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STRONGLY EP ELEMENTS AND THE SOLUTIONS OF EQUATIONS IN RINGS

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Abstract. In this paper, we present many new characterizations of strongly EP elements in rings with involution. We especially investigate the strongly EP elements by constructing certain equations and considering the solutions of equations, revealing the existence of solutions of certain equations and the general solutions of some binary equations that play a role in characterizing strongly EP elements. Proofs of relevant conclusions are also given.

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1. Introduction

Throughout this paper, R is an associative ring with 1. Let a be an element that belongs to R. Then a is said to be group invertible if there exists $a^{\#} \in R$ such that

$$aa^{\#}a = a, \quad a^{\#}aa^{\#} = a^{\#}, \quad aa^{\#} = a^{\#}a.$$

The element $a^{\#}$ is called the group inverse of a and it is uniquely determined by these equations [1]. We use $R^{\#}$ to denote the set of all group invertible elements of R.

An involution $* : a \longmapsto a^*$ in a ring R is an anti-isomorphism of degree 2, that is,

$$(a^*)^* = a, \quad (a+b)^* = a^* + b^*, \quad (ab)^* = b^*a^* \quad \text{for} \quad \text{any} \quad a, b \in R.$$

An element $a \in R$ satisfying $aa^* = a^*a$ is said to be normal.

We say the element $b = a^{\dagger}$ is the Moore-Penrose inverse (or MP-inverse) of a, if the following conditions hold (see [13]):

$$aba = a, \quad bab = b, \quad (ab)^* = ab, \quad (ba)^* = ba.$$

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There is at most one element $b(=a^\dagger)$ satisfying the above conditions (see [8]). The set of all MP-invertible elements of R will be denoted by $R^\dagger$. We say $a \in R$ is EP if a belongs to $R^\# \cap R^\dagger$ and satisfies $a^\# = a^\dagger$ [6]. The set of all EP elements of R is denoted by R^{EP}.

An element $a \in R$ satisfying $a = aa^*a$ is called a partial isometry. If $a \in R^{EP}$ is partial isometry, then a is called a strongly EP element. We respectively denote the sets of all partial isometry elements and strongly EP elements of R by R^{PI} and R^{SEP}.

EP elements have been investigated by many authors. In [11], many new characterizations of EP elements in rings with involution in purely algebraic terms are presented. At the same time, D. Mosić and D. S. Djordjević gave some equivalent conditions under which an element in R is an EP element in [12]. S. Z. Xu, J. L. Chen and J. Benítez in [14] showed that the EP elements in R can be characterized by three equations, which are $(xa)^* = xa$, $xa^2 = a$ and $ax^2 = x$. In addition, more interesting results on EP elements and partial isometries can also be found in [2,3,7,10,9,14,18].

In [4], using the generalized inverse of elements, common solutions of linear equations in a ring are discussed. Interesting research in this direction can be found in the literatures [5,16,17]. Recently, by means of the solution of constructed equations, a new kind of characterizations of generalized inverse elements are studied such as [15]. Motivated by these results above, this paper mainly considers the new characterizations of the strongly EP element, which is a special kind of EP element. A number of equivalent conditions are given to characterize these generalized inverses. Different from some existing research, we especially consider the characterizations of partial isometries and strongly EP elements from the perspective of the solutions of equations, which is a new way to study generalized inverses. This is an interesting and meaningful job.

2. Results

On the basis of the existing research results, we study the relationship between the characterization of the EP element and the solutions of certain equations. By observing the established equalities, we describe some equations, and constantly deform and extend these equations. From the perspectives of the existence of the solutions of certain equations and the expression of the general solutions of some binary equations, a series of equivalent conditions for the elements in rings with involution to become EP elements are given.

To facilitate the proof of the theorems later, we first show a useful auxiliary lemma.

Lemma 2.1. *Let $a \in R^{\#} \cap R^{\dagger}$. Then $a^{\#}a(a^{\dagger})^* = (a^{\dagger})^* = (a^{\dagger})^* aa^{\#}$.*

Proof. Since $(a^{\dagger})^* = aa^{\dagger}(a^{\dagger})^*$, we get

$$a^{\#}a(a^{\dagger})^* = a^{\#}a^2a^{\dagger}(a^{\dagger})^* = aa^{\dagger}(a^{\dagger})^* = (a^{\dagger})^*.$$

Similarly, we have $(a^{\dagger})^*aa^{\#} = (a^{\dagger})^*$. □

In the following research, we let $a \in R^{\#} \cap R^{\dagger}$ and $\chi_a = \{a, a^{\#}, a^{\dagger}, a^*, (a^{\#})^*, (a^{\dagger})^*\}$. Observing Lemma 2.1, we can construct the following equation.

$$a^{\#}ax = x. \tag{1}$$

Theorem 2.2. *Let $a \in R^{\#} \cap R^{\dagger}$. Then $a \in R^{EP}$ if and only if the equation* (1) *has at least four solutions in χ_a.*

Proof. By Lemma 2.1, we know that $a, a^{\#}$ and $(a^{\dagger})^*$ are always the solutions of the equation (1).

($\Rightarrow$) Assume that $a \in R^{EP}$, then $a^{\#} = a^{\dagger}$. This implies that $x = a^{\dagger}$ is also a solution.

($\Leftarrow$) 1) If $x = a^{\dagger}$ is a solution, then $a^{\dagger} = a^{\#}aa^{\dagger}$, this gives $a \in R^{EP}$ by [11, Theorem 2.1];

2) If $x = a^*$ is a solution, then $a^* = a^{\#}aa^*$, this infers $a \in R^{EP}$ by [11, Theorem 2.1];

3) If $x = (a^{\#})^*$ is a solution, then $(a^{\#})^* = a^{\#}a(a^{\#})^*$. Post-multiplying the equality by $(a^*)^2$, one has $a^* = a^{\#}aa^*$. Hence $a \in R^{EP}$ by [11, Theorem 2.1]. □

Remark 2.3. The general solution of the equation (1) is given by

$$x = a^{\#} + u - (1 - a^{\#}a)u,$$

where $u \in R$ is arbitrary.

Note that $a \in R^{EP}$ if and only if $a^{\#} = a^{\dagger}$. Hence Theorem 2.2 implies $a \in R^{EP}$ if and only if the general solution of the equation (1) is given by

$$x = a^{\dagger} + u - (1 - a^{\#}a)u,$$

where $u \in R$ is arbitrary.

It is well known that $a \in R^{PI}$ if and only if $a = (a^{\dagger})^*$. Thus Lemma 2.1 infers the following corollary.

Corollary 2.4. *Let $a \in R^{\#} \cap R^{\dagger}$. Then the following conditions are equivalent:*

(1) $a \in R^{PI}$;

(2) $a^{\#}a(a^{\dagger})^{*} = a$;

(3) $(a^{\dagger})^{*}aa^{\#} = a$.

Inspired by Corollary 2.4, we can construct the following equation.

$$a^{\#}x(a^{\dagger})^{*} = x. \tag{2}$$

Theorem 2.5. *Let $a \in R^{\#} \cap R^{\dagger}$. Then $a \in R^{PI}$ if and only if the equation* (2) *has at least one solution in χ_a.*

Proof. ($\Rightarrow$) Assume that $a \in R^{PI}$, then $x = a$ is a solution of the equation (2).

($\Leftarrow$) 1) If $x = a$ is a solution, then $a^{\#}a(a^{\dagger})^{*} = a$. It follows from Lemma 2.1 that $(a^{\dagger})^{*} = a$. Hence $a \in R^{PI}$;

2) If $x = a^{\#}$ is a solution, then $a^{\#}a^{\#}(a^{\dagger})^{*} = a^{\#}$. Pre-multiplying the equality by a^2, one has $aa^{\#}(a^{\dagger})^{*} = a^{\#}a(a^{\dagger})^{*} = a$, which gives $a \in R^{PI}$ by 1);

3) If $x = a^{\dagger}$ is a solution, then $a^{\#}a^{\dagger}(a^{\dagger})^{*} = a^{\dagger}$. Pre-multiplying the equality by $a^{*}a^2$, one yields $a^{\dagger}a = a^{*}a^2a^{\dagger}$. Post-multiplying the equality by $aa^{\#}a^{\dagger}$, we get $a^{\dagger} = a^{*}$. Hence $a \in R^{PI}$;

4) If $x = a^{*}$ is a solution, then $a^{\#}a^{*}(a^{\dagger})^{*} = a^{*}$. That is $a^{\#} = a^{*}$. Therefore, $a \in R^{PI}$ by [12, Theorem 2.2];

5) If $x = (a^{\#})^{*}$ is a solution, then $a^{\#}(a^{\#})^{*}(a^{\dagger})^{*} = (a^{\#})^{*}$. Applying the involution to the equality, one yields $a^{\#} = a^{\dagger}a^{\#}(a^{\#})^{*}$. Pre-multiplying the last equality by $(a^{\dagger})^{*}a^2$ and using Lemma 2.1, we have $(a^{\dagger})^{*}a = (a^{\dagger})^{*}(a^{\#})^{*}$. It follows from [12, Theorem 2.3(x)] that $a \in R^{PI}$;

6) If $x = (a^{\dagger})^{*}$ is a solution, then $a^{\#}(a^{\dagger})^{*}(a^{\dagger})^{*} = (a^{\dagger})^{*}$. Post-multiplying the equality by $a^{*}aa^{\#}$ and using Lemma 2.1, one gets $a^{\#}(a^{\dagger})^{*} = aa^{\#}$. Pre-multiplying the equality by a and again using Lemma 2.1, one has $(a^{\dagger})^{*} = a$, which shows $a \in R^{PI}$. $\square$

Remark 2.6. The equation (2) can be generalized as follows

$$a^{\#}x(a^{\dagger})^{*} - y = 0. \tag{3}$$

The general solution of the equation (3) is given by

$$x = -aua^{*} + v - aa^{\#}vaa^{\dagger},$$

$$y = -a^{\#}aua^{\dagger}a,$$

where $u, v \in R$.

Hence $a \in R^{PI}$ if and only if the general solution of the equation (3) is given by

$$x = -(a^\dagger)^* u a^* + v - a a^\# v a a^\dagger,$$
$$y = -a^\# a u a^\dagger a,$$

where $u, v \in R$.

According to [12, Theorem 2.2], we know $a \in R^{SEP}$ if and only if $a^\# = a^*$. Hence Lemma 2.1 leads to the following corollary.

Corollary 2.7. *Let $a \in R^\# \cap R^\dagger$. Then the following conditions are equivalent:*

(1) $a \in R^{SEP}$;
(2) $a^* a (a^\dagger)^* = (a^\dagger)^*$;
(3) $(a^\dagger)^* a a^* = (a^\dagger)^*$.

We then can give the following equation by Corollary 2.7.

Example 2.8. Let $R = \mathbb{Z}_2^{3\times 3}$. Then we define $a^* = a^T$ for any $a \in R$. Take $a = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, then we have $a^\# = a$, $a^\dagger = a^* = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$. Hence we get $a \in R^{PI}$ while $a \notin R^{SEP}$.

Now, we change the equation (2) into equation (4) as follows:

$$axa = x. \tag{4}$$

Obviously, we can prove that every element in χ_a is the solution of the equation (4).

Remark 2.9. If $a \in R^{PI}$, then the general solution of the equation (3) is given by

$$x = a^\dagger + u - (1 - a^\dagger a)u,$$

where $u \in R$ is arbitrary.

Observing Corollary 2.7, we can easily obtain the following lemma which gives a characterization of normal elements.

Lemma 2.10. *Let $a \in R^\# \cap R^\dagger$. Then a is normal if and only if $a^* a (a^\dagger)^* = a$.*

From Lemma 2.10, we can construct the following equation.

$$a^* x (a^\dagger)^* = x. \tag{5}$$

Clearly, the equation (5) is solvable in χ_a if and only if $x = a^\dagger a x a^\dagger a$ for each solution x in χ_a. Hence we obtain the following lemma.

Lemma 2.11. *Let $a \in R^{\#} \cap R^{\dagger}$. Then $a \in R^{EP}$ if and only if the equation $x = a^{\dagger}axa^{\dagger}a$ has at least one solution in χ_a.*

Evidently, if $a \in R^{EP}$, then $x = a^*$ is a solution of the equation (5). Therefore, Lemma 2.11 leads to the following theorem.

Theorem 2.12. *Let $a \in R^{\#} \cap R^{\dagger}$. Then $a \in R^{EP}$ if and only if the equation* (5) *has at least one solution in χ_a.*

Remark 2.13. Let $a \in R^{EP}$. Then the general solution of the equation (5) is given by

$$x = -(a^{\dagger})^* u a^* + v - aa^{\dagger}vaa^{\dagger},$$

where $u, v \in R$ are satisfying $aa^{\dagger}uaa^{\dagger} = (a^{\dagger})^* ua^* - v + aa^{\dagger}vaa^{\dagger}$.

By Lemma 2.1, we have $a^{\#}a(a^{\dagger})^* = (a^{\dagger})^* = (a^{\dagger})^* aa^{\#}$. Applying the involution to the equality, one has

$$a^{\dagger}a^*(a^{\#})^* = a^{\dagger} = (a^{\#})^* a^* a^{\dagger}. \tag{6}$$

Combining the equality (6) with the fact that $a \in R^{SEP}$ if and only if $(a^{\#})^* = a$, we have the following lemma.

Lemma 2.14. *Let $a \in R^{\#} \cap R^{\dagger}$. Then the following conditions are equivalent:*

(1) $a \in R^{SEP}$;

(2) $a^{\dagger}a^*a = a^{\dagger}$;

(3) $aa^*a^{\dagger} = a^{\dagger}$.

Note that $a^{\dagger}a^*a \in R^{EP}$ with $(a^{\dagger}a^*a)^{\dagger} = a^{\dagger}(a^{\#})^*a = (a^{\dagger}a^*a)^{\#}$. Lemma 2.14 gives the following theorem.

Theorem 2.15. *Let $a \in R^{\#} \cap R^{\dagger}$. Then the following conditions are equivalent:*

(1) $a \in R^{SEP}$;

(2) $a^{\dagger}(a^{\#})^*a = a$;

(3) $a(a^{\#})^*a^{\dagger} = a$.

Post-multiplying the equality (2) of Theorem 2.15 by $a^{\dagger}$ and pre-multiplying the equality (3) of Theorem 2.15 by $a^{\dagger}$, we have the next corollary.

Corollary 2.16. *Let $a \in R^{\#} \cap R^{\dagger}$. Then the following conditions are equivalent:*

(1) $a \in R^{SEP}$;

(2) $a^\dagger (a^\#)^* = aa^\dagger$;
(3) $(a^\#)^* a^\dagger = a^\dagger a$;
(4) $a^\# (a^\dagger)^* = aa^\dagger$;
(5) $(a^\dagger)^* a^\# = a^\dagger a$.

Post-multiplying the equality (2) of Corollary 2.16 by a^* and pre-multiplying the equality (3) of Corollary 2.16 by a^*, with the help of equality (6), we have the following corollary which appears in [12, Theorem 2.3].

Corollary 2.17. *Let $a \in R^\# \cap R^\dagger$. Then the following conditions are equivalent:*
(1) $a \in R^{SEP}$;
(2) $a^\dagger = aa^\dagger a^*$;
(3) $a^\dagger = a^* a^\dagger a$.

Condition (2) of Corollary 2.17 gives the following equation.

$$x = axa^*. \tag{7}$$

Then, the following theorem follows from [12, Theorem 2.3] and Corollary 2.17.

Theorem 2.18. *Let $a \in R^\# \cap R^\dagger$. Then $a \in R^{SEP}$ if and only if the equation* (7) *has at least one solution in χ_a.*

Note that if $a \in R^{SEP}$, then $a = (a^\dagger)^*$ and $a^* = a^\dagger$. Therefore, we can obtain the following equation from equation (7).

$$x = (a^\dagger)^* x a^\dagger. \tag{8}$$

Theorem 2.19. *Let $a \in R^\# \cap R^\dagger$. Then $a \in R^{SEP}$ if and only if the equation* (8) *has at least one solution in χ_a.*

If $a \in R^{SEP}$, by Corollary 2.17, one has $a^\dagger (a^\dagger)^* a^* = (a^\dagger)^* a^\dagger a^\dagger$. Then we can construct the following equation.

$$x(a^\dagger)^* a^* = (a^\dagger)^* x a^\dagger. \tag{9}$$

Theorem 2.20. *Let $a \in R^\# \cap R^\dagger$. Then $a \in R^{PI}$ if and only if the equation* (9) *has at least one solution in χ_a.*

Proof. ($\Rightarrow$) Assume that $a \in R^{PI}$, then $a^* = a^\dagger$. This infers $x = a$ is a solution.

($\Leftarrow$) 1) If $x = a$ is a solution, then $a(a^\dagger)^* a^* = (a^\dagger)^* aa^\dagger$. It follows that $a^2 a^\dagger = (a^\dagger)^* aa^\dagger$. Post-multiplying the equality by $aa^\#$, one obtains $a = (a^\dagger)^* aa^\#$. By Lemma 2.1, we have $a = (a^\dagger)^*$. Hence $a \in R^{PI}$;

2) If $x = a^{\#}$ is a solution, then $a^{\#}(a^{\dagger})^* a^* = (a^{\dagger})^* a^{\#} a^{\dagger}$. Post-multiplying the equality by a^2, one has $a = (a^{\dagger})^* a^{\#} a$. Hence $a \in R^{PI}$ by Lemma 2.1;

3) If $x = a^{\dagger}$ is a solution, then $a^{\dagger}(a^{\dagger})^* a^* = (a^{\dagger})^* a^{\dagger} a^{\dagger}$, that is $a^{\dagger} = (a^{\dagger})^* a^{\dagger} a^{\dagger}$. Post-multiplying the equality by a and then applying the involution to the last equation, we have $a^{\dagger} a = a^{\dagger} a (a^{\dagger})^* a^{\dagger}$. Pre-multilying the last equality by $a^{\#} a$ and using Lemma 2.1, one obtains $a^{\#} a = (a^{\dagger})^* a^{\dagger}$. Post-multiplying the equality $a^{\#} a = (a^{\dagger})^* a^{\dagger}$ by a, we have $a = (a^{\dagger})^*$. Hence $a \in R^{PI}$;

4) If $x = a^*$ is a solution, then $a^*(a^{\dagger})^* a^* = (a^{\dagger})^* a^* a^{\dagger}$, that is $a^* = a a^{\dagger} a^{\dagger}$. Post-multiplying the equality by a and then applying the involution, one has $a^* a = a^{\dagger} a^2 a^{\dagger}$. Again post-multiplying the last equality by a, we have $a^* a^2 = a^{\dagger} a^2$. Hence $a \in R^{PI}$ by [12, Theorem 2.2];

5) If $x = (a^{\#})^*$ is a solution, then $(a^{\#})^*(a^{\dagger})^* a^* = (a^{\dagger})^*(a^{\#})^* a^{\dagger}$. Note that $a a^{\dagger} a^{\#} = a^{\#}$. Then $(a^{\#})^* = (a^{\dagger})^*(a^{\#})^* a^{\dagger}$. Applying the involution to the last equality, one yields that $a^{\#} = (a^{\dagger})^* a^{\#} a^{\dagger}$. Post-multiplying the equality by a^2, we have $a = (a^{\dagger})^* a a^{\#}$. Consequently, we deduce that $a \in R^{PI}$ by Lemma 2.1;

6) If $x = (a^{\dagger})^*$ is a solution, then $(a^{\dagger})^*(a^{\dagger})^* a^* = (a^{\dagger})^*(a^{\dagger})^* a^{\dagger}$. Applying the involution to the equality, one has $a a^{\dagger} a^{\dagger} = (a^{\dagger})^* a^{\dagger} a^{\dagger}$. Pre-multiplying the last equality by a^*, one has $a^* a^{\dagger} = a^{\dagger} a^{\dagger}$. Hence $a \in R^{PI}$. $\square$

Example 2.21. Let $R = \mathbb{Z}_2^{3\times 3}$. Then we define $a^* = a^T$ for any $a \in R$. Take $a = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, then we have $a^{\#} = a$, $a^{\dagger} = a^* = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$. Then the equation (9) only has three solutions in χ_a, which are $x = a, x = a^{\#}, a = (a^{\dagger})^*$.

Next, we change the equation (9) as follows.

$$x a^* (a^{\dagger})^* = (a^{\dagger})^* x a^{\dagger}. \tag{10}$$

Theorem 2.22. *Let $a \in R^{\#} \cap R^{\dagger}$. Then $a \in R^{SEP}$ if and only if the equation* (10) *has at least one solution in χ_a.*

Proof. ($\Rightarrow$) Assume that $a \in R^{SEP}$, then $a^* = a^{\dagger} = a^{\#}$, this infers $x = a$ is a solution.

($\Leftarrow$) 1) If $x = a$ is a solution, then $a a^*(a^{\dagger})^* = (a^{\dagger})^* a a^{\dagger}$. It follows that $a = (a^{\dagger})^* a a^{\dagger}$. Hence $a^2 a^{\dagger} = (a^{\dagger})^* a a^{\dagger}$. This gives $a \in R^{PI}$ by the proof of 1) in Theorem 2.20. Now we have $a = (a^{\dagger})^* a a^{\dagger} = a^2 a^*$ because $a = (a^{\dagger})^*$ and $a^{\dagger} = a^*$. By [12, Theorem 2.3(xx)], we have $a \in R^{SEP}$;

2) If $x = a^{\#}$ is a solution, then $a^{\#} a^* (a^{\dagger})^* = (a^{\dagger})^* a^{\#} a^{\dagger}$, that is $a^{\#} = (a^{\dagger})^* a^{\#} a^{\dagger}$. Post-multiplying the equality by a^2 and using Lemma 2.1, one has $a = (a^{\dagger})^*$. Hence

$a \in R^{PI}$. Now we have $a^{\#} = (a^{\dagger})^* a^{\#} a^{\dagger} = aa^{\#} a^{\dagger}$. Multiplying the equality by a on the left, we obtain $aa^{\#} = aa^{\dagger}$. Consequently, $a \in R^{EP}$ and then $a \in R^{SEP}$;

3) If $x = a^{\dagger}$ is a solution, then $a^{\dagger} a^* (a^{\dagger})^* = (a^{\dagger})^* a^{\dagger} a^{\dagger}$, that is $a^{\dagger} a^{\dagger} a = (a^{\dagger})^* a^{\dagger} a^{\dagger}$. Post-multiplying the equality by $a^*(a^{\#})^*$ and using the equality (6), one gets $a^{\dagger} = (a^{\dagger})^* a^{\dagger} a^{\dagger}$. Hence, by the proof of 3) of Theorem 2.20, we have $a \in R^{PI}$. Note that $a^{\dagger} = a^{\dagger} a^{\dagger} a$. Then $a^{\dagger} = a^* a^{\dagger} a$, it follows from [12, Theorem 2.3(xviii)] that $a \in R^{SEP}$;

4) If $x = a^*$ is a solution, then $a^* a^* (a^{\dagger})^* = (a^{\dagger})^* a^* a^{\dagger}$, that is $a^* a^{\dagger} a = aa^{\dagger} a^{\dagger}$. Post-multiplying the equality by $a^*(a^{\#})^*$ and then using equality (6), one has $a^* = aa^{\dagger} a^{\dagger}$. By the proof of 4) of Theorem 2.20, we get $a \in R^{PI}$. Note that $a^* = aa^{\dagger} a^{\dagger} = a^* a^{\dagger} a$. Then $a^{\dagger} = aa^{\dagger} a^{\dagger} = a^* a^{\dagger} a$, this implies $a \in R^{SEP}$;

5) If $x = (a^{\#})^*$ is a solution, then $(a^{\#})^* a^* (a^{\dagger})^* = (a^{\dagger})^* (a^{\#})^* a^{\dagger}$. Post-multiplying the equality by $a^*(a^{\#})^*$ and using the equality (6), one has $(a^{\#})^* = (a^{\dagger})^* (a^{\#})^* a^{\dagger}$. By the proof of 5) of Theorem 2.20, we get $a \in R^{PI}$. Now we have $(a^{\#})^* = (a^{\dagger})^* (a^{\#})^* a^{\dagger} = (a^{\#})^* a^* (a^{\dagger})^*$, and it follows that $a^{\#} = a^{\dagger} a a^{\#}$. Hence $a \in R^{EP}$. Consequently, $a \in R^{SEP}$;

6) If $x = (a^{\dagger})^*$ is a solution, then $(a^{\dagger})^* a^* (a^{\dagger})^* = (a^{\dagger})^* (a^{\dagger})^* a^{\dagger}$. Applying the involution to the equality, one gets $a^{\dagger} = (a^{\dagger})^* a^{\dagger} a^{\dagger}$. Pre-multiplying the last equality by $a^{\#} a$ and using Lemma 2.1, one has $a^{\#} a a^{\dagger} = a^{\dagger}$, which gives $a \in R^{EP}$. This leads to $a^{\#} a = a^{\dagger} a = (a^{\dagger})^* a^{\dagger} a^{\dagger} a = (a^{\dagger})^* a^{\dagger}$. By the proof of 3) of Theorem 2.20, we know that $a \in R^{PI}$. Thus $a \in R^{SEP}$. □

We generalize the equation (8) as follows

$$x - (a^{\dagger})^* y a^{\dagger} = 0. \tag{11}$$

Proposition 2.23. *The general solution of the equation* (11) *is given by*

$$\begin{cases} x = -aa^{\dagger} p a a^{\dagger} \\ y = -a^* p a + z - a^{\dagger} a z a^{\dagger} a \end{cases}, \text{ where } p, z \in R. \tag{12}$$

Proof. First, the formula (12) is exactly the solution of the equation (11).

Next, assume that $\begin{cases} x = x_0 \\ y = y_0 \end{cases}$ is a solution of the equation (11). Thus we get $x_0 = (a^{\dagger})^* y_0 a^{\dagger}$.

Note that

$$-aa^{\dagger}((a^{\dagger})^* y_0 a^{\dagger}) aa^{\dagger} = -(a^{\dagger})^* y_0 a^{\dagger} = -x_0.$$

Hence we obtain

$$x_0 = -aa^{\dagger}(-(a^{\dagger})^* y_0 a^{\dagger}) aa^{\dagger}.$$

Moreover, we find that

$$\begin{aligned}-a^*(-x_0)a + y_0 - a^\dagger a y_0 a^\dagger a &= -a^*(-(a^\dagger)^* y_0 a^\dagger)a + y_0 - a^\dagger a y_0 a^\dagger a \\ &= a^\dagger a y_0 a^\dagger a + y_0 - a^\dagger a y_0 a^\dagger a = y_0.\end{aligned}$$

Therefore, the formula (12) is the general solution of the equation (11). $\square$

Corollary 2.24. (1) *Let $a \in R^{\#} \cap R^{\dagger}$. Then $a \in R^{EP}$ if and only if the general solution of the equation* (11) *is given by*

$$\begin{cases} x = -aa^\dagger p a a^{\#} \\ y = -a^* p a + z - a^\dagger a z a^\dagger a \end{cases}, \text{ where } p, z \in R. \tag{13}$$

(2) *Let $a \in R^{\#} \cap R^{\dagger}$. Then $a \in R^{PI}$ if and only if the general solution of the equation* (11) *is given by*

$$\begin{cases} x = -aa^\dagger p a a^{\dagger} \\ y = -a^\dagger p a + z - a^\dagger a z a^\dagger a \end{cases}, \text{ where } p, z \in R. \tag{14}$$

(3) *Let $a \in R^{\#} \cap R^{\dagger}$. Then $a \in R^{SEP}$ if and only if the general solution of the equation* (11) *is given by*

$$\begin{cases} x = -aa^\dagger p a a^{\#} \\ y = -a^\dagger p a + z - a^\dagger a z a^\dagger a \end{cases}, \text{ where } p, z \in R. \tag{15}$$

Corollary 2.25. *Let $a \in R^{\#} \cap R^{\dagger}$. Then a is normal if and only if the general solution of the equation* (11) *is given by*

$$\begin{cases} x = -aa^\dagger p a a^{\#} \\ y = -a^\dagger a^* a p a + z - a^\dagger a z a^\dagger a \end{cases}, \text{ where } p, z \in R. \tag{16}$$

Proof. ($\Rightarrow$) Assume that a is normal. Then $a \in R^{EP}$ and $a^* = a^\dagger a^* a$. Hence the formula (12) is the same as the formula (16), which is the solution of the equation (11) by Proposition 2.23.

($\Leftarrow$) If the general solution of the equation (11) is given by formula (16), then

$$-aa^\dagger p a a^{\#} - (a^\dagger)^*(-a^\dagger a^* a p a + z - a^\dagger a z a^\dagger a)a^\dagger = 0.$$

That is $aa^\dagger p a a^{\#} = (a^\dagger)^* a^\dagger a^* a p a a^\dagger$ for any $p \in R$. Especially, choose $p = 1$, we have $aa^{\#} = (a^\dagger)^* a^\dagger a^* a^2 a^\dagger$. Post-multiplying the equation by $aa^\dagger$, one yields $aa^{\#} = aa^\dagger$. Hence $a \in R^{EP}$.

Now we obtain $aa^{\#} = (a^\dagger)^* a^\dagger a^* a^2 a^\dagger = (a^\dagger)^* a^\dagger a^* a$. Thus we deduce that

$$a^* = a^* a a^\dagger = a^* a a^{\#} = a^*(a^\dagger)^* a^\dagger a^* a = a^\dagger a^* a,$$

$$aa^* = aa^\dagger a^* a = a^\dagger a a^* a = a^* a,$$

which implies a is normal. □

We still have two questions as follows:

Question 2.26. What is the expression of the general solution of the equation $xa^*(a^\dagger)^* - (a^\dagger)^* y a^\dagger = 0$?

Question 2.27. Let $a \in R^{\#} \cap R^{\dagger}$. When can $aa^\dagger - aa^{\#}$ accurately be a regular element?

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