ and $M = (\oplus \mathbb{Z}_2) \oplus M'$, where R' is a subring of R and M' is an R-submodule of M and $\mathrm{Ass}_R(M) = \{\mathbb{Z}_2 \oplus 0, 0 \oplus R'\}$.*

Proof. As in Theorem 3.7 we can assume that $r(\text{Ann}_R(M)) = 0$. Let $EG(M)$ be a star graph and let $\{V_1 = \{c\}, V_2 = \{x, y, z, \cdots\}\}$ be a partition of $V(EG(M))$. We prove that $V_i \cup \{0\}$ for $i = 1, 2$, is a prime ideal of R. By the hypotheses and the proof of Lemma 2.3 we have $c^2 = c$ and also it follows that $R = R_1 \oplus R_2$ and $M = M_1 \oplus M_2$, where R_1 and R_2 are subrings of R, M_1 and M_2 are R-submodules of M, $c = (1, 0)$ is the universal vertex of $EG(M)$, $Z_{R_1}(M_1) = Z_{R_2}(M_2) = 0$. Moreover, R_1 has characteristic 2 so $R_1 = \mathbb{Z}_2$. Hence, $Z(M) = \mathbb{Z}_2 \oplus 0 \cup 0 \oplus R_2$. Therefore, $V_1 \cup \{0\} = \mathbb{Z}_2 \oplus 0, V_2 \cup \{0\} = 0 \oplus R_2$ and $\text{Ass}_R(M) = \{\mathbb{Z}_2 \oplus 0, 0 \oplus R_2\}$. □

4. Relations between the zero divisor graph and the essential graph

In this section we will study the relations between the zero-divisor graph defined in [11] and the essential graph for modules.

Definition 4.1. [11, Definition 2.1] Let M be an R-module. The zero-divisor graph of M, denoted by $\Gamma(M)$ is a simple undirected graph whose vertex set is $Z(M) \setminus \text{Ann}_R(M)$ and two distinct vertices x and y are adjacent if and only if $xyM = 0$.

To commence, we show that the zero-divisor graph is a subgraph of the essential graph.

Lemma 4.2. *Let M be an R-module. Then $\Gamma(M)$ is a subgraph of $EG(M)$.*

Proof. Suppose that x and y are adjacent in $\Gamma(M)$. Then $xyM = 0$ and $M = \text{Ann}_M(xy)$ is an essential submodule of M. Hence, x and y are adjacent in $EG(M)$. □

Lemma 4.3. *Let M be an R-module and $x \in Z(M) \setminus r(\text{Ann}_R(M))$. If $\text{Ann}_M(x)$ is a prime submodule of M, then $N_{\Gamma(M)}(x) = N_{EG(M)}(x)$.*

Proof. Suppose that $x \in Z(M) \setminus r(\text{Ann}_R(M))$ and $\text{Ann}_M(x)$ is a prime submodule of M. It is enough to show that $N_{EG(M)}(x) \subseteq N_{\Gamma(M)}(x)$. Assume that $y \in N_{EG(M)}(x)$. Thus $\text{Ann}_M(xy)$ is an essential submodule of M. In view of [5, Theorem 5(iii)] $\text{Ann}_M(x)$ is not an essential submodule of M. Hence, there exists a nonzero submodule N of M such that $\text{Ann}_M(x) \cap N = 0$. Therefore, for some $m \in M$ we have $xym = 0$ but $xm \neq 0$ so we get that $xyM = 0$ since $\text{Ann}_M(x)$ is a prime submodule of M. Therefore, x and y are adjacent in $\Gamma(M)$ and the proof is completed. □

The following example shows that Lemma 4.3 does not hold necessarily for elements of $r(\text{Ann}_R(M))$.

Example 4.4. Consider $M = \mathbb{Z}/12\mathbb{Z}$ as a $\mathbb{Z}$-module. For $6 \in r(\mathrm{Ann}_R(M))$, $\mathrm{Ann}_M(6) = 2\mathbb{Z}/12\mathbb{Z}$ is a prime submodule of M but $N_{\Gamma(M)}(6) \neq N_{EG(M)}(6)$.

Lemma 4.5. *Let M be a Noetherian R-module. Then 0 is a prime submodule of M if and only if $EG(M)$ is a null graph. In particular, $EG(M) = \Gamma(M)$.*

Proof. Suppose that 0 is a prime submodule of M. Then $|\mathrm{MinAss}_R(M)| = 1$ and so the result follows by Theorem 3.2. □

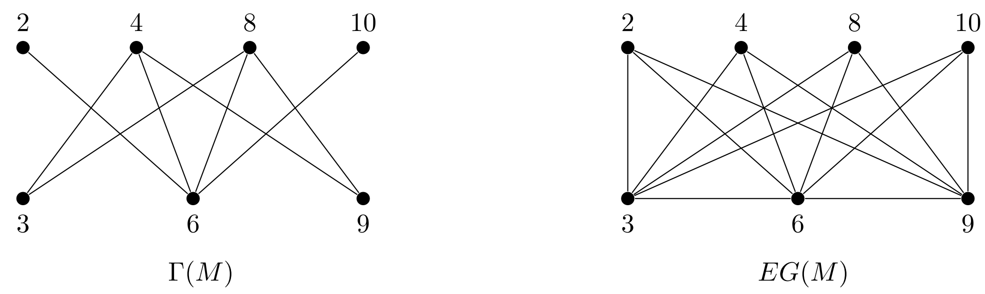
Theorem 4.6. *Let M be a Noetherian R-module with $r(\mathrm{Ann}_R(M)) = \mathrm{Ann}_R(M)$. Then $\Gamma(M) = EG(M)$.*

Proof. It is obvious that $\Gamma(M)$ is a subgraph of $EG(M)$, by Lemma 4.2. Now, it is sufficient to show that each edge of $EG(M)$ is an edge of $\Gamma(M)$. Suppose that $x - y$ is an edge of $EG(M)$. Then $\mathrm{Ann}_M(xy)$ is an essential submodule of M. By the assumption the chain $\mathrm{Ann}_M(xy) \subseteq \mathrm{Ann}_M((xy)^2) \subseteq \ldots \subseteq \mathrm{Ann}_M((xy)^n) \subseteq \ldots$ of submodules does stabilize, thus there is $n \in \mathbb{N}$ such that $\mathrm{Ann}_M((xy)^n) = \mathrm{Ann}_M((xy)^{n+i})$, for all $i \geq 0$. Assume that $m \in \mathrm{Ann}_M(xy) \cap (xy)^n M$. Thus $m = (xy)^n m'$ for some $m' \in M$. Hence $(xy)^{n+1} m' = xym = 0$, which implies that $m' \in \mathrm{Ann}_M((xy)^{n+1}) = \mathrm{Ann}_M((xy)^n)$. Then $m = 0$ and $(xy)^n M = 0$ since $\mathrm{Ann}_M(xy)$ is an essential submodule of M. Therefore, $xy \in r(\mathrm{Ann}_R(M))$ and so $xyM = 0$. □

The following examples have been presented to show that the properties of being Noetherian and $r(\mathrm{Ann}_R(M)) = \mathrm{Ann}_R(M)$ are necessary conditions in Theorem 4.6.

Example 4.7. (i) Example 2.11 shows that for the non-Noetherian $\mathbb{Z}$-module $\mathbb{Z}_{p^\infty}$ we have $r(\mathrm{Ann}_{\mathbb{Z}}(\mathbb{Z}_{p^\infty})) = \mathrm{Ann}_{\mathbb{Z}}(\mathbb{Z}_{p^\infty})$ but $EG(\mathbb{Z}_{p^\infty}) \neq \Gamma(\mathbb{Z}_{p^\infty})$.

(ii) For the Noetherian $\mathbb{Z}$-module $\mathbb{Z}/12\mathbb{Z}$, $\mathrm{Ann}_{\mathbb{Z}}(\mathbb{Z}/12\mathbb{Z})) \neq \mathrm{Ann}_{\mathbb{Z}}(\mathbb{Z}/12\mathbb{Z})$. The following figures (induced subgraphs of $\Gamma(\mathbb{Z}/12\mathbb{Z})$ and $EG(\mathbb{Z}/12\mathbb{Z})$) show that $EG(\mathbb{Z}/12\mathbb{Z}) \neq \Gamma(\mathbb{Z}/12\mathbb{Z})$.



$\Gamma(M)$ $EG(M)$

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