ON SOLUBILITY OF GROUPS WITH FINITELY MANY CENTRALIZERS

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Abstract. In this paper we present a new sufficient condition for a solubility criterion in terms of centralizers of elements. This result is a corrigendum of one of Zarrin’s results. Furthermore, we extend some of K. Khoramshahi and M. Zarrin’s results in the primitive case.

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1. Introduction

Let $G$ be a group, given $g \in G$ we define $C_G(g) = \{x \in G\mid xg = gx\}$ the centralizer of $g$ in $G$ and $Cent(G) = \{C_G(g)\mid g \in G\}$ the set of all centralizers of elements in $G$. Denote by $|W|$ the cardinal of the set $W$. If $|Cent(G)| = n \in \mathbb{N}$ we say that $G$ is a $C_n$-group or that $G$ is a $n$-centralizer group. If $G/Z(G)$ is an $n$-centralizer too, we say that $G$ is a primitive $n$-centralizer group, or simply primitive $n$-centralizer.

The study of finite groups in terms of $|Cent(G)|$ was started by Belcastro and Sherman in [3]. It is easy to see that a group is 1-centralizer if and only if it is abelian and there is no $n$-centralizer group for $n = 2, 3$. An $n$-centralizer group was constructed for each $n \neq 2, 3$ in [2]. We collect a few results in the following theorem.

Theorem 1.1. Suppose $G$ is a finite $n$-centralizer group. Then

1. $n = 4 \iff G/Z(G) \cong C_2 \times C_2$ (see [3]).
2. $n = 5 \iff G/Z(G) \cong C_3 \times C_3$ or $S_3$ (see [3]).
3. $n = 6 \Rightarrow G/Z(G) \cong D_8, A_4, C_2 \times C_2 \times C_2$ or $C_2 \times C_2 \times C_2 \times C_2$ (see [2]).
4. $n = 7 \iff G/Z(G) \cong C_5 \times C_5, D_{10}$ or $\langle x, y\mid x^5 = y^4 = 1, x^y = x^3 \rangle$ (see [1]).
5. $n = 8 \Rightarrow G/Z(G) \cong C_2 \times C_2 \times C_2, A_4$ or $D_{12}$ (see [1]).

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(6) \( n = 9 \iff G/Z(G) \cong D_{14}, C_7 \times C_7, \ Hol(C_7) \) or a non-abelian group of order 21 (see [6]).

(7) \( n = 10 \Rightarrow G/Z(G) \cong D_{16}, C_4^2, C_4 \times C_4, (C_4 \times C_2) \times C_2, C_2 \times D_8, C_5^9, C_3^3 \) or \( C_2^2 \times C_7 \) (see [7]).

(8) If \( G \) is a primitive 11-centralizer group of odd order, then \( G/Z(G) \cong (C_9 \times C_3) \times C_3 \) (see [10]).

The concept of isoclinic groups was introduced by P. Hall in [5]. Two groups \( G_1 \) and \( G_2 \) (not necessarily finite) are said to be isoclinic if there are isomorphisms \( \varphi : G_1/Z(G_1) \to G_2/Z(G_2) \) and \( \phi : G_1' \to G_2' \) such that if \( \varphi(a_1Z(G_1)) = a_2Z(G_2) \) and \( \phi(b_1Z(G_1)) = b_2(G_2) \), then \( \phi([a_1, b_1]) = [a_2, b_2] \) for each \( a_1, b_1 \in G_1 \) and \( a_2, b_2 \in G_2 \). It is easy to see that isoclinism is an equivalence relation.

As noted by P. Hall [5], every group \( G_2 \) which is isoclinic with \( G_1 \) also is isoclinic with the product \( G_1 \times A \), where \( A \) is an abelian group. Indeed, if \( G_1 \) is isoclinic with \( G_2 \) and \( G_3 \) is isoclinic with \( G_4 \), then the direct product \( G_1 \times G_3 \) is isoclinic with \( G_2 \times G_4 \). In particular if \( A \) is an abelian group, then \( A \) is isoclinic with the trivial group, say 1, and therefore \( G \) is isoclinic with \( G \times A \), for all group \( G \).

In [13] M. Zarrin establishes a relation between isoclinism and the number centralizers of elements of \( G \). He proves that if \( G_1 \) and \( G_2 \) are isoclinic, then \( |Cent(G_1)| = |Cent(G_2)| \). He also proves that if \( G \) is an arbitrary group with \( |Cent(G)| = n \), then there are only finitely many groups \( J \), up to isoclinism, with \( |Cent(J)| = n \), moreover, there exists a finite group \( K \) that is isoclinic with \( G \) and \( |Cent(G)| = |Cent(K)| \). Theorem 3.5 of the same article is an extension of Theorem 1.1 for arbitrary groups. Note that Zarrin proves in [13] the case \( |Cent(G)| \leq 8 \).

In this short paper we prove that if \( G \) is a finite \( n \)-centralizer group such that \( n \geq 4 \) and \( |G| < \frac{30n}{19} \), then \( G \) is a non-nilpotent solvable group. This fact is a correction of the proof of Theorem B (2) in [12]. Moreover, we extend the Theorem 3.5 in [8] in the primitive case.

Let \( I(G) \) be the set of all involutions of a group \( G \), that is, \( I(G) = \{a \in G \mid a = a^{-1} \} \). The problem with the proof of Theorem B (2) in [12] is that \( |I(G)| \geq \frac{|G|}{15} \) instead of \( |I(G)| > \frac{4|G|}{15} \) and we cannot apply Potter’s result, but this problem can be refined if we change the condition in the statement Theorem B (2) to \( |G| < \frac{30n}{19} \).

2. Preliminaries

We shall need the following results in [9] and [12] for the correction of Theorem B in [12]. For the convenience of the reader, we repeat the statements of the followings results.
Lemma 2.1. Let $G$ be a finite $C_n$-group. Then

$$n \leq \frac{|G| + |I(G)|}{2}.$$ 

Theorem 2.2 (Potter, 1988). Suppose $G$ admits an automorphism which inverts more than $\frac{4|G|}{15}$ elements. Then $G$ is solvable.

3. Correction

Now we are ready to prove the following theorem, which is similar to Theorem B in [12], using the same proof outline.

Theorem 3.1. If $G$ is a finite $n$-centralizer group with $n \geq 4$, then the following holds:

1. $|G| < 2n$, then $G$ is non-nilpotent.
2. $|G| < \frac{30n}{19}$, then $G$ is a non-nilpotent solvable group.

Proof. We will just prove part (2). From part (1), which is proved in Theorem B (1) in [12], we have that $G$ is non-nilpotent, since $|G| < \frac{30n}{19} < 2n$. Moreover, since $2n > \frac{19|G|}{15}$, Lemma 2.1 implies that

$$|I(G)| \geq 2n - |G| > \frac{4|G|}{15}.$$ 

Since $I(G)$ is the set of all elements of $G$ that is inverted by the identity automorphism, Theorem 2.2 completes the proof. □

The condition (2) above is better than the part (2) of Theorem B in [12]. However using a GAP check [11] we don’t know an example of a group $G$ such that $|G| < \frac{30n+15}{19}$ and $G$ is not a solvable group. It is immediate from Theorem 1.1 examples of groups where both conditions of Theorem 3.1 holds exist, for instance $G = S_3$ and $n = 5$.

4. A condition for isoclinism

We will need of a Lemma (see Lemma 3.3 in [8]).

Lemma 4.1. Let $H$ a subgroup of an arbitrary group $G$ such that $|\text{Cent}(H)| = |\text{Cent}(G)|$. Then $H \cap Z(G) = Z(H)$ and $\frac{H}{Z(H)} \cong \frac{HZ(G)}{Z(G)}$. In particular, $H$ is isoclinic with $HZ(G)$.

Using similar arguments we extend Theorem 3.5 in [8] and for the case $n = 11$, we add the hypothesis that $H$ is a primitive 11-centralizer group.
Theorem 4.2. Let $G$ be a non-abelian arbitrary group. If $H \leq G$, $|\text{Cent}(G)| = |\text{Cent}(H)| = n = 8$, then $H$ is isoclinic with $G$. This result still holds if $n = 11, H$ is primitive and $G$ is a primitive 11-centralizer of odd order.

Proof. From Lemma 4.1, $1 \neq \frac{H}{Z(H)} \cong \frac{HZ(G)}{Z(G)} \leq \frac{G}{Z(G)}$. By Zarrin’s Theorem 3.3 (2) [13] there is a finite group $K$ which is isoclinic with $G$ and $|\text{Cent}(G)| = |\text{Cent}(K)|$, so $G/Z(G) \cong K/Z(K)$. Let $|\text{Cent}(G)| = |\text{Cent}(K)| = n = 8$. From Theorem 1.1 we have that $K/Z(K) \cong G/Z(G) \cong C_2 \times C_2 \times C_2$, $A_4$ or $D_{12}$. If $\frac{HZ(G)}{Z(G)} < \frac{G}{Z(G)}$, we have that $\frac{H}{Z(H)} \cong C_2$, $C_2 \times C_2$, $C_3$, $C_6$, or $S_3$. If $\frac{H}{Z(H)}$ is cyclic, then $H$ is abelian, which is a contradiction. If $\frac{H}{Z(H)} \cong S_3$ or $C_2 \times C_2$, from Theorem 1.1, $|\text{Cent}(H)| = 5$ or 4, which is a contradiction. Therefore from Lemma 4.1 it follows that $H/Z(H) \cong \frac{HZ(G)}{Z(G)} = \frac{G}{Z(G)}$. Let $n = 11$ and suppose that $G$ is a primitive 11-centralizer group of odd order. From Theorem 1.1 we have that $G/Z(G) \cong (C_9 \times C_3) \times C_3$. If $\frac{HZ(G)}{Z(G)} < \frac{G}{Z(G)}$, we have that $\frac{H}{Z(H)} \cong C_3 \times C_3, C_9, C_3 \times C_3$, or $(C_3 \times C_3) \times C_3$. Again, $\frac{H}{Z(H)}$ can’t be cyclic. Using the GAP (see [10]), and the fact that $H$ is primitive, we can verify that if $\frac{H}{Z(H)} \cong C_3 \times C_3, C_9 \times C_3$ or $(C_3 \times C_3) \times C_3$ then $11 = |\text{Cent}(H)| = |\text{Cent}(\frac{H}{Z(H)})| = 1$ or 5, which is a contradiction. Therefore from Lemma 4.1 it follows that $H/Z(H) \cong \frac{HZ(G)}{Z(G)} = \frac{G}{Z(G)}$. In either case we obtain $\frac{H}{Z(H)} \cong \frac{HZ(G)}{Z(G)} = \frac{G}{Z(G)}$, so $HZ(G) = G$. Again by Lemma 4.1, $H$ is isoclinic with $HZ(G) = G$. □

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References


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