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THE STRUCTURE THEOREM OF HOM-HOPF BIMODULES AND ITS APPLICATIONS

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Abstract. In this paper, we give the structure theorem of Hom-Hopf bimodules. Furthermore, we give the structure theorem of Hom-comodule algebras. Finally, we consider and study the structure theorems of Hom-Hopf bicomodules and Hom-module coalgebras.

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1. Introduction and preliminaries

1.1. Introduction. The structure theorem of Hopf modules was introduced in [12]. Let H be a Hopf algebra, M a right H-Hopf module, the structure theorem for Hopf modules is given by

$$M \cong M^{coH} \otimes H$$

as Hopf modules isomorphism, where $M^{coH} = \{m \in M \mid \rho(m) = m \otimes 1_H\}$ [12, Proposition 1]. It's also used to study integrals, for example, the proof of uniqueness and existence of integrals over a finite dimensional Hopf algebra is based on it.

The structure theorem of Hopf modules can be extended to Hopf bimodules. In [20], when H is a Hopf algebra and A an H-Galois extension of a k-algebra B, one can construct an algebra $L := L(A,H) = (A \otimes A)^{coH}$, which was previously shown to be a Hopf algebra if $B = k$. The author has proved that there is a structure theorem for relative Hopf bimodules in the form of a category equivalence ${}_A\mathcal{M}_A^H \cong {}_L\mathcal{M}$.

It is well-known that any given right H-comodule algebra A has naturally a structure of Hopf bimodules, that is, $A \in {}_A\mathcal{M}_A^H$. The structure theorem for comodule

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algebras is given by

$$A \cong A^{coH} \# H$$

as an algebra isomorphism if there exists a morphism $\phi : H \to A$ of right H-comodule algebras, where $A^{coH} = \{a \in A \mid \rho(a) = a \otimes 1_H\}$ and $A^{coH} \# H$ denotes the smash product (see [16]).

In recent years, the structure theorems for Hopf modules and comodule algebras have been generalized to weak Hopf algebras, quasi-Hopf algebras, and braided Hopf algebras by many researchers (see [9,11,18,19,22,24]).

The origins of the study of Hom-algebras can be found in [10] by Hartwig, Larsson and Silvestrov in order to develop a theory of central extensions of Lie algebras. Hom-structures (algebras, coalgebras, Hopf algebras, Lie algebras) are vector spaces, endowed with an endomorphism, such that the classical definition of these algebraic structures is deformed by the endomorphism. The theory of Hom-type algebraic structures has been studied by many researchers (see [3,4,5,6,7,13,14,15,23]). Especially, in 2011, S. Caenepeel and I. Goyvaerts studied the Hom-type algebras from the point of view of monoidal categories (see [3]).

The structure theorems of Hopf bimodules and comodule algebras have been studied in braided monoidal categories (see [1,2,9,20]). The main purpose of this paper is to investigate the structure theorems of Hopf bi(co)modules, comodule algebras, and module coalgebras in the particular case of the category $\widetilde{\mathcal{H}}(\mathcal{M}_k)$.

This paper is organized as follows. In Section 1, we recall some definitions and remarks. In Section 2, we construct the structure theorem of Hom-Hopf bimodules via the coinvariant subspace, that is, for any Hom-Hopf bimodule (M, μ), there exists an isomorphism

$$M \cong M^{coH} \otimes H$$

as Hom-Hopf bimodules, where $M^{coH} = \{m \in M \mid \rho(m) = \mu^{-1}(m) \otimes 1_H\}$. In Section 3, as an application of Section 2, we give the structure theorem of Hom-comodule algebras. In Section 4 and Section 5, we give the structure theorems of Hom-Hopf bicomodules and Hom-module coalgebras.

Notations. Throughout, we always work over a fixed field k and freely use the Hopf algebra terminology for which we refer to [21]. For a coalgebra C, we write its comultiplication $\Delta(c) = c_1 \otimes c_2$, for any $c \in C$, in which we often omit the summation symbols; for a right (left) C-comodule M, we denote its coaction by $\rho(m) = m_{(0)} \otimes m_{(1)}$ ($\rho(m) = m_{(-1)} \otimes m_{(0)}$). Any unexplained definitions and notations may be found in [8,17].

1.2. Preliminaries. Let $\mathcal{M}_k = (\mathcal{M}_k, \otimes, k, a, l, r)$ be the category of k-modules. Then there is a new monoidal category $\mathcal{H}(\mathcal{M}_k)$, whose objects are couples (M, μ), where $M \in \mathcal{M}_k$ and $\mu \in Aut_k(M)$. The morphisms of $\mathcal{H}(\mathcal{M}_k)$ are morphisms $f : (M, \mu) \to (N, \nu)$ in $\mathcal{M}_k$ such that $\nu \circ f = f \circ \mu$. For any objects $(M, \mu), (N, \nu) \in \mathcal{H}(\mathcal{M}_k)$, the monoidal structure is given by

$$(M, \mu) \otimes (N, \nu) = (M \otimes N, \mu \otimes \nu) \quad \text{and} \quad (k, id).$$

Briefly speaking, all Hom-structures are objects in the monoidal category $\widetilde{\mathcal{H}}(\mathcal{M}_k) = (\mathcal{H}(\mathcal{M}_k), \otimes, (k, id), \widetilde{a}, \widetilde{l}, \widetilde{r})$ introduced in [3], where the associator $\widetilde{a}$ is given by the formula

$$\widetilde{a}_{M,N,L} = a_{M,N,L} \circ ((\mu \otimes id) \otimes \varsigma^{-1}) = (\mu \otimes (id \otimes \varsigma^{-1})) \circ a_{M,N,L},$$

for any objects $(M, \mu), (N, \nu), (L, \varsigma) \in \mathcal{H}(\mathcal{M}_k)$, and the unitors $\widetilde{l}$ and $\widetilde{r}$ are

$$\widetilde{l}_M = \mu \circ l_M = l_M \circ (id \otimes \mu), \quad \widetilde{r}_M = \mu \circ r_M = r_M \circ (\mu \otimes id).$$

The category $\widetilde{\mathcal{H}}(\mathcal{M}_k)$ is called the Hom-category associated to the monoidal category $\mathcal{M}_k$.

In what follows we recall some definitions about Hom-structures from [3, Section 2].

Definition 1.1. A *unital monoidal Hom-associative algebra* is an object (A, α) in the category $\widetilde{\mathcal{H}}(\mathcal{M}_k)$ together with an element $1_A \in A$, and linear maps

$$m : A \otimes A \longrightarrow A, \quad a \otimes b \mapsto ab, \ \alpha \in Aut(A)$$

such that

$$\alpha(ab) = \alpha(a)\alpha(b); \quad \alpha(1_A) = 1_A,$$
$$\alpha(a)(bc) = (ab)\alpha(c); \quad a1_A = 1_A a = \alpha(a),$$

for all $a, b, c \in A$.

Remark 1.2. Let (A, α) and (A', α') be two monoidal Hom-algebras. A monoidal Hom-algebra map $f : (A, \alpha) \longrightarrow (A', \alpha')$ is a linear map such that $f \circ \alpha = \alpha' \circ f$, $f(1_A) = 1_{A'}$ and $f(ab) = f(a)f(b)$ for any $a, b \in A$.

Definition 1.3. A *counital monoidal Hom-coassociative coalgebra* is an object (C, β) in the category $\widetilde{\mathcal{H}}(\mathcal{M}_k)$ together with linear maps, $\Delta : C \longrightarrow C \otimes C$, $\Delta(c) = c_1 \otimes c_2$ and $\varepsilon : C \longrightarrow k$, such that

$$\beta(c)_1 \otimes \beta(c)_2 = \beta(c_1) \otimes \beta(c_2); \quad \varepsilon \circ \beta = \varepsilon,$$
$$\beta^{-1}(c_1) \otimes \Delta(c_2) = \Delta(c_1) \otimes \beta^{-1}(c_2); \quad c_1\varepsilon(c_2) = \varepsilon(c_1)c_2 = \beta^{-1}(c),$$

for all $c \in C$.

Remark 1.4. Let (C,β) and (C',β') be two monoidal Hom-coalgebras. A monoidal Hom-coalgebra map $f:(C,\beta)\longrightarrow(C',\beta')$ is a linear map such that $f\circ\beta=\beta'\circ f$, $\Delta\circ f=(f\otimes f)\circ\Delta$ and $\varepsilon\circ f=\varepsilon$.

Definition 1.5. A *monoidal Hom-bialgebra* $H=(H,m,1_H,\Delta,\varepsilon,\gamma)$ is a bialgebra in the category $\widetilde{\mathcal{H}}(\mathcal{M}_k)$. This means that $(H,m,1_H,\gamma)$ is a monoidal Hom-algebra and $(H,\Delta,\varepsilon,\gamma)$ is a monoidal Hom-coalgebra, such that Δ and ε are morphisms of Hom-algebras, that is, for any $h,g\in H$,

$$\Delta(hg)=\Delta(h)\Delta(g);\quad \Delta(1_H)=1_H\otimes 1_H,$$
$$\varepsilon(hg)=\varepsilon(h)\varepsilon(g);\quad \varepsilon(1_H)=1.$$

A monoidal Hom-bialgebra (H,γ) is called a monoidal Hom-Hopf algebra if there exists a morphism (called the antipode) $S:H\longrightarrow H$ in $\widetilde{\mathcal{H}}(\mathcal{M}_k)$ (i.e., $S\circ\gamma=\gamma\circ S$), which is the convolution inverse of the identity morphism id_H (i.e., $S*id_H=\eta_H\circ\varepsilon_H=id_H*S$), this means for any $h\in H$,

$$S(h_1)h_2=h_1S(h_2)=\varepsilon(h)1_H.$$

The antipodes of monoidal Hom-Hopf algebras have similar properties of those of Hopf algebras such as they are morphisms of Hom-anti-(co)algebras. Since γ is bijective and commutes with the antipode S, $S\circ\gamma^{-1}=\gamma^{-1}\circ S$.

Definition 1.6. Let (A,α) be a monoidal Hom-algebra. A *left* (A,α)*-Hom-module* consist of $(M,\mu)\in\widetilde{\mathcal{H}}(\mathcal{M}_k)$ together with a morphism $\psi:A\otimes M\longrightarrow M$, $\psi(a\otimes m)=a\cdot m$, such that

$$\mu(a\cdot m)=\alpha(a)\cdot\mu(m),$$
$$\alpha(a)\cdot(b\cdot m)=(ab)\cdot\mu(m);\quad 1_A\cdot m=\mu(m),$$

for all $a,b\in A$ and $m\in M$.

Remark 1.7. Let (M,μ) and (N,ν) be two left (A,α)-Hom-modules. A morphism $f:M\longrightarrow N$ is called left A-linear if $f(a\cdot m)=a\cdot f(m)$, for every $a\in A$, $m\in M$, and $f\circ\mu=\nu\circ f$. We denote the category of left (A,α)-Hom-modules by $\widetilde{\mathcal{H}}({}_A\mathcal{M})$.

Similarly, we can define right (A,α)-Hom-modules and right (A,α)-Hom-module maps.

Definition 1.8. Let (C,β) be a monoidal Hom-coalgebra. A *right* (C,β)*-Hom-comodule* is an object (M,μ) in $\widetilde{\mathcal{H}}(\mathcal{M}_k)$ together with a k-linear map

$\rho_M : M \longrightarrow M \otimes C,\ \rho_M(m) = m_{(0)} \otimes m_{(1)}$ such that

$$\mu(m)_{(0)} \otimes \mu(m)_{(1)} = \mu(m_{(0)}) \otimes \beta(m_{(1)}),$$
$$\mu^{-1}(m_{(0)}) \otimes \Delta(m_{(1)}) = m_{(0)(0)} \otimes (m_{(0)(1)} \otimes \beta^{-1}(m_{(1)})),$$
$$m_{(0)}\varepsilon(m_{(1)}) = \mu^{-1}(m),$$

for all $m \in M$.

Remark 1.9. Let (M,μ) and (N,ν) be two right (C,β)-Hom-comodules. A morphism $g : M \longrightarrow N$ is called right (C,β)-colinear if $g \circ \mu = \nu \circ g$ and $g(m_{(0)}) \otimes m_{(1)} = g(m)_{(0)} \otimes g(m)_{(1)}$, for any $m \in M$. The category of right (C,β)-Hom-comodules is denoted by $\widetilde{\mathcal{H}}({}^C\mathcal{M})$.

Similarly, we can define left (C,β)-Hom-comodules and left (C,β)-Hom-comodule maps.

2. Hom-Hopf bimodules

Definition 2.1. Let (A,α) be a monoidal Hom-algebra and (M,μ) a left and right (A,α)-Hom-module satisfying the condition

$$(a \cdot m) \cdot \alpha(b) = \alpha(a) \cdot (m \cdot b),$$

for all $a, b \in A$ and $m \in M$. Then we call (M,μ) an (A,α)-*Hom-bimodule*.

Definition 2.2. [3, Section 3] Let (H,α) be a monoidal Hom-Hopf algebra. A *right* (H,α)-*Hom-Hopf module* (M,μ) is defined as a right (H,α)-Hom-module, which is a right (H,α)-Hom-comodule as well, satisfying the following compatibility relation:

$$\rho(m \cdot h) = m_{(0)} \cdot h_1 \otimes m_{(1)} h_2,$$

for all $h \in H$ and $m \in M$.

Similarly, we can define a left, right (H,α)-Hom-Hopf module, and a right, left (H,α)-Hom-Hopf module.

Definition 2.3. Let (H,α) be a monoidal Hom-Hopf algebra and (M,μ) an (H,α)-Hom-bimodule. An (H,α)-*Hom-Hopf bimodule* (M,μ) is defined as a left, right (H,α)-Hom-Hopf module which is a right (H,α)-Hom-Hopf module as well.

In what follows, (H,α) is always considered to be a monoidal Hom-Hopf algebra and (M,μ) is an (H,α)-Hom-Hopf bimodule.

Define a map $E : M \to M, m \mapsto m_{(0)} \cdot S(m_{(1)})$, we have $E \circ \mu = \mu \circ E$. Denote the category of (H,α)-Hom-Hopf bimodules by ${}_H\mathcal{M}_H^H$.

Lemma 2.4. *Let* $(M,\mu) \in {}_H\mathcal{M}_H^H$. *Then the following conclusions hold:*

(a) $ImE = M^{coH} = \{m \in M \mid \rho(m) = \mu^{-1}(m) \otimes 1_H\}$.

(b) E *is idempotent, i.e.*

$$E^2 = E. \tag{1}$$

(c) *For any* $m \in M, h \in H$,

$$E(m \cdot h) = \varepsilon(h)\mu(E(m)). \tag{2}$$

(d) *For any* $m \in M, h \in H$,

$$E(h \cdot m) = (h_1 \cdot \mu^{-1}(E(m))) \cdot \alpha(S(h_2)). \tag{3}$$

(e) $E(m_{(0)}) \cdot m_{(1)} = m$, *for any* $m \in M$.

Proof. The morphism E is the right-hand analogue of the idempotent ${}_X\Pi$, and the ImE of (a) is the right-hand analogue of ${}_HX$ in [1, Proposition 3.2.1]. Hence (a) and (b) are just writing the result [1, Proposition 3.2.1] in the particular case of the category $\widetilde{\mathcal{H}}(\mathcal{M}_k)$.

In addition, by (a), we can get the following equalities:

$$\rho(E(m)) = \mu^{-1}(E(m)) \otimes 1_H, \tag{4}$$

and

$$E(m) = m_{(0)} \cdot S(m_{(1)}) = \mu^{-1}(m) \cdot 1_H = m, \tag{5}$$

for any $m \in M^{coH}$.

(c) For any $m \in M, h \in H$, we get

$$\begin{aligned} E(m \cdot h) &= (m \cdot h)_{(0)} \cdot S((m \cdot h)_{(1)}) \\ &= (m_{(0)} \cdot h_1) \cdot S(m_{(1)} h_2) \\ &= (m_{(0)} \cdot h_1) \cdot (S(h_2) S(m_{(1)})) \\ &= \mu(m_{(0)}) \cdot (h_1 \alpha^{-1}(S(h_2) S(m_{(1)}))) \\ &= \mu(m_{(0)}) \cdot ((\alpha^{-1}(h_1)\alpha^{-1}(S(h_2))) S(m_{(1)})) \\ &= \varepsilon(h)\mu(m_{(0)} \cdot S(m_{(1)})) \\ &= \varepsilon(h)\mu(E(m)). \end{aligned}$$

(d) is the right-most equality [1, (20)] rewritten in $\widetilde{\mathcal{H}}(\mathcal{M}_k)$.

(e) For any $m \in M$, we have

$$E(m_{(0)}) \cdot m_{(1)} = (m_{(0)(0)} \cdot S(m_{(0)(1)})) \cdot m_{(1)}$$

$$\begin{aligned}
&= \mu(m_{(0)(0)}) \cdot (S(m_{(0)(1)})\alpha^{-1}(m_{(1)})) \\
&= m_{(0)} \cdot (S(m_{(1)1})m_{(1)2}) \\
&= m.
\end{aligned}$$

□

Lemma 2.5. *Let* $(M, \mu) \in {}_H\mathcal{M}_H^H$.

(A) *Define* $h \triangleright m = E(h \cdot m)$ *for any* $h \in H, m \in M$. *Then*

(1) *for any* $m \in M, h \in H$,

$$h \triangleright E(m) = E(h \cdot m). \tag{6}$$

(2) $(M^{coH}, \triangleright)$ *is a left* (H, α)*-Hom-module.*

(3) *for any* $m \in M, h \in H$,

$$(h_1 \triangleright E(m)) \cdot \alpha(h_2) = h \cdot E(\mu(m)). \tag{7}$$

(B) *Define* $m \triangleleft h = E(m \cdot h)$ *for any* $h \in H, m \in M$. *Then*

(1) *for any* $h \in H, m \in M$,

$$E(m) \triangleleft h = \mu(E(m))\varepsilon(h). \tag{8}$$

(2) $(M^{coH}, \triangleleft)$ *is a right* (H, α)*-Hom-module.*

(C) $(M^{coH}, \triangleright, \triangleleft)$ *is an* (H, α)*-Hom-bimodule.*

Proof. (A) (1) For any $h \in H, m \in M$, we can prove that

$$\begin{aligned}
h \triangleright E(m) &= E(h \cdot E(m)) \\
&\overset{(3)}{=} (h_1 \cdot \mu^{-1}(E^2(m))) \cdot \alpha(S(h_2)) \\
&\overset{(1)}{=} (h_1 \cdot \mu^{-1}(E(m))) \cdot \alpha(S(h_2)) \\
&\overset{(3)}{=} E(h \cdot m).
\end{aligned}$$

(2) is the left module version analogue of equality [1, (21)] in the particular case of the category $\widetilde{\mathcal{H}}(\mathcal{M}_k)$.

(3) For any $m \in M, h \in H$, we get

$$\begin{aligned}
(h_1 \triangleright E(m)) \cdot \alpha(h_2) &\overset{(6)}{=} E(h_1 \cdot m) \cdot \alpha(h_2) \\
&\overset{(3)}{=} ((h_{11} \cdot \mu^{-1}(E(m))) \cdot \alpha(S(h_{12}))) \cdot \alpha(h_2) \\
&= (\alpha(h_{11}) \cdot E(m)) \cdot (\alpha(S(h_{12}))h_2) \\
&= (h_1 \cdot E(m)) \cdot (\alpha(S(h_{21}))\alpha(h_{22}))
\end{aligned}$$

$$\begin{aligned} &= (\alpha^{-1}(h)\cdot E(m))\cdot 1_H \\ &= h\cdot E(\mu(m)). \end{aligned}$$

(B) (1) For any $h\in H, m\in M$, we can get

$$E(m)\lhd h = E(E(m)\cdot h) \overset{(2)}{=} \varepsilon(h)\mu(E(E(m))) \overset{(1)}{=} \varepsilon(h)\mu(E(m)).$$

(2) In fact, we find that the right action $\lhd$ is trivial by Lemma 2.4(a) and eq.(8). Hence $(M^{coH},\lhd)$ is a right (H,α)-Hom-module.

(C) By (A) and (B), we know $(M^{coH},\rhd)$ is a left (H,α)-Hom-module and $(M^{coH},\lhd)$ is a right (H,α)-Hom-module. In what follows, we only need to prove that $(h\rhd m)\lhd\alpha(g)=\alpha(h)\rhd(m\lhd g)$, for any $m\in M^{coH}, h,g\in H$:

$$(h\rhd m)\lhd\alpha(g) = E(h\cdot m)\lhd\alpha(g) \overset{(8)}{=} \mu(E(h\cdot m))\varepsilon(g).$$

and

$$\begin{aligned} &\alpha(h)\rhd(m\lhd g) = \alpha(h)\rhd(E(m\cdot g)) \overset{(2)}{=} (\alpha(h)\rhd\mu(E(m)))\varepsilon(g) \\ &\overset{(5)}{=} (\alpha(h)\rhd\mu(m))\varepsilon(g) = E(\alpha(h)\cdot\mu(m))\varepsilon(g) \\ &= E(\mu(h\cdot m))\varepsilon(g) = \mu(E(h\cdot m))\varepsilon(g). \end{aligned}$$

□

Lemma 2.6. *Let (M,μ) be a left (H,α)-Hom-module. Then $M\otimes H$ is an (H,α)-Hom-Hopf bimodule via the following actions and coaction:*

$$\begin{aligned} &\blacktriangleright: H\otimes M\otimes H\to M\otimes H, \quad h\otimes(m\otimes g)\mapsto h_1\cdot m\otimes h_2 g, \\ &\blacktriangleleft: M\otimes H\otimes H\to M\otimes H, \quad (m\otimes h)\otimes g\mapsto \mu(m)\otimes h\alpha^{-1}(g), \\ &\rho_{M\otimes H}: M\otimes H\to M\otimes H\otimes H, \quad m\otimes h\mapsto(\mu^{-1}(m)\otimes h_1)\otimes\alpha(h_2), \end{aligned}$$

for any $m\in M, h,g\in H$.

Proof. It is easy to show that $M\otimes H$ is a left (H,α)-Hom-module, right (H,α)-Hom-module and right (H,α)-Hom-comodule via $\blacktriangleright,\blacktriangleleft$ and $\rho_{M\otimes H}$, respectively.

In the following, we have only to check that $M\otimes H$ is an (H,α)-Hom-bimodule, right (H,α)-Hom-Hopf module and left, right (H,α)-Hom-Hopf module.

Indeed, for any $m\in M, h,g,l\in H$, we have

$$\begin{aligned} (h\blacktriangleright(m\otimes l))\blacktriangleleft\alpha(g) &= (h_1\cdot m\otimes h_2 l)\blacktriangleleft\alpha(g) \\ &= \mu(h_1\cdot m)\otimes(h_2 l)g \\ &= \mu(h_1\cdot m)\otimes\alpha(h_2)(l\alpha^{-1}(g)) \end{aligned}$$

$$\begin{aligned} &= \alpha(h) \blacktriangleright (\mu(m) \otimes l\alpha^{-1}(g)) \\ &= \alpha(h) \blacktriangleright ((m \otimes l) \blacktriangleleft g), \end{aligned}$$

$$\begin{aligned} \rho_{M\otimes H}((m \otimes l) \blacktriangleleft h) &= \rho_{M\otimes H}(\mu(m) \otimes l\alpha^{-1}(h)) \\ &= m \otimes l_1\alpha^{-1}(h_1) \otimes \alpha(l_2)h_2 \\ &= (\mu^{-1}(m) \otimes l_1) \blacktriangleleft h_1 \otimes \alpha(l_2)h_2 \\ &= (m \otimes l)_{(0)} \blacktriangleleft h_1 \otimes (m \otimes l)_{(1)}h_2, \end{aligned}$$

and

$$\begin{aligned} \rho_{M\otimes H}(h \blacktriangleright (m \otimes l)) &= \rho_{M\otimes H}(h_1 \cdot m \otimes h_2 l) \\ &= \mu^{-1}(h_1 \cdot m) \otimes h_{21}l_1 \otimes \alpha(h_{22}l_2) \\ &= h_{11} \cdot \mu^{-1}(m) \otimes h_{12}l_1 \otimes h_2\alpha(l_2) \\ &= h_1 \blacktriangleright (\mu^{-1}(m) \otimes l_1) \otimes h_2\alpha(l_2) \\ &= h_1 \blacktriangleright (m \otimes l)_{(0)} \otimes h_2(m \otimes l)_{(1)}. \end{aligned}$$

So $M \otimes H$ is an (H, α)-Hom-Hopf bimodule. □

In what follows, we give the structure theorem for (H, α)-Hom-Hopf bimodules.

Theorem 2.7. *Let $(M, \mu) \in {}_H\mathcal{M}_H^H$. Consider M^{coH} as a left (H, α)-Hom-module with action $\triangleright$ as in Lemma 2.5. Then $M^{coH} \otimes H \in {}_H\mathcal{M}_H^H$ and $(M^{coH} \otimes H, \mu_{|M^{coH}} \otimes \alpha) \cong (M, \mu)$ as (H, α)-Hom-Hopf bimodules.*

Proof. It is [1, Proposition 3.6.3] in the particular case of $\widetilde{\mathcal{H}}(\mathcal{M}_k)$.

By Lemma 2.6, we know that $M^{coH} \otimes H$ is an (H, α)-Hom-Hopf bimodule.

We recall two maps as follows:

$$\begin{aligned} &\lambda : M^{coH} \otimes H \to M, m \otimes h \mapsto m \cdot h, \\ &\nu : M \to M^{coH} \otimes H, m \mapsto E(m_{(0)}) \otimes m_{(1)}. \end{aligned}$$

λ and ν are well-defined mutually inverse isomorphism of (H, α)-Hom-Hopf bimodules. □

3. The structure theorem of Hom-comodule algebras

In this section, as an application of Section 2, we give the structure theorem of Hom-comodule algebras. In what follows, we assume that (H, α) is a monoidal Hom-Hopf algebra.

Definition 3.1. Let (H,α) be a monoidal Hom-Hopf algebra. A monoidal Hom-algebra (B,β) is called a *right* (H,α)*-Hom-comodule algebra* [14, Definition 3.3] if (B,β) is a right (H,α)-Hom-comodule with coaction ρ obeying the following axioms:

$$\rho(ab)=a_{(0)}b_{(0)}\otimes a_{(1)}b_{(1)},\quad \rho(1_B)=1_B\otimes 1_H,$$

for all $a,b\in B$.

Let (M,μ) and (N,ν) be two right (H,α)-Hom-comodule algebras. A morphism $f:M\longrightarrow N$ is called an (H,α)-Hom-comodule algebra map, if f is right (H,α)-comodule map and algebra map such that $f\circ\mu=\nu\circ f$.

Assume that (B,β) is a right (H,α)-Hom-comodule algebra with coinvariant monoidal Hom-subalgebra $A=B^{coH}$ and there exists a right (H,α)-Hom-comodule algebra map $\phi:H\to B$, where (H,α) is a right (H,α)-Hom-comodule algebra via comultiplication Δ.

Define two (H,α)-actions on B: $h\rightharpoonup b=\phi(h)b$, $b\leftharpoonup h=b\phi(h)$. It is easy to check that $(B,\rightharpoonup,\leftharpoonup,\rho_B)$ is an (H,α)-Hom-Hopf bimodule.

Definition 3.2. Let (H,α) be a monoidal Hom-bialgebra. A monoidal Hom-algebra (B,β) is called *left* (H,α)*-Hom-module algebra* [14, Definition 2.6] if (B,β) is a left (H,α)-Hom-module with action $\cdot$ obeying the following axioms:

$$h\cdot(ab)=(h_1\cdot a)(h_2\cdot b),\quad h\cdot 1_B=\varepsilon(h)1_B,$$

for all $a,b\in B,h\in H$.

Lemma 3.3. *If there exists a right* (H,α)*-Hom-comodule algebra map* $\phi:H\to B$*, then* $(A,\triangleright,1_B)$ *is a left* (H,α)*-Hom-module algebra.*

Proof. It is the rephrasing of part of the result [9, Proposition 3.5] in the particular case of the category $\widetilde{\mathcal{H}}(\mathcal{M}_k)$. □

By the proof of Theorem 2.7, we know that the map $\Phi:A\otimes H\to B,a\otimes h\mapsto a\phi(h)$ provides an isomorphism of (H,α)-Hom-Hopf bimodules, with inverse $\Phi^{-1}:B\to A\otimes H,b\mapsto E(b_{(0)})\otimes b_{(1)}$. In what follows, we will show that Φ is a Hom-algebra map.

Definition 3.4. Let (H,α) be a monoidal Hom-bialgebra, (A,β) a left (H,α)-Hom-module algebra. The *Hom-smash product* [14, Definition 2.7] $(A\#H,\beta\otimes\alpha)$ of (A,β) and (H,α) is defined as follows, for all $a,b\in A,h,g\in H$:

(1) as a k-space, $A\#H=A\otimes H$;

(2) whose Hom-multiplication is given by

$$(a\#h)(b\#g) = a(h_1 \cdot \beta^{-1}(b))\#\alpha(h_2)g.$$

Note that $(A\#H, \beta\#\alpha)$ is a monoidal Hom-algebra with unit $1_A\#1_H$.

Lemma 3.5. *Assume that there exists a right (H, α)-Hom-comodule algebra map $\phi : H \to B$. Then*

$$\Phi : A\#H \to B, \quad a\#h \mapsto a\phi(h)$$

is a Hom-algebra map, where $(A\#H, \beta \otimes \alpha)$ is a Hom-smash product algebra.

Proof. For any $a \otimes h, b \otimes g \in A\#H$, we obtain

$$\begin{aligned}
& \Phi((a\#h)(b\#g)) \\
= \ & \Phi\Big(a\Big(h_1 \rhd \beta^{-1}(b)\Big)\#\alpha(h_2)g\Big) \\
= \ & \Big(a\Big(h_1 \rhd \beta^{-1}(b)\Big)\Big)\phi(\alpha(h_2)g) \\
= \ & \beta(a)\Big(\Big(h_1 \rhd \beta^{-1}(b)\Big)\Big(\phi(h_2)\phi(\alpha^{-1}(g))\Big)\Big) \\
= \ & \beta(a)\Big(\beta^{-1}\Big(\big(h_1 \rhd \beta^{-1}(b)\big)\beta\big(\phi(h_2)\big)\Big)\phi(g)\Big) \\
\overset{(7)}{=} \ & \beta(a)\Big(\beta^{-1}\Big(\phi(h)b\Big)\phi(g)\Big) \\
= \ & \beta(a)\Big(\phi(h)\Big(\beta^{-1}\big(b\phi(g)\big)\Big)\Big) \\
= \ & \Phi(a \otimes h)\Phi(b \otimes g),
\end{aligned}$$

and it is not difficult to verify that $\Phi(1_A \otimes 1_H) = 1_B$ and $\Phi \circ (\beta \otimes \alpha) = \beta \circ \Phi$. □

When $(A \otimes H, \beta \otimes \alpha)$ is a Hom-smash product, with the coaction in Lemma 2.6, we can check that $(A \otimes H, \beta \otimes \alpha)$ is a Hom-comodule algebra and Φ is a Hom-comodule map.

So, by Lemma 3.5 and Theorem 2.7, we obtain the main result:

Theorem 3.6. *There exists an isomorphism of Hom-comodule algebras as follows:*

$$(A\#H, \beta \otimes \alpha) \cong (B, \beta).$$

Here $(A\#H, \beta \otimes \alpha)$ is a Hom-smash product.

Lemma 3.5 and Theorem 3.6 are the rephrasing of [9, Proposition 3.5] in the particular case of the category $\widetilde{\mathcal{H}}(\mathcal{M}_k)$.

Example 3.7. Let (H,α) be a monoidal Hom-Hopf algebra, then $(H\otimes H,\alpha\otimes\alpha,\rho_{H\otimes H})$ is a Hom-comodule algebra, where comodule structure map is defined by $\rho_{H\otimes H}(h\otimes g)=\alpha^{-1}(h)\otimes g_1\otimes\alpha(g_2)$, for any $h,g\in H$. It is obvious that $\Delta: H\to H\otimes H$ is a Hom-comodule algebra map. By Theorem 3.6, there exists a Hom-algebra isomorphism:

$$(H\otimes H)^{coH}\# H\cong H\otimes H.$$

4. Hom-Hopf bicomodules

Definition 4.1. Let (C,β) be a monoidal Hom-coalgebra and (M,μ) a left and right (C,β)-Hom-comodule satisfying the condition

$$(m_{(0)(-1)}\otimes m_{(0)(0)})\otimes\beta^{-1}(m_{(1)})=\beta^{-1}(m_{(-1)})\otimes(m_{(0)(0)}\otimes m_{(0)(1)}) \tag{9}$$

for all $m\in M$. Then we call (M,μ) a *(C,β)-Hom-bicomodule.*

Definition 4.2. If (M,μ) is both a right (H,α)-Hom-module and a left (H,α)-Hom-comodule such that for any $m\in M, h\in H$,

$$\rho_M(m\cdot h)=m_{(-1)}h_1\otimes m_{(0)}\cdot h_2 \tag{10}$$

then M is called a *right, left (H,α)-Hom-Hopf module.*

Definition 4.3. Let (M,μ) be an (H,α)-Hom-bicomodule. An (H,α)-Hom-Hopf bicomodule (M,μ) is defined as *a right, left (H,α)-Hom-Hopf module* which is a right (H,α)-Hom-Hopf module as well.

In this section, we consider the structure theorem of Hom-Hopf bicomodules and we always assume that (H,α) is a monoidal Hom-Hopf algebra.

Assume that (M,μ) is an (H,α)-Hom-Hopf bicomodule, we denote the category of (H,α)-Hom-Hopf bicomodules by ${}^H\mathcal{M}^H_H$. Denote left, right Hom-comodule coaction by ρ^- and ρ^+, respectively.

Lemma 4.4. *Let $(M,\mu)\in{}^H\mathcal{M}^H_H$. Define the map*

$$\delta: M\to H\otimes M, m\mapsto\rho^-_M(E(m))=E(m)_{(-1)}\otimes E(m)_{(0)}.$$

Then $(id\otimes E)\delta(m)=\delta(m)$, for any $m\in M$.

Proof. For any $m\in M$, we have

$$\begin{aligned}\delta(m) &= E(m)_{(-1)}\otimes E(m)_{(0)}\\ &= (m_{(0)}\cdot S(m_{(1)}))_{(-1)}\otimes(m_{(0)}\cdot S(m_{(1)}))_{(0)}\\ &\overset{(10)}{=} m_{(0)(-1)}S(m_{(1)2})\otimes m_{(0)(0)}\cdot S(m_{(1)1}),\end{aligned}$$

so

$$\begin{aligned}
(id\otimes E)\delta(m) &= (id\otimes E)(m_{(0)(-1)}S(m_{(1)2})\otimes m_{(0)(0)}\cdot S(m_{(1)1}))\\
&\overset{(1)}{=} m_{(0)(-1)}S(m_{(1)2})\otimes \mu(E(m_{(0)(0)}))\varepsilon(m_{(1)1})\\
&= m_{(0)(-1)}\alpha^{-1}(S(m_{(1)}))\otimes \mu(E(m_{(0)(0)}))\\
&= m_{(0)(-1)}\alpha^{-1}(S(m_{(1)}))\otimes \mu(m_{(0)(0)(0)}\cdot S(m_{(0)(0)(1)}))\\
&\overset{(9)}{=} \alpha(m_{(0)(0)(-1)})\alpha^{-1}(S(m_{(1)}))\otimes \mu(m_{(0)(0)(0)}\cdot \alpha^{-1}(S(m_{(0)(1)})))\\
&= m_{(0)(-1)}S(m_{(1)2})\otimes m_{(0)(0)}\cdot S(m_{(1)1}).
\end{aligned}$$

So we can get $(id\otimes E)\delta(m)=\delta(m)$. □

Note that by the above proof we have

$$E(m)_{(-1)}\otimes E(m)_{(0)} = E(m)_{(-1)}\otimes E(E(m)_{(0)}) \tag{11}$$

and

$$E(m)_{(-1)}\otimes E(m)_{(0)} = m_{(0)(-1)}\alpha^{-1}(S(m_{(1)}))\otimes \mu(E(m_{(0)(0)})). \tag{12}$$

It is obvious that (H,α) is an (H,α)-Hom-Hopf bicomodule via its multiplication and comultiplication.

Let $H^+ = Ker\varepsilon$, $\overline{M} = M/M\cdot H^+$ and $(M\cdot H^+, \mu_{|M\cdot H^+})\in \widetilde{\mathcal{H}}(\mathcal{M}_k)$. Then μ induce an automorphism $\overline{\mu}$ of $\overline{M}$ and $(\overline{M},\overline{\mu})\in \widetilde{\mathcal{H}}(\mathcal{M}_k)$. Define $\pi:(M,\mu)\to(\overline{M},\overline{\mu})$, $m\mapsto \overline{m}$. It is easy to see that π is a morphism in the category $\widetilde{\mathcal{H}}(\mathcal{M}_k)$.

Proposition 4.5. *The following conclusions hold:*

(1) *For any $\overline{m}\in\overline{M}, h\in H$,*

$$\overline{m\cdot h} = \overline{\mu(m)}\varepsilon(h). \tag{13}$$

(2) $M\cdot H^+\subseteq KerE$.

(3) *Define* $\overline{\rho}^-:\overline{M}\to H\otimes\overline{M}$, $\overline{m}\mapsto \overline{m}_{(-1)}\otimes\overline{m}_{(0)} = E(m)_{(-1)}\otimes\pi(E(m)_{(0)})$. *Then* $(\overline{M},\overline{\mu},\overline{\rho}^-)$ *is a left* (H,α)*-Hom-comodule.*

(4) *Define* $\overline{\rho}^+:\overline{M}\to\overline{M}\otimes H$, $\overline{m}\mapsto\overline{m}_{(0)}\otimes\overline{m}_{(1)} = \pi(E(m)_{(0)})\otimes E(m)_{(1)} = \pi(\mu^{-1}(E(m)))\otimes 1_H$. *Then* $(\overline{M},\overline{\mu},\overline{\rho}^+)$ *is a right* (H,α)*-Hom-comodule.*

(5) $(\overline{M},\overline{\mu},\overline{\rho}^+,\overline{\rho}^-)$ *is an* (H,α)*-Hom-bicomodule.*

Proof. (1) Since $\varepsilon(h-\varepsilon(h)1_H)=0$, $h-\varepsilon(h)1_H\in Ker\varepsilon = H^+$.

For any $h\in H, m\in M$, $m\cdot(h-\varepsilon(h)1_H)\in M\cdot H^+$, so $m\cdot h-\mu(m)\varepsilon(h)\in M\cdot H^+$ and hence $\overline{m\cdot h}=\overline{\mu(m)\varepsilon(h)}=\overline{\mu(m)}\varepsilon(h)$.

(2) For any $h\in H^+, m\in M$, we have $E(m\cdot h)\overset{(1)}{=}\varepsilon(h)\mu(E(m))=0$, so $M\cdot H^+\subseteq KerE$.

(3) Firstly, it is easy to see that $\overline{\rho}^-$ is well-defined. Next, we will prove that $(\overline{M}, \overline{\rho}^-)$ is a left (H,α)-Hom-comodule.

As a matter of fact, for any $\overline{m} \in \overline{M}$,

$$\begin{aligned}
(\varepsilon \otimes id)\overline{\rho}^-(\overline{m}) &= (\varepsilon \otimes id)(E(m)_{(-1)} \otimes \pi(E(m)_{(0)}))\\
&= \varepsilon(E(m)_{(-1)})\pi(E(m)_{(0)}) = \pi(\mu^{-1}(E(m)))\\
&= \overline{\mu}^{-1}(\pi(E(m))) = \overline{\mu}^{-1}(\overline{E(m)}) = \overline{\mu}^{-1}(\overline{m_{(0)} \cdot S(m_{(1)})})\\
&\overset{(13)}{=} \overline{\mu}^{-1}(\overline{\mu(m_{(0)})})\varepsilon(S(m_{(1)})) = \overline{\mu}^{-1}(\overline{m}).
\end{aligned}$$

Note that by the above proof we have

$$\pi(E(m)) = \overline{E(m)} = \overline{m}. \tag{14}$$

$$\begin{aligned}
(\alpha^{-1} \otimes \overline{\rho}^-)\overline{\rho}^-(\overline{m}) &= \alpha^{-1}(E(m)_{(-1)}) \otimes \overline{\rho}^-\pi(E(m)_{(0)})\\
&= \alpha^{-1}(E(m)_{(-1)}) \otimes \overline{\rho}^-\overline{E(m)_{(0)}}\\
&= \alpha^{-1}(E(m)_{(-1)}) \otimes (id \otimes \pi)\rho^-_M(E(E(m)_{(0)}))\\
&\overset{(11)}{=} \alpha^{-1}(E(m)_{(-1)}) \otimes (id \otimes \pi)\rho^-_M(E(m)_{(0)})\\
&= \alpha^{-1}(E(m)_{(-1)}) \otimes (id \otimes \pi)(E(m)_{(0)(-1)} \otimes E(m)_{(0)(0)})\\
&= E(m)_{(-1)1} \otimes (id \otimes \pi)(E(m)_{(-1)2} \otimes \mu^{-1}(E(m)_{(0)}))\\
&= E(m)_{(-1)1} \otimes E(m)_{(-1)2} \otimes \pi\mu^{-1}(E(m)_{(0)})\\
&= E(m)_{(-1)1} \otimes E(m)_{(-1)2} \otimes \overline{\mu}^{-1}\pi(E(m)_{(0)})\\
&= (\Delta \otimes \overline{\mu}^{-1})\overline{\rho}^-(\overline{m}).
\end{aligned}$$

It is easy to check that $\overline{\rho}^-(\overline{\mu}(\overline{m})) = \alpha(\overline{m}_{(-1)}) \otimes \overline{\mu}(\overline{m}_{(0)})$.

(4) In fact the comodule coaction $\overline{\rho}^+$ is trivial by eq.(14). Hence $(\overline{M}, \overline{\mu}, \overline{\rho}^+)$ is a right (H,α)-Hom-comodule.

(5) By (3) and (4), we can check that $\overline{m}_{(0)(-1)} \otimes \overline{m}_{(0)(0)} \otimes \alpha^{-1}(\overline{m}_{(1)}) = \alpha^{-1}(\overline{m}_{(-1)}) \otimes (\overline{m}_{(0)(0)} \otimes \overline{m}_{(0)(1)})$. Hence $(\overline{M}, \overline{\mu}, \overline{\rho}^+, \overline{\rho}^-)$ is an (H,α)-Hom-bicomodule. □

Lemma 4.6. *Let (M,μ) be an (H,α)-Hom-Hopf bicomodule. Then $(\overline{M} \otimes H, \rho^-_{\overline{M}\otimes H}, \rho^+_{\overline{M}\otimes H}, \overline{\varphi})$ is an (H,α)-Hom-Hopf bicomodule via the following action and coactions:*

$\overline{\varphi} : (\overline{M} \otimes H) \otimes H \to \overline{M} \otimes H, \quad (\overline{m} \otimes h) \otimes g \mapsto \overline{\mu}(\overline{m}) \otimes h\alpha^{-1}(g);$

$\rho^+_{\overline{M}\otimes H} : \overline{M} \otimes H \to (\overline{M} \otimes H) \otimes H, \quad \overline{m} \otimes h \mapsto (\overline{\mu}^{-1}(\overline{m}) \otimes h_1) \otimes \alpha(h_2);$

$\rho^-_{\overline{M}\otimes H} : \overline{M} \otimes H \to H \otimes (\overline{M} \otimes H), \quad \overline{m} \otimes h \mapsto E(m)_{(-1)}h_1 \otimes \pi(E(m)_{(0)}) \otimes h_2.$

Theorem 4.7. *Let $(M,\mu) \in {}^H\mathcal{M}_H^H$. By Lemma 4.6 we know that $\overline{M} \otimes H$ is an (H,α)-Hom-Hopf bicomodule, then we have $(\overline{M} \otimes H, \overline{\mu} \otimes \alpha) \cong (M,\mu)$ as (H,α)-Hom-Hopf bicomodules.*

Proof. We define two maps as follows:

$$\lambda : \overline{M} \otimes H \to M, \quad \overline{m} \otimes h \mapsto E(m) \cdot h,$$

$$\theta : M \to \overline{M} \otimes H, \quad m \mapsto \overline{m_{(0)}} \otimes m_{(1)}.$$

λ and θ are well-defined mutually inverse isomorphism of (H,α)-Hom-Hopf bicomodules.

The theorem is a dual form of [1, Proposition 3.6] in the particular case of $\widetilde{\mathcal{H}}(\mathcal{M}_k)$. □

5. The structure theorem of Hom-module coalgebras

In this section, we give the structure theorem of Hom-module coalgebras as an application of Section 4.

Definition 5.1. [14, Definition 3.4] Let (H,α) be a monoidal Hom-bialgebra. A monoidal Hom-coalgebra (C,β) is called a *right* (H,α)*-Hom-module coalgebra* if (C,β) is a right (H,α)-Hom-module with action $\cdot$ obeying the following axioms:

$$\Delta(c \cdot h) = c_1 \cdot h_1 \otimes c_2 \cdot h_2, \ \varepsilon(c \cdot h) = \varepsilon(c)\varepsilon(h),$$

for all $c \in C, h \in H$.

In this section, (H,α) is always considered to be a monoidal Hom-Hopf algebra with antipode S, and (C,β) a right (H,α)-Hom-module coalgebra. Then $C \cdot H^+$ is a Hom-coideal [3, Section 7] of C, since $H^+ = Ker\varepsilon$ is a Hom-coideal of (H,α), and

$$\Delta(C \cdot H^+) = \Delta(C)\Delta(H^+) \subseteq C \otimes C \cdot H^+ + C \cdot H^+ \otimes C, \varepsilon(C \cdot H^+) = 0.$$

Hence $\overline{C} = (C/C \cdot H^+, \overline{\beta})$ is a Hom-coalgebra such that $\pi : C \to \overline{C}$ is a Hom-coalgebra map [3, Section 7].

Suppose that there exists a right (H,α)-Hom-module coalgebra map $\phi : C \to H$, where (H,α) is a right (H,α)-Hom-module coalgebra via its multiplication.

Define two (H,α)-coactions on (C,β) as follows:

$$\rho_C^- : C \to H \otimes C, c \mapsto \phi(c_1) \otimes c_2;$$

$$\rho_C^+ : C \to C \otimes H, c \mapsto c_1 \otimes \phi(c_2).$$

Then one easily checks that $(C, \rho_C^-, \rho_C^+, \cdot)$ is an (H,α)-Hom-Hopf bicomodule.

Definition 5.2. [14, Definition 3.1] Let (H,α) be a monoidal Hom-bialgebra. A monoidal Hom-coalgebra (C,β) is called a *left* (H,α)*-Hom-comodule coalgebra* if (C,β) is a left (H,α)-Hom-comodule with coaction $\rho(c)=c_{(-1)}\otimes c_{(0)}$ obeying the following axioms:

$$c_{(-1)}\otimes \Delta(c_{(0)})=c_{1(-1)}c_{2(-1)}\otimes c_{1(0)}\otimes c_{2(0)},\ c_{(-1)}\varepsilon(c_{(0)})=\varepsilon(c)1_H,$$

for all $c\in C$.

Lemma 5.3. *If there exists a right (H,α)-Hom-module coalgebra map $\phi: C\to H$, then*

(1) $\phi(E(c))=\varepsilon(c)1_H$.
(2) $(\overline{C},\overline{\beta},\overline{\rho}^-,\overline{\Delta},\overline{\varepsilon})$ *is a left (H,α)-Hom-comodule coalgebra, where $(\overline{C},\overline{\beta})$ is a monoidal Hom-coalgebra together with comultiplication $\overline{\Delta}$ and counit $\overline{\varepsilon}$ of $\overline{C}$, and its comodule structure map $\overline{\rho}^-$ is given by Proposition 4.5.*

Proof. (1) For any $c\in C$, we have

$$\begin{aligned}\phi(E(c)) &= \phi(c_{(0)}\cdot S(c_{(1)}))=\phi(c_{(0)})S(c_{(1)})\\ &= \phi(c_1)S(\phi(c_2))=\phi(c)_1S(\phi(c)_2)\\ &= \varepsilon(\phi(c))1_H=\varepsilon(c)1_H.\end{aligned}$$

(2) By Proposition 4.5(3), $\overline{C}$ is a left (H,α)-Hom-comodule. To see that $(\overline{C},\overline{\beta},\overline{\Delta},\overline{\varepsilon})$ is a left (H,α)-Hom-comodule coalgebra, we have only to prove that: $\phi(E(c_1)_1)$ $\phi(E(c_2)_1)\otimes\pi(E(c_1)_2)\otimes\pi(E(c_2)_2)=(id\otimes\overline{\Delta})\overline{\rho}^-(\overline{c})$, and $(id\otimes\overline{\varepsilon})\overline{\rho}^-(\overline{c})=\overline{\varepsilon}(\overline{c})1_H$, for any $c\in C,\overline{c}\in\overline{C}$.

As a matter of fact, we have

$$\begin{aligned}&\phi(E(c_1)_1)\phi(E(c_2)_1)\otimes\pi(E(c_1)_2)\otimes\pi(E(c_2)_2)\\ =\ &\phi\Big(\big(c_{1(0)}\cdot S(c_{1(1)})\big)_1\Big)\phi\Big(\big(c_{2(0)}\cdot S(c_{2(1)})\big)_1\Big)\otimes\pi\Big(\big(c_{1(0)}\cdot S(c_{1(1)})\big)_2\Big)\\ &\otimes\pi\Big(\big(c_{2(0)}\cdot S(c_{2(1)})\big)_2\Big)\\ =\ &\phi\Big(\big(c_{11}\cdot S(\phi(c_{12}))\big)_1\Big)\phi\Big(\big(c_{21}\cdot S(\phi(c_{22}))\big)_1\Big)\otimes\pi\Big(\big(c_{11}\cdot S(\phi(c_{12}))\big)_2\Big)\\ &\otimes\pi\Big(\big(c_{21}\cdot S(\phi(c_{22}))\big)_2\Big)\\ =\ &\phi\Big(c_{111}\cdot S\big(\phi(c_{122})\big)\Big)\phi\Big(c_{211}\cdot S\big(\phi(c_{222})\big)\Big)\otimes\pi\Big(c_{112}\cdot S\big(\phi(c_{121})\big)\Big)\\ &\otimes\pi\Big(c_{212}\cdot S\big(\phi(c_{221})\big)\Big)\\ =\ &\Big(\phi(c_{111})S\big(\phi(c_{122})\big)\Big)\Big(\phi(c_{211})S\big(\phi(c_{222})\big)\Big)\otimes\overline{c_{112}\cdot S(\phi(c_{121}))}\\ &\otimes\overline{c_{212}\cdot S(\phi(c_{221}))}\end{aligned}$$

$$
\begin{aligned}
&\stackrel{(13)}{=} \Big(\phi(c_{111})S\big(\phi(c_{122})\big)\Big)\Big(\phi(c_{211})S\big(\phi(c_{222})\big)\Big)\otimes\overline{\beta(c_{112})}\varepsilon(c_{121})\otimes\overline{\beta(c_{212})}\varepsilon(c_{221})\\
&= \bigg(\phi(c_{111})S\Big(\phi\big(\beta^{-1}(c_{12})\big)\Big)\bigg)\bigg(\phi(c_{211})S\Big(\phi\big(\beta^{-1}(c_{22})\big)\Big)\bigg)\otimes\overline{\beta(c_{112})}\otimes\overline{\beta(c_{212})}\\
&= \bigg(\phi(c_{111})S\Big(\phi\big(c_{121}\big)\Big)\bigg)\bigg(\phi(\beta(c_{1221}))S\Big(\phi\big(\beta^{-2}(c_{2})\big)\Big)\bigg)\otimes\overline{\beta(c_{112})}\otimes\overline{\beta^{2}(c_{1222})}\\
&= \bigg(\phi(c_{111})S\Big(\phi\big(\beta(c_{1211})\big)\Big)\bigg)\bigg(\phi(\beta(c_{1212}))S\Big(\phi\big(\beta^{-2}(c_{2})\big)\Big)\bigg)\otimes\overline{\beta(c_{112})}\\
&\quad\otimes\overline{\beta(c_{122})}\\
&= \alpha(\phi(c_{111}))\Bigg(\alpha^{-1}\bigg(S\Big(\phi(\beta(c_{1211}))\Big)\phi\Big(\beta(c_{1221})\Big)\bigg)S\Big(\phi\big(\beta^{-2}(c_{2})\big)\Big)\Bigg)\otimes\overline{\beta(c_{112})}\\
&\quad\otimes\overline{\beta(c_{122})}\\
&= \alpha(\phi(c_{111}))S(\phi(\beta^{-1}(c_{2})))\otimes\overline{\beta(c_{112})}\otimes\overline{c_{12}}\\
&= \phi(\beta(c_{111}))S(\phi(\beta^{-1}(c_{2})))\otimes\overline{\beta(c_{112})}\otimes\overline{c_{12}}\\
&= \phi\bigg(\beta(c_{111})\cdot S\Big(\phi\big(\beta^{-1}(c_{2})\big)\Big)\bigg)\otimes\overline{\beta(c_{112})}\otimes\overline{c_{12}}\\
&= \phi\bigg(c_{11}\cdot S\Big(\phi\big(\beta^{-1}(c_{2})\big)\Big)\bigg)\otimes\overline{\beta(c_{121})}\otimes\overline{\beta(c_{122})}\\
&= \phi\bigg(c_{11}\cdot S\Big(\phi\big(\beta^{-1}(c_{2})\big)\Big)\bigg)\otimes\overline{\Delta}(\overline{\beta(c_{12})})\\
&= \phi\bigg(c_{11}\cdot S\Big(\phi(c_{22})\Big)\bigg)\otimes\overline{\Delta}(\overline{\beta(c_{12})}\varepsilon(c_{21}))\\
&\stackrel{(13)}{=} (id\otimes\overline{\Delta})(\phi(c_{11}\cdot S(\phi(c_{22})))\otimes\overline{c_{12}\cdot S(\phi(c_{21}))})\\
&= (id\otimes\overline{\Delta})(\phi(E(c)_1)\otimes\overline{E(c)_2})\\
&= (id\otimes\overline{\Delta})\overline{\rho}^{-}(\overline{c}).
\end{aligned}
$$

So we can get $\phi(E(c_1)_1)\phi(E(c_2)_1)\otimes\pi(E(c_1)_2)\otimes\pi(E(c_2)_2)=(id\otimes\overline{\Delta})\overline{\rho}^{-}(\overline{c})$. It is easy to check that $(id\otimes\overline{\varepsilon})\overline{\rho}^{-}(\overline{c})=\overline{\varepsilon}(\overline{c})1_H$. □

By Theorem 4.7, we know that the map

$$\Phi: C\to\overline{C}\otimes H, c\mapsto\overline{c_1}\otimes\phi(c_2)$$

is a bijection with inverse

$$\Phi^{-1}:\overline{C}\otimes H\to C, \overline{c}\otimes h\mapsto E(c)\cdot h.$$

Definition 5.4. Let (H,α) be a monoidal Hom-bialgebra, (C,β) a left (H,α)-Hom-comodule coalgebra. The *Hom-smash coproduct* [14, Proposition 3.2] $(C\times H,\beta\otimes\alpha)$ of (C,β) and (H,α) is defined as follows, for all $c\in C, h\in H$:

(1) as a k-space, $C \times H = C \otimes H$;
(2) whose Hom-comultiplication is given by

$$\Delta(c \times h) = c_1 \otimes c_{2(-1)}\alpha^{-1}(h_1) \otimes \beta(c_{2(0)}) \otimes h_2.$$

Note that $(C \times H, \beta \otimes \alpha)$ is a monoidal Hom-coalgebra with counit $\varepsilon_{C\times H}$.

Lemma 5.5. *If there exists a right (H, α)-Hom-module coalgebra map $\phi : C \to H$. Then the map $\Phi : C \to \overline{C} \times H, c \mapsto \overline{c_1} \times \phi(c_2)$ is a Hom-coalgebra map, where $(\overline{C} \times H, \overline{\beta} \otimes \alpha)$ is a Hom-smash coproduct.*

Proof. For any $c \in C$, we have

$$\begin{aligned}
& \Delta\Phi(c) = \Delta(\overline{c_1} \otimes \phi(c_2)) \\
= \;& \overline{c_{11}} \otimes \overline{c_{12}}_{(-1)}\alpha^{-1}\Big(\phi(c_2)_1\Big) \otimes \overline{\beta}\Big(\overline{c_{12}}_{(0)}\Big) \otimes \phi(c_2)_2 \\
= \;& \overline{c_{11}} \otimes \phi\Big(E(c_{12})_1\Big)\alpha^{-1}\Big(\phi(c_{21})\Big) \otimes \overline{\beta}\Big(\overline{E(c_{12})_2}\Big) \otimes \phi(c_{22}) \\
= \;& \overline{c_{11}} \otimes \phi\bigg(\Big(c_{121} \cdot S\Big(\phi(c_{122})\Big)\Big)_1\bigg)\alpha^{-1}(\phi(c_{21})) \otimes \overline{\beta}\bigg(\overline{\Big(c_{121} \cdot S\Big(\phi(c_{122})\Big)\Big)_2}\bigg) \\
& \otimes\phi(c_{22}) \\
= \;& \overline{c_{11}} \otimes \phi\Big(c_{1211} \cdot S\Big(\phi(c_{1222})\Big)\Big)\alpha^{-1}(\phi(c_{21})) \otimes \overline{\beta}\Big(\overline{c_{1212} \cdot S\Big(\phi(c_{1221})\Big)}\Big) \\
& \otimes\phi(c_{22}) \\
= \;& \overline{c_{11}} \otimes \Big(\phi(c_{1211})S\Big(\phi(c_{1222})\Big)\Big)\alpha^{-1}(\phi(c_{21})) \otimes \overline{\beta}\Big(\overline{c_{1212} \cdot S\Big(\phi(c_{1221})\Big)}\Big) \\
& \otimes\phi(c_{22}) \\
= \;& \overline{c_{11}} \otimes \alpha(\phi(c_{1211}))\Big(S\Big(\phi(c_{1222})\Big)\alpha^{-2}\Big(\phi(c_{21})\Big)\Big) \otimes \overline{\beta}\Big(\overline{c_{1212} \cdot S\Big(\phi(c_{1221})\Big)}\Big) \\
& \otimes\phi(c_{22}) \\
= \;& \overline{c_{11}} \otimes \alpha^2\Big(\phi(c_{12111})\Big)\Big(S\Big(\phi(c_{1221})\Big)\phi(c_{1222})\Big) \otimes \overline{\beta}\Big(\overline{\beta(c_{12112}) \cdot S\Big(\phi(c_{1212})\Big)}\Big) \\
& \otimes\alpha^{-1}(\phi(c_2)) \\
\overset{(13)}{=} \;& \overline{c_{11}} \otimes \alpha^2\Big(\phi(c_{12111})\Big)\varepsilon(c_{122})1_H \otimes \overline{\beta}\Big(\overline{\beta^2(c_{12112})}\varepsilon(c_{1212})\Big) \otimes \alpha^{-1}\Big(\phi(c_2)\Big) \\
= \;& \overline{c_{11}} \otimes \alpha^2\Big(\phi(c_{1211})\Big)\varepsilon(c_{122}) \otimes \overline{\beta}\Big(\overline{\beta^2(c_{12121})}\Big)\varepsilon(c_{12122}) \otimes \alpha^{-1}\Big(\phi(c_2)\Big) \\
= \;& \overline{c_{11}} \otimes \alpha^2\Big(\phi(c_{1211})\Big)\varepsilon(c_{122}) \otimes \overline{\beta^2(c_{1212})} \otimes \alpha^{-1}\Big(\phi(c_2)\Big) \\
= \;& \overline{c_{11}} \otimes \alpha\Big(\phi(c_{121})\Big) \otimes \overline{\beta(c_{122})} \otimes \alpha^{-1}\Big(\phi(c_2)\Big) \\
= \;& \overline{c_{11}} \otimes \phi(c_{12}) \otimes \overline{c_{21}} \otimes \phi(c_{22})
\end{aligned}$$

$$
\begin{aligned}
&= (\Phi \otimes \Phi)(c_1 \otimes c_2) \\
&= (\Phi \otimes \Phi)\Delta(c).
\end{aligned}
$$

$\varepsilon_{\overline{C}\otimes H}\Phi(c) = \varepsilon_{\overline{C}\otimes H}(\overline{c}_1 \times \phi(c_2)) = \varepsilon(c_1)\varepsilon(c_2) = \varepsilon(c)$, and it is easy to check that $\Phi \circ \beta = (\overline{\beta} \otimes \alpha) \circ \Phi$. □

Remark 5.6. Let (H, α) be a monoidal Hom-Hopf algebra and (C, β) a left Hom-comodule coalgebra. Then the Hom-smash coproduct $(C \times H, \Delta, \varepsilon)$ is a right (H, α)-Hom-module coalgebra, where the right (H, α)-action on $C \times H$ is given by $\varphi : C \times H \otimes H \to C \times H, c \times h \otimes g \mapsto \beta(c) \otimes h\alpha^{-1}(g)$. Furthermore, we can check that $\Phi : C \to \overline{C} \times H, c \mapsto \overline{c_1} \times \phi(c_2)$ is a Hom-module coalgebra map.

So, by Lemma 5.5 and Remark 5.6, we obtain the main result:

Theorem 5.7. *There exists an isomorphism of Hom-module coalgebras as follows:*

$$(\overline{C} \times H, \overline{\beta} \otimes \alpha) \cong (C, \beta),$$

where $(\overline{C} \times H, \overline{\beta} \otimes \alpha)$ *is a Hom-smash coproduct.*

Example 5.8. Let D be a right Hom-ideal subcoalgebra of (H, α) (that is D is a right Hom-ideal and also a subcoalgebra of (H, α)). Then D is a right (H, α)-Hom-module coalgebra, and there exists a right(H, α)-Hom-module coalgebra map $\phi : D \hookrightarrow H$. By Theorem 5.7, there exists a Hom-coalgebra isomorphism:

$$\overline{D} \times H \cong D.$$

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