ANNIHILATOR CONDITIONS WITH GENERALIZED SKEW DERIVATIONS AND LIE IDEALS OF PRIME RINGS

Vincenzo De Filippis, Nadeem ur Rehman and Giovanni Scudo

Received: 2 September 2021; Revised: 5 April 2022; Accepted: 5 April 2022

Communicated by Tuğçe Pekacar Çalıç

Abstract. Let \( R \) be a prime ring, \( Q \) its right Martindale quotient ring, \( L \) a non-central Lie ideal of \( R \), \( n \geq 1 \) a fixed integer, \( F \) and \( G \) two generalized skew derivations of \( R \) with the same associated automorphism, \( p \in R \) a fixed element. If \( p(F(x)F(y) - G(y)x)^n = 0 \), for any \( x, y \in L \), then there exist \( a, c \in Q \) such that \( F(x) = ax \) and \( G(x) = cx \), for any \( x \in R \), with \( pa = pc = 0 \), unless when \( R \) satisfies the standard polynomial identity \( s_4(x_1, \ldots, x_4) \).

Mathematics Subject Classification (2020): 16W25, 16N60

Keywords: Generalized skew derivation, prime ring

1. Introduction

This work is devoted to consider some related problems concerning annihilators of power values of some appropriate identities involving additive maps in prime rings. Throughout this paper \( R \) always denotes a prime ring, \( Z(R) \) the center of \( R \), \( Q \) the right Martindale quotient ring of \( R \) and \( C = Z(Q) \), the center of \( Q \), \( (C \) is usually called the extended centroid of \( R \). We introduce on \( R \) an additive mapping \( d \) which satisfies the following rule:

\[
d(xy) = d(x)y + \alpha(x)d(y)
\]

for all \( x, y \in R \). The map \( d \) is said to be a skew derivation of \( R \) and \( \alpha \) is called the associated automorphism of \( d \). Consequently, let us also define the concept of a generalized skew derivation \( F \) of \( R \), that is an additive mapping \( F \) such that

\[
F(xy) = F(x)y + \alpha(x)d(y)
\]

for all \( x, y \in R \), where \( d \) is a skew derivation of \( R \) and \( \alpha \) is the associated automorphism of \( d \). The map \( d \) is called an associated skew derivation of \( F \). The automorphism \( \alpha \) is called the associated automorphism of \( F \).

Nilpotent values of skew derivations and generalized skew derivations of prime rings were recently studied by several authors.
In [2], J.-C. Chang shows that if \( F \) is a generalized skew derivation of \( R \), \( L \) is a non-commutative Lie ideal of \( R \) and \( n \geq 1 \) a fixed integer such that \( F(x)^n = 0 \), for all \( x \in L \), then \( F(x) = 0 \), for all \( x \in R \). Later, in [20], a generalization of the previous cited result involving an annihilator condition is given. More precisely, the main result in [20] proves that if \( F \) is a generalized skew derivation of \( R \), \( L \) is a non-commutative Lie ideal of \( R \), \( n \geq 1 \) a fixed integer and \( a \in R \) is a fixed element such that \( aF(x)^n = 0 \), for all \( x \in L \), then \( aF(x) = 0 \), for all \( x \in R \), unless \( R \) satisfies the standard identity \( s_4 \).

This last result has recently been further improved as follows: let \( 0 \neq p \) be an element of \( R \), \( F \) and \( G \) generalized skew derivations with the same associated skew derivation \( d \) of a prime ring \( R \), \( L \) a non-commutative Lie ideal of \( R \), \( l_1, \ldots, l_k, n \) nonnegative integers with \( l_1 \neq 0 \) and \( n > 0 \). If

\[
p \left( F(u)^{l_1}G(u)^{l_2}F(u)^{l_3}G(u)^{l_4} \cdots G(u)^{l_k} \right)^n = 0 \quad \forall u \in L,
\]

then \( d = 0 \) and there exist \( a, c \in Q_r \) such that \( F(x) = ax \) and \( G(x) = cx \), for any \( x \in R \). Moreover either \( pa = 0 \) or \( c = 0 \), unless \( R \) satisfies \( s_4 \) (see [14, Main Theorem]).

Further nil-power conditions have been investigated in another recent paper (see [19]) and the following result was proved: If \( R \) is a prime ring, \( F \) is a generalized skew derivation of \( R \), \( L \) a non-commutative Lie ideal of \( R \) and \( n \geq 1 \) is a fixed integer such that \( (F(x)F(y) - yx)^n = 0 \), for any \( x, y \in L \), then \( \text{char}(R) = 2 \) and \( R \subseteq M_2(C) \), the \( 2 \times 2 \) matrix ring over \( C \).

Following this line of investigation, the aim of this paper is to generalize the result in [19] to the case when two different generalized skew derivations act on the non-central Lie ideal \( L \), also introducing an annihilating condition. To be more precise, we will prove the following:

**Theorem 1.1.** Let \( R \) be a prime ring, \( Q_r \) its right Martindale quotient ring, \( L \) a non-central Lie ideal of \( R \), \( n \geq 1 \) a fixed integer, \( F \) and \( G \) two generalized skew derivations of \( R \) with the same associated automorphism, \( p \in R \) a fixed element. If \( p(F(x)F(y) - G(y)x)^n = 0 \), for any \( x, y \in L \), then there exist \( a, c \in Q_r \) such that \( F(x) = ax \) and \( G(x) = cx \), for any \( x \in R \), with \( pa = pc = 0 \), unless when \( R \) satisfies the standard polynomial identity \( s_4(x_1, \ldots, x_4) \).

Let us recall some well known results and notations which will be useful in the sequel.

We will denote by \( SDer(Q_r) \) the set of all skew-derivations of \( Q_r \) and by \( SD_{\text{int}}(Q_r) \) the \( C \)-subspace of \( SDer(Q_r) \) consisting of all inner skew-derivations of \( Q_r \).
Two different skew derivations \(d\) and \(\delta\) are said to be \(C\)-linearly dependent modulo \(SD_{\text{int}}\), if there exist \(\lambda, \mu \in C\), \(a \in Q_r\) and \(\alpha \in Aut(Q)\) such that \(\lambda d(x) + \mu \delta(x) = ax - \alpha(x)a\) for all \(x \in R\).

If \(d\) and \(\delta\) are \(C\)-linearly independent skew derivations modulo \(SD_{\text{int}}\), associated with the same automorphism \(\alpha\), such that \(\Phi(x_i, d(x_j), \delta(x_k))\) is a skew-differential identity on \(R\), then \(\Phi(x_i, y_j, z_k)\) is a generalized polynomial identity of \(R\), where \(x_i, y_j, z_k\) are distinct indeterminates (it follows from main results in [4,5,6]).

It is known that, if \(I\) is a two-sided ideal \(I\) of \(R\), then \(I, R, \) and \(Q_r\) satisfy the same generalized polynomial identities with coefficients in \(Q_r\) (see [3]). Furthermore, \(I, R, \) and \(Q_r\) satisfy the same generalized polynomial identities with automorphisms (see [5, Theorem 1]).

2. The result for inner generalized derivations

We start by proving the main theorem in case both \(F\) and \(G\) are generalized inner derivations of \(R\) and \([R,R] \subseteq L\). In this sense we assume that there are \(a, b, c, q \in Q_r\) such that \(F(x) = ax + xb\) and \(G(x) = cx + xq\), for any \(x \in R\). Hence, by our assumption, \(R\) satisfies the generalized polynomial identity

\[
\Psi(x_1, x_2, y_1, y_2) = p\left\{ (a[x_1, x_2] + [x_1, x_2]b)(a[y_1, y_2] + [y_1, y_2]b) - (c[y_1, y_2] + [y_1, y_2]q)[x_1, x_2] \right\}^n \tag{1}
\]

For brevity we denote \(X = [x_1, x_2], Y = [y_1, y_2]\) and

\[
\Psi(X, Y) = p\left\{ (aX + Xb)(aY + Yb) - (cY + Yq)X \right\}^n \tag{2}
\]

Lemma 2.1. Assume \(p \neq 0\). Either \(\Psi(X, Y)\) is a non-trivial generalized polynomial identity for \(R\) or \(b, q \in C\) with \(p(a + b) = p(c + q) = 0\).

Proof. Assume that \(\Psi(X, Y)\) is a trivial generalized polynomial identity for \(R\). Let \(T = Q_r \ast_C C\{X\}\) be the free product over \(C\) of the \(C\)-algebra \(Q_r\) and the free \(C\)-algebra \(C\{X\}\), with \(X\) the set consisting of non-commuting indeterminates \(x_1, x_2, y_1, y_2\).

Now consider the generalized polynomial \(\Psi(X, Y) \in Q_r \ast_C C\{X\}\).
By our hypothesis,

\[
\Psi(X,Y) = p\left\{ (aX + Xb)(aY + Yb) - (cY + Yq)X \right\}^n
\]

\[
= p\left\{ (aX + Xb)(aY + Yb) - (cY + Yq)X \right\}^{n-1} \cdot \left\{ (aX + Xb)aY - (cY + Yq)X + (aX + Xb)Yb \right\}
\]

\[
= 0 \in T. \tag{3}
\]

Suppose firstly \( b \not\in C \), that is \( \{b,1\} \) is linearly \( C \)-independent. Therefore, since \( \Psi(X,Y) = 0 \in T \),

\[
p\left\{ (aX + Xb)(aY + Yb) - (cY + Yq)X \right\}^{n-1} \cdot (aX + Xb)Yb = 0 \in T
\]

implying

\[
p\left\{ (aX + Xb)(aY + Yb) - (cY + Yq)X \right\}^{n-1} \cdot (aX + Xb) = 0 \in T
\]

that is

\[
p\left\{ (aX + Xb)(aY + Yb) - (cY + Yq)X \right\}^{n-1} \cdot Xb = 0 \in T.
\]

Thus

\[
p\left\{ (aX + Xb)(aY + Yb) - (cY + Yq)X \right\}^{n-1} = 0 \in T.
\]

Continuing this process, we get

\[
p\left\{ (aX + Xb)(aY + Yb) - (cY + Yq)X \right\} = 0 \in T
\]

which means that

\[
p(aX + Xb)Yb = 0 \in T.
\]

Hence the contradiction \( pXb = 0 \) follows. Thus \( \{b,1\} \) is linearly \( C \)-dependent, that is \( b \in C \).

Analogously, by (3) and \( a' = a + b \), it follows that

\[
p\left\{ a'Xa'Y - (cY + Yq)X \right\}^{n-1} \cdot (a'Xa')Y -
\]

\[
p\left\{ a'Xa'Y - (cY + Yq)X \right\}^{n-1} \cdot (cY + Yq)X = 0 \in T
\]

that is, both

\[
p\left\{ a'Xa'Y - (cY + Yq)X \right\}^{n-1} \cdot (a'Xa')Y = 0 \in T
\]
and
\[ p\left\{ a'Xa'Y - (cY + Yq)X \right\}^{n-1} \cdot (cY + Yq)X = 0 \in T. \]  \hspace{1cm} (4)

In particular, (4) implies
\[ p\left\{ a'Xa'Y - (cY + Yq)X \right\}^{n-1} \cdot (cY + Yq) = 0 \in T. \]

Hence, if we suppose \( q \notin C \), it follows that
\[ p\left\{ a'Xa'Y - (cY + Yq)X \right\}^{n-1} \cdot Yq = 0 \in T \]
which implies
\[ p\left\{ a'Xa'Y - (cY + Yq)X \right\}^{n-1} = 0 \in T. \]

As above, continuing this process we get
\[ p\left\{ a'Xa'Y - (cY + Yq)X \right\} = 0 \in T \]
and arrive at the contradiction \( pYq = 0. \)

Therefore \( q \in C \) and, for \( c' = c + q \), we write relation (3) as follows
\[ p\left\{ a'Xa'Y - c'YX \right\} \cdot \left\{ a'Xa'Y - c'YX \right\}^{n-1} = 0 \in T. \]  \hspace{1cm} (5)

It is easy to see that if either \( pa' = 0 \) or \( pc' = 0 \) then both \( pa' \) and \( pc' \) must be zero.

Then we finally assume \( pa' \neq 0 \) and \( pc' \neq 0 \). In case there exists \( 0 \neq \lambda \in C \) such that \( pc' = \lambda pa' \neq 0 \), then (5) implies
\[ \left\{ a'Xa'Y - c'YX \right\}^{n-1} = 0 \in T \]  \hspace{1cm} (6)
which is a contradiction, since \( a' \neq 0 \) and \( c' \neq 0 \).

Hence \( \{ pc', pa' \} \) is linearly independent and by (5) we get
\[ pa'Xa'Y \cdot \left\{ a'Xa'Y - c'YX \right\}^{n-1} = 0 \in T. \]

Once again relation (6) holds and we are done. \( \Box \)

**Lemma 2.2.** Assume that \( R \) is a primitive ring, which is isomorphic to a dense ring of linear transformations on some vector space \( V \) over a division ring \( D \), \( \dim_D V \geq 2 \), \( f \in \text{End}(V) \) and \( a \in R \). If \( av = 0 \), for any \( v \in V \) such that \( \{ v, f(v) \} \) is linearly \( D \)-independent, then \( a = 0 \), unless \( \dim_D V = 2 \) and \( \text{char}(R) = 2 \).
Proof. We fix a vector \( v \in V \) such that \( \{v, f(v)\} \) is linearly \( D \)-independent, then \( av = 0 \). Let \( w \in V \) be such that \( \{w, v\} \) is linearly \( D \)-dependent. Then both \( aw = 0 \) and \( w \in \text{Span}\{v, f(v)\} \) follow trivially.

Let now \( w \in V \) such that \( \{w, v\} \) is linearly \( D \)-independent and \( aw \neq 0 \). By the hypothesis it follows that \( \{w, f(w)\} \) is linearly \( D \)-dependent, as are \( \{w + v, f(w + v)\} \) and \( \{w - v, f(w - v)\} \). Therefore there exist \( \lambda_w, \lambda_{w+v}, \lambda_{w-v} \in D \) such that

\[
f(w) = w\lambda_w, \quad f(w + v) = (w + v)\lambda_{w+v}, \quad f(w - v) = (w - v)\lambda_{w-v}.
\]

In other words we have

\[
w\lambda_w + f(v) = w\lambda_{w+v} + v\lambda_{w+v} \quad (7)
\]

and

\[
w\lambda_w - f(v) = w\lambda_{w-v} - v\lambda_{w-v}. \quad (8)
\]

Assume \( \dim D V \geq 3 \). It is easy to see that \( w \in \text{Span}\{v, f(v)\} \), otherwise (7) forces a contradiction. Therefore, for any choice of \( w \in V \), we have \( w \in \text{Span}\{v, f(v)\} \), that is \( V = \text{Span}\{v, f(v)\} \), a contradiction.

In order to complete the proof, we then consider the case \( \dim D V = 2 \) and assume \( \text{char}(R) \neq 2 \), if not we are finished.

By comparing (7) with (8) we get both

\[
w(2\lambda_w - \lambda_{w+v} - \lambda_{w-v}) + v(\lambda_{w-v} - \lambda_{w+v}) = 0 \quad (9)
\]

and

\[
2f(v) = w(\lambda_{w+v} - \lambda_{w-v}) + v(\lambda_{w+v} + \lambda_{w-v}). \quad (10)
\]

By (9) and since \( \{w, v\} \) is \( D \)-independent and \( \text{char}(R) \neq 2 \), we have \( \lambda_w = \lambda_{w+v} = \lambda_{w-v} \). Thus by (10) it follows \( 2f(v) = 2v\lambda_w \). Since \( \{f(v), v\} \) is \( D \)-independent, the conclusion \( \lambda_w = \lambda_{w+v} = 0 \) follows, that is \( f(w) = 0 \) and \( f(w + v) = 0 \), which implies the contradiction \( f(v) = 0 \). Thus, if \( \dim D V = 2 \) and \( \text{char}(R) \neq 2 \), it follows that \( aw = 0 \), for any choice of \( w \in V \), that is \( aV = 0 \). Therefore \( a = 0 \) follows. \( \Box \)

**Proposition 2.3.** If \( R \) satisfies (1) then \( b, q \in C \) and \( p(a + b) = p(c + q) = 0 \), unless when \( \text{char}(R) = 2 \) and \( R \) satisfies \( s_4 \).

**Proof.** We of course suppose \( p \neq 0 \). In light of Lemma 2.1, we may assume that the generalized polynomial \( \Psi(x_1, x_2, y_1, y_2) \) is a non-trivial generalized polynomial identity for \( R \). By [3] it follows that \( \Psi(x_1, x_2) \) is a non-trivial generalized polynomial identity for \( Q_r \). In view of [13, Theorem 2.5 and Theorem 3.5], we know that both \( Q_r \) and \( Q_r \otimes_C \overline{C} \) are centrally closed, where \( \overline{C} \) is the algebraic closure of \( C \). We may
replace \( Q_r \) by itself or \( Q_r \otimes_C \overline{C} \) according as \( C \) is finite or infinite. Therefore we may assume that \( Q_r \) is centrally closed over \( C \) which is either finite or algebraically closed. By Martindale’s theorem [18], \( Q_r \) is a primitive ring having a non-zero socle \( H \), with \( C \) as the associated division ring. In light of Jacobson’s theorem [16, page 75], \( Q_r \) is isomorphic to a dense ring of linear transformations on some vector space \( V \) over \( C \). Since \( R \) is not commutative, we have \( \dim_C V \geq 2 \). Moreover, if \( \dim_C V = 2 \) we would assume \( \text{char}(R) \neq 2 \), if not we are done.

We divide the proof in several steps.

**Step 1.** \( b \in C \):

Suppose \( b \notin C \) and let \( v \in V \) be such that \( \{v, bv\} \) is linearly \( C \)-independent. Since \( \dim_C V \geq 2 \) and by the density of \( Q_r \), there exist \( r_1, r_2, s_1, s_2 \in Q_r \) such that

\[
\begin{align*}
    r_1 v &= 0 & r_2 v &= v & r_1(bv) &= -v & r_2(bv) &= 0 \\
    s_1 v &= 0 & s_2 v &= v & s_1(bv) &= -v & s_2(bv) &= 0.
\end{align*}
\]

By (1) we get

\[
0 = p \left\{ (a[r_1, r_2] + [r_1, r_2]b)(a[s_1, s_2] + [s_1, s_2]b) - (c[s_1, s_2] + [s_1, s_2]q)[r_1, r_2] \right\}^n v = pv.
\]

Hence we have proved that \( pv = 0 \) for any vector \( v \in V \) such that \( \{v, bv\} \) is linearly independent. By Lemma 2.2, \( p = 0 \) follows. This contradiction says that \( b \) must be a central element of \( Q_r \) and (1) reduces to

\[
0 = p \left\{ a'[x_1, x_2][a'[y_1, y_2] - (c[y_1, y_2] + [y_1, y_2]q)[x_1, x_2]] \right\}^n v = pv.
\]

where \( a' = a + b \).

**Step 2.** \( q \in C \):

Assume now \( q \notin C \) and let \( v \in V \) be such that \( \{v, qv\} \) is linearly \( C \)-independent. As above, there are \( r_1, r_2, s_1, s_2 \in Q_r \) such that

\[
\begin{align*}
    r_1 v &= 0 & r_2 v &= qv & r_1(qv) &= v \\
    s_1 v &= 0 & s_2 v &= v & s_1(qv) &= v & s_2(qv) &= 0.
\end{align*}
\]

By (11) we get

\[
0 = p \left\{ a'[r_1, r_2][a'[s_1, s_2] - (c[s_1, s_2] + [s_1, s_2]q)[r_1, r_2]] \right\}^n v = pv.
\]
Thus \( pv = 0 \) for any vector \( v \in V \) such that \( \{v, qv\} \) is linearly independent. As above the contradiction \( p = 0 \) follows.

Therefore both \( b \in C \) and \( q \in C \), that is \( Q_r \) satisfies

\[
p\left( a'[x_1, x_2]a'[y_1, y_2] - c'[y_1, y_2][x_1, x_2] \right)^n
\]

where \( a' = a + b \) and \( c' = c + q \).

**Step 3. Either \( pa' = 0 \) or \( a' \in C \):**

If \( a' \notin C \) then there is \( v \in V \) such that \( \{v, a'v\} \) is linearly \( C \)-independent. By the density of \( Q_r \), there are \( r_1, r_2, s_1, s_2 \in Q_r \) such that

\[
r_1(a'v) = 0 \quad r_2(a'v) = v \quad r_1v = v
\]

\[
s_1v = 0 \quad s_2v = v \quad s_1(a'v) = -v \quad s_2(a'v) = 0.
\]

By (12) it follows

\[
0 = p\left( a'[r_1, r_2]a'[s_1, s_2] - c'[s_1, s_2][r_1, r_2] \right)^n a'v = pa'v.
\]

Thus \( pa'v = 0 \) for any vector \( v \in V \) such that \( \{v, a'v\} \) is linearly independent, implying \( pa' = 0 \).

**Step 4. Let \( \dim_C V \geq 3 \), then either \( pc' = 0 \) or \( c' \in C \):**

If \( c' \notin C \) then there is \( v \in V \) such that \( \{v, c'v\} \) is linearly \( C \)-independent. Moreover, since \( \dim_C V \geq 3 \), there exists \( w \in V \) such that \( \{v, c'v, w\} \) is linearly \( C \)-independent.

Again by the density of \( Q_r \), there are \( r_1, r_2, s_1, s_2 \in Q_r \) such that

\[
r_1(c'v) = 0 \quad r_2(c'v) = v \quad r_1v = v
\]

\[
s_1v = 0 \quad s_2v = w \quad s_1w = v \quad s_1(c'v) = 0 \quad s_2(c'v) = c'v.
\]

Relation (12) implies

\[
0 = p\left( a'[r_1, r_2]a'[s_1, s_2] - c'[s_1, s_2][r_1, r_2] \right)^n c'v = (-1)^n pc'v.
\]

Hence, \( pc'v = 0 \) for any vector \( v \in V \) such that \( \{v, c'v\} \) is linearly independent, that is \( pc' = 0 \).

**Step 5. Let \( \dim_C V \geq 3 \). If \( pa' = 0 \), then \( pc' = 0 \):**

If \( c' \notin C \) the conclusion follows from Step 4. Moreover, if \( a' \in C \) then \( p = 0 \), which is not possible. Hence we assume \( c' \in C \) and \( a' \notin C \). Therefore \( Q_r \) satisfies

\[
pc'[y_1, y_2][x_1, x_2] \left\{ a'[x_1, x_2]a'[y_1, y_2] - c'[y_1, y_2][x_1, x_2] \right\}^{n-1}.
\]
Since \( a' \not\in C \), there is \( v \in V \) such that \( \{v, a'v\} \) is linearly \( C \)-independent. Moreover, since \( \dim_C V \geq 3 \), there exists \( w \in V \) such that \( \{v, a'v, w\} \) is linearly \( C \)-independent. By the density of \( Q_r \), there are \( r_1, r_2, s_1, s_2 \in Q_r \) such that
\[
\begin{align*}
r_1v &= 0 & r_2v &= a'v & r_1(a'v) &= a'v \\
s_1v &= 0 & s_2v &= v & s_1(a'v) &= 0 & s_2(a'v) &= w & s_1w &= v.
\end{align*}
\]
Relation (13) implies
\[
0 = pc'[s_1, s_2][r_1, r_2]\left\{a'[r_1, r_2]a'[s_1, s_2] - c'[s_1, s_2][r_1, r_2]\right\}^{n-1} v = (-c')^n pv.
\]
Hence \( (-c')^n pv = 0 \) for any vector \( v \in V \) such that \( \{v, a'v\} \) is linearly independent, that is \( pc' = 0 \) (we remark that, since we assume \( p \neq 0 \), this implies \( c' = 0 \)).

**Step 6. Let** \( \dim_C V \geq 3 \). **If** \( pc' = 0 \), **then** \( pa' = 0 \):

The proof of this step is quite similar to the previous one and we omit it for brevity.

**Step 7. If** \( \dim_C V \geq 3 \), **then both** \( pa' = 0 \) **and** \( pc' = 0 \):

In light of the previous argument, to complete the proof of this step we may assume both \( a' \in C \) and \( c' \in C \). In this case \( Q_r \) satisfies
\[
p\left\{a'^2[x_1, x_2][y_1, y_2] - c'[y_1, y_2][x_1, x_2]\right\}^n v = (-1)^n pa'^2 v.
\]
Let \( \{v, w\} \) be a set of linearly independent vectors of \( V \) and \( r_1, r_2, s_1, s_2, r'_1, r'_2, s'_1, s'_2 \in Q_r \) such that
\[
\begin{align*}
r_1v &= 0 & r_2v &= v & s_1v &= 0 & s_2v &= w & s_1w &= w & r_1w &= v & r_2w &= 0 \\
\end{align*}
\]
and
\[
\begin{align*}
s'_1v &= 0 & s'_2v &= v & r'_1v &= 0 & r'_2v &= w & r'_1w &= w & s'_1w &= v & s'_2w &= 0.
\end{align*}
\]
Thus (14) implies both
\[
p\left\{a'^2[r_1, r_2][s_1, s_2] - c'[s_1, s_2][r_1, r_2]\right\}^n v = (-1)^n pa'^2 v
\]
and
\[
p\left\{a'^2[r'_1, r'_2][s'_1, s'_2] - c'[s'_1, s'_2][r'_1, r'_2]\right\}^n v = pc'^n v.
\]
As above we may conclude that \( pa' = 0 \) and \( pc' = 0 \), as required.

Finally, in all that follows we assume \( \dim_C V = 2 \), that is \( Q_r \cong M_2(C) \), with \( \text{char}(C) \neq 2 \). Firstly we notice that (12) reduces to
\[
p\left\{a'[x_1, x_2]a'[y_1, y_2] - c'[y_1, y_2][x_1, x_2]\right\}^2.
\]
\[\text{(15)}\]
We resume our proof starting from the Step 3, so we know that either \( pa' = 0 \) or \( a' \in C \).

**Step 8. If \( Q_r \cong M_2(C) \) and \( pa' = 0 \) then \( pc' = 0 \):**

Under this assumption \( Q_r \) satisfies

\[
pc'[y_1, y_2][x_1, x_2] \left\{ a'[x_1, x_2]a'[y_1, y_2] - c'[y_1, y_2][x_1, x_2] \right\}.
\]

(16)

Of course we may assume that \( a' \) is not a scalar matrix, if not \( p = 0 \) follows.

We firstly suppose \( C \) is an infinite field. By [9, Lemma 1] there exists an \( C \)-automorphism \( \varphi \) of \( M_2(C) \) such that \( \varphi(a') \) has all non-zero entries. Clearly \( \varphi(a') \), \( \varphi(c') \) and \( \varphi(p) \) must satisfy the condition (16) that is

\[
\varphi(pc')[y_1, y_2][x_1, x_2] \left\{ \varphi(a')[x_1, x_2]\varphi(a')[y_1, y_2] - \varphi(c')[y_1, y_2][x_1, x_2] \right\}
\]

(17)

is an identity for \( M_2(C) \). Let \( e_{ij} \) denote the matrix unit with 1 in \( (i, j) \)-entry and zero elsewhere. Thus, for \( [x_1, x_2] = e_{12} \) and \( [y_1, y_2] = e_{21} \) in (17), and right multiplying by \( e_{11} \) we get

\[
\varphi(pc')e_{22}\varphi(a')e_{12}\varphi(a')e_{21} = 0.
\]

Since \( \varphi(a') \) has all non-zero entries, it follows that both \( (1, 2) \)-entry and \( (2, 2) \)-entry of the matrix \( \varphi(pc') \) must be zero. Similarly, for \( [x_1, x_2] = e_{21} \) and \( [y_1, y_2] = e_{12} \) in (17), and right multiplying by \( e_{22} \) we have that both \( (2, 1) \)-entry and \( (1, 1) \)-entry of the matrix \( \varphi(pc') \) must be zero. Therefore \( \varphi(pc') = 0 \), that is \( pc' = 0 \).

Now let \( K \) be an infinite field which is an extension of the field \( C \) and let \( \overline{Q_r} = M_2(K) \cong Q_r \otimes C K \). Consider the generalized polynomial

\[
P(x_1, x_2, x_3, x_4) = pc'[x_3, x_4][x_1, x_2] \left\{ a'[x_1, x_2]a'[x_3, x_4] - c'[x_3, x_4][x_1, x_2] \right\}
\]

which is a generalized polynomial identity for \( Q_r \). Moreover it is multi-homogeneous of multi-degree \( (2, 2, 2) \) in the indeterminates \( x_1, x_2, x_3, x_4 \).

Hence the complete linearization of \( P(x_1, x_2, x_3, x_4) \) is a multilinear generalized polynomial \( \Theta(x_1, \ldots, x_4, z_1, \ldots, z_4) \) in 8 indeterminates, moreover

\[
\Theta(x_1, \ldots, x_4, z_1, \ldots, z_4) = 2^4P(x_1, x_2, x_3, x_4).
\]

Clearly the multilinear polynomial \( \Theta(x_1, \ldots, x_4, z_1, \ldots, z_4) \) is a generalized polynomial identity for \( Q_r \) and \( \overline{Q_r} \) too. Since \( char(C) \neq 2 \) we obtain \( P(r_1, r_2, r_3, r_4) = 0 \), for all \( r_1, \ldots, r_4 \in \overline{Q_r} \), and the conclusion \( pc' = 0 \) follows from the above argument.

**Step 9. If \( Q_r \cong M_2(C) \) and \( a' \in C \) then \( a' = 0 \) and \( pc' = 0 \):**
In this final case $Q_r$ satisfies

$$p\left\{c'^2[x_1,x_2][y_1,y_2] - c'[y_1,y_2]x_1,x_2\right\}^2. \quad (18)$$

For $[x_1,x_2] = e_{12}$ and $[y_1,y_2] = e_{21}$ in (18), and right multiplying by $e_{11}$ we get $a'^4p e_{11} = 0$, implying that both $(2,1)$-entry and $(1,1)$-entry of the matrix $a'^4p$ must be zero. Once again, for $[x_1,x_2] = e_{21}$ and $[y_1,y_2] = e_{12}$ in (18), and right multiplying by $e_{22}$ we have $a'^4p e_{22} = 0$, that is both $(2,2)$-entry and $(1,2)$-entry of the matrix $a'^4p$ must be zero. Therefore $a'^4p = 0$, that is $a' = 0$. Hence (18) reduces to

$$p\left\{c'[y_1,y_2]x_1,x_2\right\}^2. \quad (19)$$

Notice that, if $c' \in C$ it follows that $c'^2p[x_1,x_2]^4$ is an identity for $Q_r$. In this case it is well known that $c'^2p = 0$, that is $c' = 0$. On the other hand, if we assume that $c' \notin C$, there is $v \in V$ such that $\{v,c'v\}$ is linearly $C$-independent. By the density of $Q_r$, there are $r_1,r_2,s_1,s_2 \in Q_r$ such that

$$r_1(c'v) = 0 \quad r_2(c'v) = v \quad r_1v = v$$

$$s_1v = 0 \quad s_2v = c'v \quad s_1(c'v) = v.$$

Thus, relation (19) implies

$$0 = p\left\{c'[s_1,s_2][r_1,r_2]\right\}^2c'v = pc'v.$$

as above, this last relation implies $pc' = 0$, as required. \qed

3. The case of inner generalized skew derivations

In this section we consider the case when the maps have the following forms:

$$F(x) = ax + \alpha(x)b, \quad G(x) = cx + \alpha(x)u$$

for all $x \in R$, for suitable fixed elements $p,a,b,c,u \in Q_r$ and $\alpha \in Aut(Q_r)$. Moreover we suppose that $Q_r$ satisfies

$$p\left\{\left(a[x_1,x_2] + \alpha([x_1,x_2])b\right)\left(a[y_1,y_2] + \alpha([y_1,y_2])b\right)
- \left(c[y_1,y_2] + \alpha([y_1,y_2])u\right)x_1,x_2\right\}^n. \quad (20)$$

In light of Proposition 2.3 we may always assume $\alpha \neq I_R$, the identity map on $R$.  

Lemma 3.1. Assume that $R$ is isomorphic to a dense ring of linear transformations on some vector space $V$ over a division ring $D$, containing non-zero linear transformations of finite rank. If $R$ satisfies (20) then there exist $a', c' \in Q_v$ such that $F(x) = a'x$ and $G(x) = c'x$, for any $x \in R$, with $pa' = pc' = 0$, unless when $\dim_D V \leq 2$.

Proof. We suppose $\dim_D V \geq 3$.

Since $R$ is a primitive ring with non-zero socle, by [16, p. 79], there exists a semi-linear automorphism $T \in \text{End}(V)$ such that $\alpha(x) = TxT^{-1}$ for all $x \in R$. Hence, $R$ satisfies

$$p \left\{ \left( a[x_1, x_2] + T[x_1, x_2]T^{-1}b \right) \left( a[y_1, y_2] + T[y_1, y_2]T^{-1}b \right) 
- \left( c[y_1, y_2] + T[y_1, y_2]T^{-1}u \right)[x_1, x_2] \right\}^n = 0. \tag{21}$$

Assume there exists $v \in V$ such that $\{v, T^{-1}bv\}$ is linearly $D$-independent. Since $\dim_D V \geq 3$, there exists $w \in V$ such that $\{w, v, T^{-1}bv\}$ is linearly $D$-independent. Moreover, by the density of $R$, there exist $r_1, r_2, s_1, s_2 \in R$ such that

$$r_1v = 0 \quad r_2v = v \quad r_1w = T^{-1}v \quad r_1T^{-1}bv = 0 \quad r_2T^{-1}bv = w$$

$$s_1v = 0 \quad s_2v = v \quad s_1w = T^{-1}v \quad s_1T^{-1}bv = 0 \quad s_2T^{-1}bv = w$$

and we get

$$0 = p \left\{ \left( a[r_1, r_2] + T[r_1, r_2]T^{-1}b \right) \left( a[s_1, s_2] + T[s_1, s_2]T^{-1}b \right) 
- \left( c[s_1, s_2] + T[s_1, s_2]T^{-1}u \right)[r_1, r_2] \right\}^n v = pv.$$

Hence, for any $v \in V$ such that $\{v, T^{-1}bv\}$ is linearly $D$-independent, it follows $pv = 0$. By Lemma 2.2 we get $p = 0$, which is a contradiction.

Therefore, for any $v \in V$, there exists $\lambda_v \in D$ such that $T^{-1}bv = v\lambda_v$. In this case, it is well known that there exists a unique $\lambda \in D$ such that $T^{-1}bv = v\lambda$, for all $v \in V$ (see for example Lemma 1 in [7]). Thus

$$\left( ax + \alpha(x)b \right)v = \left( ax + TxT^{-1}b \right)v = axv + T(xv\lambda) =$$

$$axv + T((xv)\lambda) = axv + T(T^{-1}bxv) =$$

$$axv + bxv = (a + b)xv.$$

Hence, for all $v \in V$,

$$\left( ax + \alpha(x)b - (a + b)x \right)v = 0$$
which implies \( F(x) = ax + \alpha(x)b = (a + b)x \), for all \( x \in R \), since \( V \) is faithful. Therefore we have that \( R \) satisfies

\[
p\left\{ (a + b)[x_1, x_2](a + b)[y_1, y_2] - \left( c[y_1, y_2] + T[y_1, y_2]T^{-1}u\right)[x_1, x_2] \right\}^n. \quad (22)
\]

Now assume there exists \( v \in V \) such that \( \{v, T^{-1}uv\} \) is linearly \( D \)-independent. As above there exists \( w \in V \) such that \( \{w, v, T^{-1}uv\} \) is linearly \( D \)-independent and there exist \( r_1, r_2, s_1, s_2 \in R \) such that

\[
\begin{align*}
    r_1v &= 0 \quad r_2v = w \quad r_1w = v \\
    s_1v &= 0 \quad s_2v = v \quad s_1w = T^{-1}v \quad s_1T^{-1}u = 0 \quad s_2T^{-1}uv = w.
\end{align*}
\]

From (22) it follows that

\[
0 = p\left\{ (a+b)[r_1, r_2](a+b)[s_1, s_2] - \left( c[s_1, s_2] + T[s_1, s_2]T^{-1}u\right)[r_1, r_2] \right\}^n v = (-1)^n pv.
\]

Once again, since \( p \) is not zero, by Lemma 2.2 we obtain a contradiction. Thus, there exists a unique \( \mu \in D \) such that \( T^{-1}uv = v\mu \), for all \( v \in V \). This implies \( G(x) = cx + \alpha(x)u = (c + u)x \), for all \( x \in R \).

Therefore, we have proved that, if \( \text{dim}_D V \geq 3 \), both \( F \) and \( G \) are inner generalized derivations. The required conclusion then follows from Proposition 2.3. \qed

**Proposition 3.2.** If \( R \) satisfies (20) then there exist \( a', c' \in Q_r \) such that \( F(x) = a'x \) and \( G(x) = c'x \), for any \( x \in R \), with \( pa' = pc' = 0 \), unless when \( R \) satisfies \( s_4 \).

**Proof.** Suppose firstly \( \alpha \) is an \( X \)-inner automorphism of \( R \). Thus assume \( \alpha(x) = qxq^{-1} \), for all \( x \in R \), that is

\[
F(x) = ax + qxq^{-1}b, \quad G(x) = cx + qxq^{-1}u
\]

for all \( x \in R \), where \( q \) is an invertible element of \( Q_r \). Under our assumption, \( R \) satisfies

\[
p\left\{ \left( a[x_1, x_2] + q[x_1, x_2]q^{-1}b \right) \left( a[y_1, y_2] + q[y_1, y_2]q^{-1}b \right) \\
- \left( c[y_1, y_2] + q[y_1, y_2]q^{-1}u\right)[x_1, x_2] \right\}^n. \quad (23)
\]

Since \( \alpha \) is not the identity map on \( R \), we consider the case \( q \notin C \). Moreover, notice that if both \( q^{-1}b \in C \) and \( q^{-1}u \in C \), then \( F \) and \( G \) are inner generalized derivations defined respectively as follows

\[
F(x) = (a + b)x, \quad G(x) = (c + u)x \quad \forall x \in R.
\]
and the conclusion follows again from Proposition 2.3.
On the other hand, if either $q^{-1}b \notin C$ or $q^{-1}u \notin C$, the identity (23) is a non-trivial generalized polynomial identity for $R$ as well as for $Q_r$. In light of the same arguments set out in Proposition 2.3, we may assume that $Q_r$ is a primitive ring having a non-zero socle $H$, with $C$ as the associated division ring. Moreover $Q_r$ is isomorphic to a dense ring of linear transformations on some vector space $V$ over $C$. By Lemma 3.1 we conclude that $\dim_C V \leq 2$, that is $Q_r$ satisfies $s_4$, as required.

Then we now consider the case $\alpha$ is not an inner automorphism of $R$. Since $\alpha \neq I_R$, by [4] $R$ is a GPI-ring and $Q_r$ is also GPI-ring by [3]. Once again $Q_r$ is isomorphic to a dense ring of linear transformations on some vector space $V$ and its associated division ring $D$ is finite-dimensional over $C$. Thus, by Lemma 3.1, one of the following holds:

1. there exist $a', c' \in Q_r$ such that $F(x) = a'x$ and $G(x) = c'x$, for any $x \in R$, with $pa' = pc' = 0$ (in this case we are done)
2. $\dim_D V \leq 2$.

To complete the proof we have to study this last case. Since $\dim_D V \leq 2$ and by our main hypothesis, $Q_r$ satisfies

$$
p\left\{ \left( a[x_1, x_2] + \alpha([x_1, x_2])b \right) \left( a[y_1, y_2] + \alpha([y_1, y_2])b \right) - \left( c[y_1, y_2] + \alpha([y_1, y_2])u \right)[x_1, x_2] \right\}^2
$$

(24)

Here we divide the argument into the following three cases.

**Case 1: Assume char($R$) = 0 or char($R$) = $p \geq 3$.**

By [5, Theorem 3] and (24), it follows that

$$
p\left\{ \left( a[x_1, x_2] + [t_1, t_2]b \right) \left( a[y_1, y_2] + [z_1, z_2]b \right) - \left( c[y_1, y_2] + [z_1, z_2]u \right)[x_1, x_2] \right\}^2
$$

(25)

is a generalized polynomial identity for $Q_r$. In particular $Q_r$ satisfies the blended component

$$
p\left\{ [t_1, t_2]b[z_1, z_2]b \right\}^2
$$

(26)

which implies easily $b = 0$, since we suppose $p \neq 0$.

Analogously, for $b = 0$ and $y_1 = y_2 = 0$ in (25), we have that $Q_r$ also satisfies

$$
p\left\{ [z_1, z_2]u[x_1, x_2] \right\}^2
$$

(27)
that is \( u = 0 \). Therefore \( F(x) = ax \) and \( G(x) = cx \), for any \( x \in R \), and \( pa = pc = 0 \) follows from Proposition 2.3, unless \( Q_r \) satisfies \( s_4 \).

**Case 2: Assume the automorphism \( \alpha \) is not Frobenius.**

Also in this case, by (24) and [5, Theorem 2], one can see that \( Q_r \) satisfies (25), and we conclude as above.

**Case 3: Assume the automorphism \( \alpha \) is Frobenius and \( \text{char}(R) = 2 \).**

Hence there exists a fixed integer \( h \) such that \( \alpha(x) = x^{2^h} \), for all \( x \in C \). In particular, there is \( x \in C \) such that \( x^{2^h} \neq x \). Moreover we assume \( C \) is infinite, otherwise \( D \) should be a finite division ring, that is \( D \) is a field and we are done.

Let \( 0 \neq \lambda \in C \) be such that \( \lambda^{2^h} \neq \lambda \). In (24) replace \( y_1 \) by \( \lambda y_1 \) and get

\[
p\left\{ \left( a[x_1,x_2] + \alpha([x_1,x_2])b \right) \left( a[y_1,y_2] + \lambda^{2^h-1}\alpha([y_1,y_2])b \right) - \left( c[y_1,y_2] + \lambda^{2^h-1}\alpha([y_1,y_2])u[x_1,x_2] \right) \right\}^2.
\]

If denote

\[
\Phi_1(x_1,x_2,y_1,y_2) = a[x_1,x_2]a[y_1,y_2] + \alpha([x_1,x_2])ba[y_1,y_2] - c[y_1,y_2][x_1,x_2]
\]

and

\[
\Phi_2(x_1,x_2,y_1,y_2) = a[x_1,x_2]\alpha([y_1,y_2])b + \alpha([x_1,x_2])ba\alpha([y_1,y_2])b - \alpha([y_1,y_2])u[x_1,x_2]
\]

it follows that

\[
p\left\{ \Phi_1(r_1,r_2,r_3,r_4) + \gamma \Phi_2(r_1,r_2,r_3,r_4) \right\}^2 = 0
\]

for all \( r_1, r_2, r_3, r_4 \in Q_r \), with \( \gamma = \lambda^{2^h-1} \neq 1 \). Expanding the latter relation, we get

\[
p\left\{ \Phi_1^2 + \gamma(\Phi_1 \Phi_2 + \Phi_2 \Phi_1) + \gamma^2 \Phi_2^2 \right\} = 0.
\]

For the sake of clearness, let us denote \( t_0 = p\Phi_1^2 \), \( t_1 = p(\Phi_1 \Phi_2 + \Phi_2 \Phi_1) \) and \( t_2 = p\Phi_2^2 \). Then we can write

\[
t_0 + \gamma t_1 + \gamma^2 t_2 = 0.
\]

(29)

Replacing in the previous argument \( \gamma \) successively by 1, \( \gamma \), \( \gamma^2 \), the equation (29) gives the system of equations

\[
t_0 + t_1 + t_2 = 0
\]

\[
t_0 + \gamma t_1 + \gamma^2 t_2 = 0
\]

(30)

\[
t_0 + \gamma^2 t_1 + \gamma^4 t_2 = 0.
\]
Moreover, since $C$ is infinite, there exist infinitely many $\lambda \in C$ such that $\lambda^{(2^h-1)} \neq 1$ for $i = 1, \ldots, 4$, that is there exist infinitely many $\gamma = \lambda^{2^h-1} \in C$ such that $\gamma^i \neq 1$ for $i = 1, \ldots, 4$. Hence, the Vandermonde determinant (associated with the system (30))

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & \gamma & \gamma^2 \\ 1 & \gamma^2 & \gamma^4 \end{vmatrix} = \prod_{0 \leq i < j \leq 4} (\gamma_i - \gamma_j)$$

is not zero. Thus, we can solve the above system (30) and obtain $t_i = 0$ ($i = 0, 1, 2$).

In particular $t_0 = 0$ and $t_2 = 0$, that is

$$p\left\{ a[x_1, x_2]a[y_1, y_2] + \alpha([x_1, x_2])ba[y_1, y_2] - c[y_1, y_2][x_1, x_2] \right\}^2 \quad (31)$$

and

$$p\left\{ a[x_1, x_2]a([y_1, y_2]) + \alpha([x_1, x_2])ba([y_1, y_2]) - \alpha([y_1, y_2])u[x_1, x_2] \right\}^2 \quad (32)$$

are satisfied by $Q_r$.

In (31) replace $x_1$ by $\lambda x_1$ and get

$$p\left\{ a[x_1, x_2]a[y_1, y_2] + \lambda^{2^h-1} \alpha([x_1, x_2])ba[y_1, y_2] - c[y_1, y_2][x_1, x_2] \right\}^2 \quad (33)$$

Now we denote

$$\Omega_1(x_1, x_2, y_1, y_2) = a[x_1, x_2]a[y_1, y_2] - c[y_1, y_2][x_1, x_2]$$

and

$$\Omega_2(x_1, x_2, y_1, y_2) = \alpha([x_1, x_2])ba[y_1, y_2]$$

obtaining

$$p\left\{ \Omega_1(r_1, r_2, r_3, r_4) + \gamma \Omega_2(r_1, r_2, r_3, r_4) \right\}^2 = 0$$

for all $r_1, r_2, r_3, r_4 \in Q_r$, with $\gamma = \lambda^{2^h-1} \neq 1$. Thus, as above, for $z_0 = p\Omega_1^2$, $z_1 = p(\Omega_1\Omega_2 + \Omega_2\Omega_1)$ and $z_2 = p\Omega_2^2$, one has

$$z_0 + \gamma z_1 + \gamma^2 z_2 = 0. \quad (34)$$

By the same above Vandermonde determinant argument, we arrive at $z_0 = 0$, that is $Q_r$ satisfies

$$p\left\{ a[x_1, x_2]a[y_1, y_2] - c[y_1, y_2][x_1, x_2] \right\}^2 \quad (35)$$
Application of Proposition 2.3 to (35) leads to the conclusion $pa = pc = 0$, unless $Q_r$ satisfies $s_4$.

On the other hand, if we replace $x_1$ by $\lambda x_1$ in (32), then $Q_r$ satisfies

$$p\left\{a[x_1, x_2]a([y_1, y_2])b + \lambda^{2^n-1}a([x_1, x_2])b\alpha([y_1, y_2])b - \alpha([y_1, y_2])u[x_1, x_2]\right\}^2$$  \hfill (36)

Once again, we denote

$$\Psi_1(x_1, x_2, y_1, y_2) = a[x_1, x_2]a([y_1, y_2])b - \alpha([y_1, y_2])u[x_1, x_2]$$

and

$$\Psi_2(x_1, x_2, y_1, y_2) = \alpha([x_1, x_2])b\alpha([y_1, y_2])b$$

obtaining

$$p\left\{\Psi_1(r_1, r_2, r_3, r_4) + \gamma \Psi_2(r_1, r_2, r_3, r_4)\right\}^2 = 0$$

for all $r_1, r_2, r_3, r_4 \in Q_r$, with $\gamma = \lambda^{2^n-1} \neq 1$. Therefore, for $w_0 = p\Psi_1^2$, $w_1 = p(\Psi_1\Psi_2 + \Psi_2\Psi_1)$ and $w_2 = p\Psi_2^2$, it follows that

$$w_0 + \gamma w_1 + \gamma^2 w_2 = 0.$$  \hfill (37)

Similarly to what we saw previously, we get $w_0 = 0$ and $w_2 = 0$, that is both

$$p\left\{a[x_1, x_2]a([y_1, y_2])b - \alpha([y_1, y_2])u[x_1, x_2]\right\}^2$$  \hfill (38)

and

$$p\left\{\alpha([x_1, x_2])b\alpha([y_1, y_2])b\right\}^2$$  \hfill (39)

are identities for $Q_r$. We remark that (39) means that

$$p\left\{[r_1, r_2]b[s_1, s_2]b\right\}^2 = 0 \quad \forall r_1, r_2, s_1, s_2 \in Q_r$$

implying $b = 0$ (since $p \neq 0$). Then (38) reduces to

$$p\left\{\alpha([y_1, y_2])u[x_1, x_2]\right\}^2$$

that is $u = 0$.

Hence we have proved that either $Q_r$ satisfies $s_4$, or $F(x) = ax$ and $G(x) = cx$, for any $x \in R$, with $pa = pc = 0$, as required. \hfill $\square$
4. The proof of Theorem 1.1

In this final section we consider the more general situation and write $F(x) = ax + d(x)$, $G(x) = cx + \delta(x)$ for all $x \in R$, where $a, c \in Q_r$ and $d, \delta$ are skew derivations of $R$. Let $\alpha$ be the automorphism associated with $d$ and $\delta$. Thus, for any $x, y \in R$,

$$d(xy) = d(x)y + \alpha(x)d(y)$$

and

$$\delta(xy) = \delta(x)y + \alpha(x)\delta(y).$$

To prove our main result, we always assume that $R$ does not satisfy the standard identity $s_4$. Under this assumption, and since $L$ is not central, there exists a non-zero ideal $I$ of $R$ such that $0 \neq [I, R] \subseteq L$ ([15, pages 4-5], [12, Lemma 2 and Proposition 1], [17, Theorem 4]). Therefore we have that there exists a non-central ideal $I$ of $R$ such that

$$p\{F(u)F(v) - G(v)u\}^n = 0 \quad \forall u, v \in [I, I].$$

Since $R$ and $I$ satisfy the same generalized differential identities with automorphisms, we may assume that

$$p\{F([x_1, x_2])F([y_1, y_2]) - G([y_1, y_2])[x_1, x_2]\}^n$$

is an identity for $R$. In other words $R$ satisfies

$$p\left\{a[x_1, x_2] + d([x_1, x_2])(a[y_1, y_2]) + d([y_1, y_2]) - \left(c[y_1, y_2] + \delta([y_1, y_2])\right)[x_1, x_2]\right\}^n.$$  \hspace{1cm} (41)

The following results which will be useful in the sequel:

**Fact 4.1.** ([10, Lemma 3.2]) Let $R$ be a prime ring, $\alpha, \beta \in Aut(Q_r)$ and $d : R \to R$ be a skew derivation, associated with the automorphism $\alpha$. If there exist $0 \neq \theta \in C$, $0 \neq \eta \in C$ and $u, b \in Q_r$ such that

$$d(x) = \theta\left(u\alpha(x) + \eta\left(b\alpha(x) - \beta(x)b\right)\right), \quad \forall x \in R$$

then $d$ is an inner skew derivation of $R$. More precisely, either $b = 0$ or $\alpha = \beta$.

**Fact 4.2.** ([11, Fact 4.2]) Let $R$ be a prime ring, $\alpha, \beta \in Aut(Q_r)$ and $d, \delta : R \to R$ be skew derivations, associated with the automorphism $\alpha$. If there exist $0 \neq \eta \in C$
and \( p \in Q_r \) such that
\[
\delta(x) = \eta d(x) + \left( px - \beta(x)p \right), \quad \forall x \in R
\]
then either \( \alpha = \beta \) or \( px - \beta(x)p = 0 \) and \( \delta(x) = \eta d(x) \), for any \( x \in R \).

**Remark 4.3.** If we assume that both \( F \) and \( G \) are inner generalized skew derivations, then we may write
\[
d(x) = bx - \alpha(x)b \quad \text{and} \quad F(x) = ax + bx - \alpha(x)b \quad \forall x \in R
\]
and
\[
\delta(x) = ux - \alpha(x)u \quad \text{and} \quad G(x) = cx + ux - \alpha(x)u \quad \forall x \in R
\]
where \( a, b, c, u \in Q_r \) and \( \alpha \in \text{Aut}(R) \).

We would like to point out that, in case \( R \) satisfies (41) and by Proposition 3.2, we may conclude that one of the following holds:

1. \( d = \delta = 0 \) and \( pa = pc = 0 \);
2. \( R \) satisfies \( s_4 \).

**Proof of Theorem 1.1.** By Propositions 2.3 and 3.2 we may assume that \( d, \delta \) are not simultaneously inner skew derivations. In particular \( d, \delta \) are not simultaneously zero. In all that follows we may also suppose that \( R \) does not satisfy \( s_4 \).

By (41), \( R \) satisfies
\[
p\left\{ \left( a[x_1, x_2] + d(x_1)x_2 + \alpha(x_1)d(x_2) - d(x_2)x_1 - \alpha(x_2)d(x_1) \right) \cdot \left( a[y_1, y_2] + d(y_1)y_2 + \alpha(y_1)d(y_2) - d(y_2)y_1 - \alpha(y_2)d(y_1) \right) - c[y_1, y_2][x_1, x_2] - \left( \delta(y_1)y_2 + \alpha(y_1)\delta(y_2) - \delta(y_2)y_1 - \alpha(y_2)\delta(y_1) \right) \right\}^n.
\]

Let \( d \neq 0 \) and \( \delta \neq 0 \) be \( C \)-linearly independent modulo \( SD_{\text{int}} \).

In this case, by (43), \( R \) satisfies
\[
p\left\{ \left( a[x_1, x_2] + t_1x_2 + \alpha(x_1)t_2 - t_2x_1 - \alpha(x_2)t_1 \right) \cdot \left( a[y_1, y_2] + z_1y_2 + \alpha(y_1)z_2 - z_2y_1 - \alpha(y_2)z_1 \right) - c[y_1, y_2][x_1, x_2] - \left( w_1y_2 + \alpha(y_1)w_2 - w_2y_1 - \alpha(y_2)w_1 \right) \right\}^n.
\]
In particular, for \( x_1 = t_2 = y_1 = z_2 = 0 \), \( R \) satisfies
\[
p \left\{ \left( t_1x_2 - \alpha(x_2)t_1 \right) \cdot \left( z_1y_2 - \alpha(y_2)z_1 \right) \right\}^n. \tag{45}
\]

If \( \alpha \) is the identity map, then \( R \) satisfies \( p[x_1, x_2]^{2n} \), which forces \( p = 0 \), a contradiction. Thus \( \alpha \) is not the identity on \( R \). Since (45) is a non-trivial generalized identity also for \( Q_r \), then \( Q_r \) is isomorphic to a dense subring of the ring of linear transformations of a vector space \( V \) over a division ring \( D \), containing non-zero linear transformations of finite rank and, as above, there exists a semi-linear automorphism \( T \in {\text{End}}(V) \) such that \( \alpha(x) = T x T^{-1} \) for all \( x \in Q_r \).

Hence, \( Q_r \) satisfies
\[
p \left\{ \left( t_1x_2 - T x_2 T^{-1} t_1 \right) \cdot \left( z_1y_2 - T y_2 T^{-1} z_1 \right) \right\}^n. \tag{46}
\]

Let \( \text{dim}_D V \geq 2 \) and suppose that, for any \( v \in V \), there exists \( \lambda_v \in D \) such that \( T^{-1}v = v\lambda_v \). As mentioned above, there exists a unique \( \lambda \in D \) such that \( T^{-1}v = v\lambda \), for all \( v \in V \). In this case \( \alpha \) is the identity, a contradiction.

Therefore, there exists \( v \in V \) such that \( \{ v, T^{-1}v \} \) is linearly \( D \)-independent. By the density of \( Q_r \), there exist \( r_1, r_2, s_1, s_2 \in Q_r \) such that
\[
s_1v = 0 \quad s_2v = T^{-1}v \quad s_1T^{-1}v = v \quad r_1v = 0 \quad r_2v = T^{-1}v \quad r_1T^{-1}v = v
\]
and, by (46), we get
\[
p \left\{ \left( r_1r_2 - T r_2 T^{-1} r_1 \right) \cdot \left( s_1s_2 - T s_2 T^{-1} s_1 \right) \right\}^n v = pv. \tag{47}
\]

As above, application of Lemma 2.2 and since \( p \neq 0 \), it follows \( \text{dim}_D V = 2 \) and \( Q_r \) satisfies
\[
p \left\{ \left( t_1x_2 - \alpha(x_2)t_1 \right) \cdot \left( z_1y_2 - \alpha(y_2)z_1 \right) \right\}^2. \tag{48}
\]

On the other hand, if \( \text{dim}_D V = 1 \), \( Q_r \) is a domain satisfying
\[
p \left\{ \left( t_1x_2 - \alpha(x_2)t_1 \right) \cdot \left( z_1y_2 - \alpha(y_2)z_1 \right) \right\}.
\]

Therefore, more generally we may assume that (48) is an identity for \( Q_r \). In particular, for \( t_1 = z_1 \) and \( x_2 = y_2 \), \( Q_r \) satisfies \( p \left( z_1y_2 - \alpha(y_2)z_1 \right)^2 \). Since \( p \neq 0 \), this last relation implies \( r_1r_2 - \alpha(r_2)r_1 = 0 \), for any \( r_1, r_2 \in Q_r \) (see [1, Theorem B and Corollary]). It is easy to see that this case may occur only if \( R \) is commutative and \( \alpha \) is the identity, a contradiction.
Let \( d \neq 0 \) and \( \delta \neq 0 \) be \( C \)-linearly dependent modulo \( SD_{\text{int}} \).

Here we assume that there exist \( \lambda, \mu \in C \), \( c' \in Q_r \) and \( \gamma \in \text{Aut}(R) \) such that 

\[
\lambda d(x) + \mu \delta(x) = c' x - \gamma(x)c' \quad \text{for all} \quad x \in R.
\]

- We firstly study the case \( 0 \neq \lambda \in C \) and \( 0 \neq \mu \in C \).

Denote \( \eta = -\mu^{-1}\lambda \) and \( p' = \mu^{-1}c' \). So \( \delta(x) = \eta d(x) + p' x - \gamma(x)p' \) for all \( x \in R \). 

By Fact 4.2, we know that either \( \delta(x) = \eta d(x) \) for all \( x \in R \), or \( \gamma = \alpha \).

In case \( \gamma = \alpha \), one has \( \delta(x) = \eta d(x) + p' x - \alpha(x)p' \) for all \( x \in R \). Therefore by (43), \( Q_r \) satisfies

\[
P \left\{ \left( a[x_1, x_2] + d(x_1)x_2 + \alpha(x_1)d(x_2) - d(x_2)x_1 - \alpha(x_2)d(x_1) \right), \right.
\]

\[
\cdot \left( a[y_1, y_2] + d(y_1)y_2 + \alpha(y_1)d(y_2) - d(y_2)y_1 - \alpha(y_2)d(y_1) \right)
\]

\[
- c[y_1, y_2][x_1, x_2] - \left( \eta d(y_1)y_2 + \alpha(y_1)\eta d(y_2) - \eta d(y_2)y_1 - \alpha(y_2)\eta d(y_1) \right)[x_1, x_2]
\]

\[
- \left( p'[y_1, y_2] - [\alpha(y_1), \alpha(y_2)]p' \right)[x_1, x_2] \right\}^n.
\]

Applying Fact 4.1 we may assume that \( d \) is not inner. By (49) \( Q_r \) satisfies

\[
P \left\{ \left( a[x_1, x_2] + t_1x_2 + \alpha(x_1)t_2 - t_2x_1 - \alpha(x_2)t_1 \right), \right.
\]

\[
\cdot \left( a[y_1, y_2] + z_1y_2 + \alpha(y_1)z_2 - z_2y_1 - \alpha(y_2)z_1 \right)
\]

\[
- c[y_1, y_2][x_1, x_2] - \left( \eta z_1y_2 + \alpha(y_1)\eta z_2 - \eta z_2y_1 - \alpha(y_2)\eta z_1 \right)[x_1, x_2]
\]

\[
- \left( p'[y_1, y_2] - [\alpha(y_1), \alpha(y_2)]p' \right)[x_1, x_2] \right\}^n.
\]

In particular, for \( x_1 = t_2 = y_1 = z_2 = 0 \) in (50), it follows that \( Q_r \) satisfies again relation (45), so that a contradiction follows as above.

Analogously, for \( \delta = \eta d \), the relation (49) reduces to

\[
P \left\{ \left( a[x_1, x_2] + d(x_1)x_2 + \alpha(x_1)d(x_2) - d(x_2)x_1 - \alpha(x_2)d(x_1) \right), \right.
\]

\[
\cdot \left( a[y_1, y_2] + d(y_1)y_2 + \alpha(y_1)d(y_2) - d(y_2)y_1 - \alpha(y_2)d(y_1) \right)
\]

\[
- c[y_1, y_2][x_1, x_2] - \left( \eta d(y_1)y_2 + \alpha(y_1)\eta d(y_2) - \eta d(y_2)y_1 - \alpha(y_2)\eta d(y_1) \right)[x_1, x_2]
\]

\[
- \left( p'[y_1, y_2] - [\alpha(y_1), \alpha(y_2)]p' \right)[x_1, x_2] \right\}^n.
\]

It is easy to see that \( Q_r \) satisfies again (45) and we conclude as above.
Assume now $\lambda = 0$.

Hence $\delta(x) = p'x - \gamma(x)p'$ for all $x \in R$, where $p' = \mu^{-1}c'$ and $d$ is not inner.

Then, by relation (43), $Q_r$ satisfies

$$p\left\{ \left( a[x_1, x_2] + t_1x_2 + \alpha(x_1)t_2 - t_2x_1 - \alpha(x_2)t_1 \right) \right. + \left. \left( a[y_1, y_2] + z_1y_2 + \alpha(y_1)z_2 - z_2y_1 - \alpha(y_2)z_1 \right) \right. - c[y_1, y_2][x_1, x_2] - \left( p'[y_1, y_2] - [\gamma(y_1), \gamma(y_2)]p' \right)[x_1, x_2] \right\}^n.$$  \hspace{1cm} (52)

Also in this case, for $x_1 = t_2 = y_1 = z_2 = 0$ in (52), $Q_r$ satisfies (45) and we are done.

The case $\mu = 0$

In this case, $d(x) = p'x - \gamma(x)p'$ for all $x \in R$, where $p' = \lambda^{-1}c'$ and $\delta$ is not inner. Moreover $\alpha = \gamma$ (as a reduction of Fact 4.2). Relation (43) implies that $Q_r$ satisfies

$$p\left\{ \left( a[x_1, x_2] + p'[x_1, x_2] - [\alpha(x_1), \alpha(x_2)]p' \right) \left( a[y_1, y_2] + p'[y_1, y_2] - [\alpha(y_1), \alpha(y_2)]p' \right) - c[y_1, y_2][x_1, x_2] - \left( \delta(y_1)y_2 + \alpha(y_1)\delta(y_2) - \delta(y_2)y_1 - \alpha(y_2)\delta(y_1) \right)[x_1, x_2] \right\}^n.$$ \hspace{1cm} (53)

Since $\delta$ is not inner, $Q_r$ satisfies

$$p\left\{ \left( a[x_1, x_2] + p'[x_1, x_2] - [\alpha(x_1), \alpha(x_2)]p' \right) \left( a[y_1, y_2] + p'[y_1, y_2] - [\alpha(y_1), \alpha(y_2)]p' \right) - c[y_1, y_2][x_1, x_2] - \left( z_1y_2 + \alpha(y_1)z_2 - z_2y_1 - \alpha(y_2)z_1 \right)[x_1, x_2] \right\}^n.$$ \hspace{1cm} (54)

For $z_1 = z_2 = 0$ in (54), it follows that

$$p\left\{ \left( a[x_1, x_2] + p'[x_1, x_2] - [\alpha(x_1), \alpha(x_2)]p' \right) \left( a[y_1, y_2] + p'[y_1, y_2] - [\alpha(y_1), \alpha(y_2)]p' \right) - c[y_1, y_2][x_1, x_2] \right\}^n.$$ \hspace{1cm} (55)

is an identity for $Q_r$. Application of Proposition 3.2 implies $p'x - \alpha(x)p' = 0$, for any $x \in Q_r$, that is $d = 0$, which is a contradiction.
The case $\delta = 0$

Here we have to consider the only case when $0 \neq d$ is an outer skew derivation. By (43), $R$ satisfies

$$p\left\{ (a[x_1, x_2] + d(x_1)x_2 + \alpha(x_1)d(x_2) - d(x_2)x_1 - \alpha(x_2)d(x_1)) \cdot (a[y_1, y_2] + d(y_1)y_2 + \alpha(y_1)d(y_2) - d(y_2)y_1 - \alpha(y_2)d(y_1)) - c[y_1, y_2][x_1, x_2] \right\}^n.$$  \hspace{1cm} (56)

Then, since $0 \neq d$ is outer, $R$ satisfies

$$p\left\{ (a[x_1, x_2] + t_1x_2 + \alpha(x_1)t_2 - t_2x_1 - \alpha(x_2)t_1) \cdot (a[y_1, y_2] + z_1y_2 + \alpha(y_1)z_2 - z_2y_1 - \alpha(y_2)z_1) - c[y_1, y_2][x_1, x_2] \right\}^n.$$  \hspace{1cm} (57)

As above, for $x_1 = t_2 = y_1 = z_2 = 0$ in (57), (45) is an identity for $R$ and we are done again.

The case $d = 0$

In this final case, relation (43) reduces to

$$p\left\{ a[x_1, x_2]a[y_1, y_2] - c[y_1, y_2][x_1, x_2] - \left( \delta(y_1)y_2 + \alpha(y_1)\delta(y_2) - \delta(y_2)y_1 - \alpha(y_2)\delta(y_1) \right)[x_1, x_2] \right\}^n.$$  \hspace{1cm} (58)

Moreover, we may assume that $0 \neq \delta$ is not inner. Therefore (58) implies that $R$ satisfies

$$p\left\{ a[x_1, x_2]a[y_1, y_2] - c[y_1, y_2][x_1, x_2] - \left( z_1y_2 + \alpha(y_1)z_2 - z_2y_1 - \alpha(y_2)z_1 \right)[x_1, x_2] \right\}^n.$$  \hspace{1cm} (59)

and in particular, for $y_1 = z_2 = 0$ in (59), it follows that

$$p\left\{ z_1y_2 - \alpha(y_2)z_1 \right\}^n$$  \hspace{1cm} (60)

is satisfied by $R$, as well as by $Q_r$.

Now let’s fix any two elements $r_1, r_2 \in Q_r$ and denote $w = r_1r_2 - \alpha(r_2)r_1$. By (60)
we have that
\[ p \left\{ w[x_1, x_2] \right\}^n \]
is an identity for \( Q_r \). This last implies \( pw = 0 \) (see for instance [8, Theorem]). By the arbitrariness of \( r_1, r_2 \in Q_r \), it follows that \( Q_r \) satisfies the generalized identity
\[ p \left\{ z_1 y_2 - \alpha(y_2) z_1 \right\}. \]
Since \( p \neq 0 \), as above we get \( (r_1 r_2 - \alpha(r_2) r_1) = 0 \), for any \( r_1, r_2 \in Q_r \) (see [1, Theorem B and Corollary]). Once again, since \( R \) is not commutative, a contradiction follows. \( \square \)

**Availability of data and material.** No datasets were generated or analysed during the current study.

**References**


Vincenzo De Filippis (Corresponding Author) and Giovanni Scudo
Department of Engineering
University of Messina
98166 Messina, Italy
e-mails: defilippis@unime.it (V. De Filippis)
gscudo@unime.it (G. Scudo)

Nadeem ur Rehman
Department of Mathematics
Aligarh Muslim University
Aligarh, 202002 India
e-mail: rehman100@gmail.com (N. Rehman)