

ROTA—BAXTER OPERATORS ON $\text{Cur}(\mathfrak{sl}_2(\mathbb{C}))$

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ABSTRACT. We classify all Rota—Baxter operators on the simple Lie conformal algebra $\text{Cur}(\mathfrak{sl}_2(\mathbb{C}))$ and clarify which of them arise from the solutions to the conformal classical Yang—Baxter equation due to the connection discovered by Y. Hong and C. Bai in 2020.

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1. Introduction

A linear operator R defined on an algebra A is called a Rota—Baxter operator, if the following relation holds for all $a, b \in A$,

$$R(a)R(b) = R(R(a)b + aR(b) + kab).$$

Here k is a fixed scalar from the ground field, which is called a weight of R . Nowadays, we observe the growing interest to such operators. Defined by G. Baxter in 1960 [5] as an abstract generalization of the integral operator, Rota—Baxter operators showed their importance due to close connection with the Yang—Baxter equation [1,6], pre- and postalgebras [3,15] (among them pre-Lie algebras (also called left-symmetric algebras) are of the most interest), double Lie algebras [13,14]. For more details see the monograph of L. Guo [17].

A notion of Lie conformal algebra introduced by V.G. Kac in [19] is an important tool to study vertex operator algebras. In turn, vertex algebras describe algebraic properties of the operator product expansion (OPE) in the two-dimensional conformal field theory developed by A.A. Belavin, A.M. Polyakov and A.B. Zamolodchikov [7] in 1983. In 1986, R. Borcherds confirmed the deep connection between

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vertex algebras and the Monster group [9]. At the moment, vertex algebras form an actively studied area related to many others such as representation theory and mathematical physics, see [12,22].

In 2008, J. Liberati introduced [23] a notion of Lie conformal bialgebra and suggested the so called conformal classical Yang—Baxter equation as a source of (coboundary) Lie conformal bialgebras. In 2012, this theory was extended by C. Boyallian and J. Liberati for Lie pseudobialgebras [10].

In 2020, Y. Hong and C. Bai introduced [18] a notion of Rota—Baxter operator on a Lie conformal algebra and showed that every skew-symmetric solution to the conformal classical Yang—Baxter equation on a Lie conformal algebra L endowed with an invariant bilinear nondegenerate form gives rise to a Rota—Baxter operator on L . This result reproduces the well-known connection between Rota—Baxter operators and solutions to the classical Yang—Baxter equation on a finite-dimensional semisimple Lie algebra [6]. A connection between associative algebras and the associative Yang—Baxter equation, which does not involve any kind of form, was found by M. Aguiar [1].

Pseudoalgebras defined by B. Bakalov, A. D’Andrea and V.G. Kac in [4] serve as a natural generalization of conformal algebras involving a cocommutative Hopf algebra H . For $H = \{e\}$, we get ordinary algebras and for $H = F[\partial]$ we obtain exactly conformal algebras. A notion of Rota—Baxter operator on $(H-)$ pseudoalgebras was suggested by L. Liu and S. Wang in 2020 [24]. Note that this definition applied to conformal algebras differs from the one given by Y. Hong and C. Bai.

Properties and cohomologies of Rota—Baxter operators on Lie conformal algebras were recently studied in [25,28]. In [27], cohomologies of associative conformal Rota—Baxter algebras were defined.

In [2], A. D’Andrea and V.G. Kac proved the following structure result: Every simple Lie conformal algebra of finite type is isomorphic either to the Virasoro conformal algebra Vir or to the current Lie conformal algebra $\text{Cur}(\mathfrak{g})$ associated to a simple finite-dimensional Lie algebra \mathfrak{g} . In [18], the authors showed that there are no nontrivial Rota—Baxter operators on Vir .

The main goal of the current work is to describe all Rota—Baxter operators on the remaining simple Lie conformal algebra $\text{Cur}(\mathfrak{sl}_2(\mathbb{C}))$ of rank 3. We solve this problem completely, dealing with zero and nonzero cases separately. Since the classification of Rota—Baxter operators on $\mathfrak{sl}_2(\mathbb{C})$ was known [20,21,26], we can trivially extend them onto $\text{Cur}(\mathfrak{sl}_2(\mathbb{C}))$. However, we find other Rota—Baxter operators, that depend on the choice of a polynomial $q(\partial)$.

In [16], all solutions to the classical Yang—Baxter equation on $\text{Cur}(\mathfrak{sl}_2(\mathbb{C}))$ were described. Hence, we may apply the connection established by Y. Hong and C. Bai to get Rota—Baxter operators of weight 0 on $\text{Cur}(\mathfrak{sl}_2(\mathbb{C}))$ corresponding to these solutions. Doing this, we get all Rota—Baxter operators of weight zero that are defined with an odd polynomial $q(\partial)$.

Let us give a short outline of the work. In §2, we list the required preliminaries on Lie conformal algebras, the conformal classical Yang—Baxter equation, and Rota—Baxter operators. For our purposes, we prove more explicit classification of Rota—Baxter operators of nonzero weight on $\mathfrak{sl}_2(\mathbb{C})$. In §3, we firstly describe Rota—Baxter operators of weight 0 on $\text{Cur}(\mathfrak{sl}_2(\mathbb{C}))$ and then show which of them come from the solutions to the conformal classical Yang—Baxter equation. In §4, we classify Rota—Baxter operators of nonzero weight on $\text{Cur}(\mathfrak{sl}_2(\mathbb{C}))$. We also provide examples of Rota—Baxter operators defined on any finite-dimensional semisimple Lie algebra over \mathbb{C} .

Throughout the work, all algebras and vector spaces are considered over the field of complex numbers.

2. Preliminaries

2.1. Rota—Baxter operators on Lie algebras. Let \mathfrak{g} be a Lie algebra. A linear operator R on \mathfrak{g} is called a Rota—Baxter operator (RB-operator, for short) of weight $k \in \mathbb{C}$ if

$$[R(a), R(b)] = R([R(a), b] + [a, R(b)] + k[a, b]) \tag{1}$$

for all $a, b \in \mathfrak{g}$.

Note that given an RB-operator R of weight 0 on a Lie algebra \mathfrak{g} , the linear operator αR is again an RB-operator of weight 0 on \mathfrak{g} for any $\alpha \in \mathbb{C}$.

Given a Lie algebra L , we call the operators $R = 0$ and $R = -k \cdot \text{id}$ as trivial Rota—Baxter operators of weight k on L .

Lemma 2.1 ([17]). *Let R be an RB-operator of weight k on a Lie algebra \mathfrak{g} . Then $\phi(R) = -(R + \lambda \text{id})$ is again an RB-operator of weight k on \mathfrak{g} .*

Lemma 2.2 ([8]). *Let \mathfrak{g} be a Lie algebra, φ be an automorphism of \mathfrak{g} and R be an RB-operator of weight k on \mathfrak{g} . Then $R^{(\varphi)} = \varphi^{-1}R\varphi$ is an RB-operator of weight k on \mathfrak{g} .*

Let a Lie algebra \mathfrak{g} split as a vector space into the direct sum of two subalgebras \mathfrak{g}_1 and \mathfrak{g}_2 . An operator R defined as

$$R(a_1 + a_2) = -ka_2, \quad a_1 \in \mathfrak{g}_1, \quad a_2 \in \mathfrak{g}_2, \quad (2)$$

is an RB-operator of weight k on \mathfrak{g} called a splitting RB-operator [17].

Example 2.3. Let \mathfrak{g} be a finite-dimensional semisimple Lie algebra over \mathbb{C} with a root system Φ . Let a linear operator R act on \mathfrak{g} as follows, $R(\mathfrak{h}) = 0$ for the fixed Cartan subalgebra \mathfrak{h} of \mathfrak{g} , $R(e_\lambda) = -e_\lambda$, when $\lambda \in \Phi_+$, and $R(e_\lambda) = 0$ for all $\lambda \in \Phi_-$. Then R is a splitting Rota—Baxter operator of weight 1 on \mathfrak{g} .

Throughout this paper, let us fix the standard basis e, f, h of $\mathfrak{sl}_2(\mathbb{C})$ such that

$$[e, f] = h, \quad [h, e] = 2e, \quad [h, f] = -2f.$$

Proposition 2.4 ([20, Prop. 4]). *Up to conjugation with an automorphism of $\mathfrak{sl}_2(\mathbb{C})$ and up to a scalar multiple, we have that a Rota—Baxter operator R of weight 0 on $\mathfrak{sl}_2(\mathbb{C})$ is one of the following:*

- a) $R \equiv 0$,
- b) $R(e) = 0, R(f) = te - h, R(h) = 2e, t \in \mathbb{C}$,
- c) $R(e) = 0, R(f) = 0, R(h) = h$,
- d) $R(e) = 0, R(f) = h, R(h) = 0$,
- e) $R(e) = 0, R(f) = e, R(h) = 0$.

The following result was actually proven in [21] and in [26]. Since it was not stated in the form in which we are interested, let us prove it directly.

Proposition 2.5 ([21, 26]). *Up to conjugation with an automorphism, all nontrivial RB-operators of weight 1 on $\mathfrak{sl}_2(\mathbb{C})$ are the following:*

- a) $R(e) = -e, R(f) = R(h) = 0$,
- b) $R(e) = -(e + h), R(f) = R(h) = 0$,
- c) $R(e) = -e, R(f) = 0, R(h) = th, t \in \mathbb{C} \setminus \{0\}$.

Proof. Let R be an RB-operator of weight 1 on $\mathfrak{sl}_2(\mathbb{C})$. If either $\ker(R) = (0)$ or $\ker(R + \text{id}) = (0)$, then R is trivial [11].

Suppose that R is splitting, i. e. $\mathfrak{sl}_2(\mathbb{C}) = A \oplus B$, where A and B are subalgebras of $\mathfrak{sl}_2(\mathbb{C})$ and $A = \ker R, B = \ker(R + \text{id})$. We may assume that $\dim A = 2$ and $\dim B = 1$. It is known that up to action of $\text{Aut}(\mathfrak{sl}_2(\mathbb{C}))$, $A = \text{Span}\{f, h\}$. Thus, $B = \text{Span}\{e + kf + lh\}$ for some $k, l \in \mathbb{C}$. Let $k = 0$, then for $l = 0$ it is a). When $l \neq 0$ we conjugate with $\psi \in \text{Aut}(\mathfrak{sl}_2(\mathbb{C}))$ defined as follows, $\psi(e) = (1/l)e$, $\psi(f) = lf$, $\psi(h) = h$, and get b) for $R^{(\psi)}$.

Suppose that $k \neq 0$. Consider $\psi \in \text{Aut}(\mathfrak{sl}_2(\mathbb{C}))$ defined as follows,

$$\psi(e) = e - (k - \alpha l)f - (\alpha/2)h, \quad \psi(f) = f, \quad \psi(h) = h + \alpha f, \quad \alpha = 2(-l + \sqrt{l^2 + k}).$$

Then for the RB-operator $R' = \psi R \psi^{-1}$, we have $\ker R' = \text{Span}\{f, h\}$ and $\ker(R' + \text{id}) = \text{Span}\{e + \sqrt{l^2 + kh}\}$, it is the already considered case.

Suppose that R is not splitting, then $\dim(\ker R) = \dim(\ker(R + \text{id})) = 1$. It is known (see, e. g. [11]) that the space $\mathfrak{sl}_2(\mathbb{C})$ under the new product

$$\langle x, y \rangle := [R(x), y] + [x, R(y)] + [x, y] \tag{3}$$

is again a Lie algebra, and $R, R + \text{id}$ are homomorphisms from $(\mathfrak{sl}_2(\mathbb{C}), \langle, \rangle)$ to $(\mathfrak{sl}_2(\mathbb{C}), [,])$. Hence, $\ker R, \ker(R + \text{id})$ are ideals in $(\mathfrak{sl}_2(\mathbb{C}), \langle, \rangle)$.

Since we work over \mathbb{C} , let us consider all possible cases of the Jordan form of R . Let $e_2, e_3 \in \mathfrak{sl}_2(\mathbb{C})$ be such that $R(e_2) = 0$ and $R(e_3) = -e_3$.

CASE I: $\text{Spec}(R) = \{0, -1\}$. Up to the action of ϕ , we may assume that there exists e_1 such that $R(e_1) = e_2$. Since both kernels are ideals in $(\mathfrak{sl}_2(\mathbb{C}), \langle, \rangle)$, we derive by (3),

$$\langle e_1, e_2 \rangle = [e_1, e_2] = \mu e_2, \quad \langle e_1, e_3 \rangle = [e_2, e_3] = \lambda e_3$$

for some (nonzero) $\lambda, \mu \in \mathbb{C}$. Suppose that $[e_1, e_3] = \alpha e_1 + \beta e_2 + \gamma e_3$. Then, due to the Jacobi identity, we have

$$\begin{aligned} 0 &= [[e_1, e_2], e_3] + [[e_2, e_3], e_1] + [[e_3, e_1], e_2] \\ &= \mu \lambda e_3 - \lambda(\alpha e_1 + \beta e_2 + \gamma e_3) - \mu \alpha e_2 + \lambda \gamma e_3 = -\lambda \alpha e_1 - (\lambda \beta + \mu \alpha) e_2 + \mu \lambda e_3. \end{aligned}$$

We get $\mu \lambda = 0$, a contradiction.

CASE II: $\text{Spec}(R) = \{0, -1, t\}$, where $t \neq 0, -1$. Let e_1 be an eigenvector corresponding to the eigenvalue t . Again, we write down,

$$\langle e_1, e_2 \rangle = (t + 1)[e_1, e_2] = \mu e_2, \quad \langle e_1, e_3 \rangle = t[e_1, e_3] = \lambda e_3$$

for some nonzero $\lambda, \mu \in \mathbb{C}$. Suppose that $[e_2, e_3] = \alpha e_1 + \beta e_2 + \gamma e_3$. Then the Jacobi identity implies $\beta = \gamma = 0$ and $\lambda = -t\mu/(t + 1)$. Considering $e_2/\sqrt{\alpha}$ and $e_3/\sqrt{\alpha}$ instead of e_2 and e_3 , we get the following multiplication table:

$$[e_1, e_2] = \Lambda e_2, \quad [e_1, e_3] = -\Lambda e_3, \quad [e_2, e_3] = e_1,$$

where $\Lambda = \mu/(t + 1)$. Also, $R(e_1) = te_1$, $R(e_2) = 0$ and $R(e_3) = -e_3$. Then R is the RB-operator from case c) up to conjugation with $\psi \in \text{Aut}(\mathfrak{sl}_2(\mathbb{C}))$ defined by the rule $\psi(e_1) = (\Lambda/2)h$, $\psi(e_2) = \sqrt{\Lambda/2}e$, $\psi(e_3) = \sqrt{\Lambda/2}f$. □

2.2. Lie conformal algebras. A (free) $\mathbb{C}[\partial]$ -module L is called a conformal algebra if there is a λ -bracket on L ,

$$[\cdot, \lambda \cdot]: L \otimes L \rightarrow \mathbb{C}[\lambda] \otimes L,$$

satisfying the identities,

$$[\partial a_\lambda b] = -\lambda[a_\lambda b], \quad [a_\lambda \partial b] = (\lambda + \partial)[a_\lambda b].$$

A conformal algebra L is called a Lie conformal algebra if L satisfies the following conformal analogues of anticommutativity and the Jacobi identity:

$$[a_\lambda b] = -[b_{-\lambda-\partial} a], \quad [a_\lambda [b_\mu c]] - [b_\mu [a_\lambda c]] = [[a_\lambda b]_{\lambda+\mu} c].$$

The Virasoro Lie conformal algebra Vir is defined as follows:

$$\text{Vir} = \mathbb{C}[\partial]L, \quad [L_\lambda L] = (\partial + 2\lambda)L.$$

Given a Lie algebra \mathfrak{g} , the current Lie conformal algebra $\text{Cur}(\mathfrak{g})$ on the space $\mathbb{C}[\partial]\mathfrak{g}$ is defined by the formula

$$[f(\partial)a_\lambda g(\partial)b] = f(-\lambda)g(\lambda + \partial)[a, b], \quad a, b \in \mathfrak{g}. \tag{4}$$

A left module M over a Lie conformal algebra L is a left $\mathbb{C}[\partial]$ -module endowed with a \mathbb{C} -linear map $\cdot_\lambda \cdot: C \otimes M \rightarrow M[\lambda]$ satisfying the identities

$$\partial a_\lambda v = -\lambda a_\lambda v, \quad a_\lambda \partial v = (\partial + \lambda)a_\lambda v, \quad [a_\lambda b]_{\lambda+\mu} v = a_\lambda(b_\mu v) - b_\mu(a_\lambda v)$$

for all $a, b \in L, v \in M$.

Given a Lie conformal algebra L , the space $L^{\otimes n}$ is a left L -module under the action

$$a_\lambda(a_1 \otimes \cdots \otimes a_n) = \sum_{i=1}^n a_1 \otimes \cdots \otimes [a_\lambda a_i] \otimes \cdots \otimes a_n,$$

where $a, a_1, \dots, a_n \in L$.

A $\mathbb{C}[\partial]$ -submodule I of L is called an ideal of L if $[I_\lambda L] \subset \mathbb{C}[\lambda] \otimes I$. A Lie conformal algebra L is of finite type if L is finitely-generated as $\mathbb{C}[\partial]$ -module. A Lie conformal algebra L is called simple if $[L_\lambda L] \neq (0)$ and there are only two ideals of L : (0) and L .

Recall [2] that every simple Lie conformal algebra of finite type is isomorphic either to Vir or to $\text{Cur}(\mathfrak{g})$ associated to a simple finite-dimensional Lie algebra \mathfrak{g} .

Given a Lie algebra \mathfrak{g} and an automorphism φ of \mathfrak{g} , we can extend it to an automorphism of $\text{Cur}(\mathfrak{g})$ as a ∂ -linear operator by the formula $\varphi(f(\partial)a) = f(\partial)\varphi(a)$, $a \in \mathfrak{g}$.

2.3. Conformal classical Yang—Baxter equation. Let L be a Lie conformal algebra and $r = \sum a_i \otimes b_i \in L \otimes L$. Set $\partial_{\otimes 1} = \partial \otimes 1 \otimes 1$, $\partial_{\otimes 2} = 1 \otimes \partial \otimes 1$, $\partial_{\otimes 3} = 1 \otimes 1 \otimes \partial$, and $\partial^{\otimes 3} = \partial_{\otimes 1} + \partial_{\otimes 2} + \partial_{\otimes 3}$. The following equation,

$$\begin{aligned} \llbracket r, r \rrbracket := \sum & ([a_i \lambda a_j] \otimes b_i \otimes b_j|_{\lambda=\partial_{\otimes 2}} - a_i \otimes [a_j \lambda b_i] \otimes b_j|_{\lambda=\partial_{\otimes 3}} \\ & - a_i \otimes a_j \otimes [b_j \lambda b_i]|_{\lambda=\partial_{\otimes 2}}) = 0 \pmod{\partial^{\otimes 3}}, \end{aligned} \tag{5}$$

holding in $L^{\otimes 3}$, is called the conformal classical Yang—Baxter equation (shortly, CCYBE) [23] and $r \in L \otimes L$ satisfying (5) is called a solution to CCYBE.

The following equation fulfilled for all $a \in L$ is called the weak CCYBE:

$$a_\mu \llbracket r, r \rrbracket = 0 \pmod{\mu = -\partial^{\otimes 3}}. \tag{6}$$

A solution r to CCYBE (or the weak one) is called L -invariant, if the following equality holds for every $a \in L$,

$$a_\lambda(r + \tau(r))|_{\lambda=-\partial^{\otimes 2}} = 0, \tag{7}$$

where $\partial^{\otimes 2} = \partial \otimes 1 + 1 \otimes \partial$ and $\tau: L \otimes L \rightarrow L \otimes L$ is defined as follows, $\tau(a \otimes b) = b \otimes a$. A solution r to the (weak) CCYBE is called skew-symmetric if $r + \tau(r) = 0$.

In [23], J. Liberati proved that given a Lie conformal algebra L and $r \in L \otimes L$, the map $\delta(a) = a_\lambda r|_{\lambda=-\partial^{\otimes 2}}$ is a cocommutator of a Lie conformal bialgebra structure on L if and only if r is an L -invariant solution to the weak CCYBE on L .

2.4. Rota—Baxter operators on Lie conformal algebras. Given a Lie conformal algebra L , a ∂ -linear map R on L is called a Rota—Baxter operator (RB-operator, for short) of weight $k \in \mathbb{C}$ [18] if

$$[R(a)_\lambda R(b)] = R([R(a)_\lambda b] + [a_\lambda R(b)] + k[a_\lambda b]) \tag{8}$$

for all $a, b \in L$.

Given a Lie conformal algebra L , let us call the operators $R = 0$ and $R = -k \cdot \text{id}$ as trivial Rota—Baxter operators of weight k on L .

In [18], it was shown that there are only trivial RB-operators of weight 0 on Vir . It is easy to extend this result to the case of RB-operators of nonzero weight.

Example 2.6 ([18]). Let $L = \mathbb{C}[\partial]a \oplus \mathbb{C}[\partial]b$ be a Lie conformal algebra of rank 2 with the λ -bracket given by the formulas,

$$[a_\lambda a] = (\partial + 2\lambda)a, \quad [a_\lambda b] = (\partial + \lambda)b, \quad [b_\lambda b] = 0.$$

Then any Rota—Baxter operator of weight 0 on L is one of the following two forms:

- (i) $R(a) = -\mu(a + b)$, $R(b) = \mu(a + b)$, where $\mu \in \mathbb{C} \setminus \{0\}$;
- (ii) $R(a) = g(\partial)b$, $R(b) = 0$, where $g(\partial) \in \mathbb{C}[\partial]$.

Given a Lie algebra \mathfrak{g} and a ∂ -linear map R on $L = \text{Cur}(\mathfrak{g})$, define a linear operator R_0 on \mathfrak{g} by the rule

$$R_0(e_i) = R(e_i)|_{\partial=0},$$

where $\{e_i \mid i \in I\}$ is a linear basis of \mathfrak{g} . Note that the definition of R_0 does not depend on choice of a linear basis $\{e_i\}$.

Lemma 2.7. *Let $L = \text{Cur}(\mathfrak{g})$ be a current Lie conformal algebra.*

- a) *Suppose that R is an RB-operator of weight k on L , then R_0 is an RB-operator of weight k on \mathfrak{g} .*
- b) *Suppose that P is an RB-operator of weight k on \mathfrak{g} , then the extension of P on L by the rule $P(f(\partial)a) = f(\partial)P(a)$, where $a \in \mathfrak{g}$, is an RB-operator of weight k on L .*

Proof. a) Given $a \in \mathfrak{g}$, we may present $R(a)$ as follows,

$$R(a) = R_0(a) + \sum_{k \geq 1} \partial^k R_k(a),$$

where $R_k(a) \in \mathfrak{g}$. Then the identity (8) written down for $a, b \in \mathfrak{g}$ and $\lambda = \partial = 0$ gives exactly (1).

b) It follows directly from the definitions of RB-operators and λ -product in the current algebra. \square

Analogously to Lemma 2.2, a conjugation $R^{(\psi)}$ of an RB-operator R on a Lie conformal algebra L with an automorphism ψ of L is again an RB-operator of the same weight [28].

Corollary 2.8. *Let $L = \text{Cur}(\mathfrak{g})$ be a current Lie conformal algebra, let R be an RB-operator of weight k on L , and let ψ_0 be an automorphism of \mathfrak{g} . Then $(R^{(\psi)})_0 = (R_0)^{(\psi_0)}$, where ψ is a ∂ -invariant extension of ψ_0 to an automorphism of L .*

3. Rota—Baxter operators of weight 0 on $\text{Cur}(\mathfrak{sl}_2(\mathbb{C}))$

Let us give a general example of RB-operators of weight 0 on the current algebra $\text{Cur}(\mathfrak{g})$ of a finite-dimensional semisimple Lie algebra \mathfrak{g} . We will see further that this construction implies one of two RB-operators on $\text{Cur}(\mathfrak{sl}_2(\mathbb{C}))$ which are not ∂ -linear extensions of RB-operators on $\mathfrak{sl}_2(\mathbb{C})$.

Example 3.1. Let \mathfrak{g} be a finite-dimensional semisimple Lie algebra over \mathbb{C} . Let a linear operator R act on $L = \text{Cur}(\mathfrak{g})$ as follows, $R(\mathfrak{h}) \subset \mathbb{C}[\partial]\mathfrak{h}$ for a Cartan

subalgebra \mathfrak{h} of \mathfrak{g} , and R maps all other weighted subspaces to zero. Then R is a Rota—Baxter operator of weight 0 on L .

3.1. Classification. Let R be an RB-operator of weight 0 on $\text{Cur}(\mathfrak{sl}_2(\mathbb{C}))$. Introduce

$$\begin{aligned} R(e) &= a_e(\partial)e + a_f(\partial)f + a_h(\partial)h, \\ R(f) &= b_e(\partial)e + b_f(\partial)f + b_h(\partial)h, \\ R(h) &= c_e(\partial)e + c_f(\partial)f + c_h(\partial)h. \end{aligned} \tag{9}$$

Now, we compute by (4),

$$\begin{aligned} [R(h)_\lambda R(h)] &= [(c_e(\partial)e + c_f(\partial)f + c_h(\partial)h)_\lambda (c_e(\partial)e + c_f(\partial)f + c_h(\partial)h)] \\ &= 2(c_e(\lambda + \partial)c_h(-\lambda) - c_e(-\lambda)c_h(\lambda + \partial))e \\ &\quad - 2(c_f(\lambda + \partial)c_h(-\lambda) - c_f(-\lambda)c_h(\lambda + \partial))f \\ &\quad + (c_e(-\lambda)c_f(\lambda + \partial) - c_e(\lambda + \partial)c_f(-\lambda))h; \end{aligned}$$

$$\begin{aligned} R([R(h)_\lambda h] + [h_\lambda R(h)]) &= R([(c_e(\partial)e + c_f(\partial)f + c_h(\partial)h)_\lambda h] + [h_\lambda(c_e(\partial)e + c_f(\partial)f + c_h(\partial)h)]) \\ &= 2R((c_e(\lambda + \partial) - c_e(-\lambda))e + (c_f(-\lambda) - c_f(\lambda + \partial))f) \\ &= 2(c_e(\lambda + \partial) - c_e(-\lambda))(a_e(\partial)e + a_f(\partial)f + a_h(\partial)h) \\ &\quad + 2(c_f(-\lambda) - c_f(\lambda + \partial))(b_e(\partial)e + b_f(\partial)f + b_h(\partial)h). \end{aligned}$$

Thus, we obtain the following equations,

$$\begin{aligned} c_e(\lambda + \partial)c_h(-\lambda) - c_e(-\lambda)c_h(\lambda + \partial) &= a_e(\partial)(c_e(\lambda + \partial) - c_e(-\lambda)) + b_e(\partial)(c_f(-\lambda) - c_f(\lambda + \partial)), \end{aligned} \tag{10}$$

$$\begin{aligned} c_f(-\lambda)c_h(\lambda + \partial) - c_f(\lambda + \partial)c_h(-\lambda) &= a_f(\partial)(c_e(\lambda + \partial) - c_e(-\lambda)) + b_f(\partial)(c_f(-\lambda) - c_f(\lambda + \partial)), \end{aligned} \tag{11}$$

$$\begin{aligned} c_e(-\lambda)c_f(\lambda + \partial) - c_e(\lambda + \partial)c_f(-\lambda) &= 2a_h(\partial)(c_e(\lambda + \partial) - c_e(-\lambda)) + 2b_h(\partial)(c_f(-\lambda) - c_f(\lambda + \partial)). \end{aligned} \tag{12}$$

Analogously, we get from (8) applied for $[R(e)_\lambda R(e)]$ and $[R(f)_\lambda R(f)]$ the equalities

$$\begin{aligned} & 2a_e(\lambda + \partial)a_h(-\lambda) - 2a_e(-\lambda)a_h(\lambda + \partial) \\ &= c_e(\partial)(a_f(\lambda + \partial) - a_f(-\lambda)) + 2a_e(\partial)(a_h(-\lambda) - a_h(\lambda + \partial)), \end{aligned} \quad (13)$$

$$\begin{aligned} & 2a_f(-\lambda)a_h(\lambda + \partial) - 2a_f(\lambda + \partial)a_h(-\lambda) \\ &= c_f(\partial)(a_f(\lambda + \partial) - a_f(-\lambda)) + 2a_f(\partial)(a_h(-\lambda) - a_h(\lambda + \partial)), \end{aligned} \quad (14)$$

$$\begin{aligned} & a_e(-\lambda)a_f(\lambda + \partial) - a_e(\lambda + \partial)a_f(-\lambda) \\ &= c_h(\partial)(a_f(\lambda + \partial) - a_f(-\lambda)) + 2a_h(\partial)(a_h(-\lambda) - a_h(\lambda + \partial)). \end{aligned} \quad (15)$$

$$\begin{aligned} & 2b_e(\lambda + \partial)b_h(-\lambda) - 2b_e(-\lambda)b_h(\lambda + \partial) \\ &= c_e(\partial)(b_e(-\lambda) - b_e(\lambda + \partial)) + 2b_e(\partial)(b_h(\lambda + \partial) - b_h(-\lambda)), \end{aligned} \quad (16)$$

$$\begin{aligned} & 2b_f(-\lambda)b_h(\lambda + \partial) - 2b_f(\lambda + \partial)b_h(-\lambda) \\ &= c_f(\partial)(b_e(-\lambda) - b_e(\lambda + \partial)) + 2b_f(\partial)(b_h(\lambda + \partial) - b_h(-\lambda)), \end{aligned} \quad (17)$$

$$\begin{aligned} & b_e(-\lambda)b_f(\lambda + \partial) - b_e(\lambda + \partial)b_f(-\lambda) \\ &= c_h(\partial)(b_e(-\lambda) - b_e(\lambda + \partial)) + 2b_h(\partial)(b_h(\lambda + \partial) - b_h(-\lambda)). \end{aligned} \quad (18)$$

Let us compare

$$\begin{aligned} [R(e)_\lambda R(f)] &= [(a_e(\partial)e + a_f(\partial)f + a_h(\partial)h)_\lambda(b_e(\partial)e + b_f(\partial)f + b_h(\partial)h)] \\ &= 2(a_h(-\lambda)b_e(\lambda + \partial) - a_e(-\lambda)b_h(\lambda + \partial))e \\ &\quad - 2(a_h(-\lambda)b_f(\lambda + \partial) - a_f(-\lambda)b_h(\lambda + \partial))f \\ &\quad + (a_e(-\lambda)b_f(\lambda + \partial) - a_f(-\lambda)b_e(\lambda + \partial))h; \end{aligned}$$

$$\begin{aligned} & R([R(e)_\lambda f] + [e_\lambda R(f)]) \\ &= R([(a_e(\partial)e + a_f(\partial)f + a_h(\partial)h)_\lambda f] + [e_\lambda(b_e(\partial)e + b_f(\partial)f + b_h(\partial)h)]) \\ &= R(-2b_h(\lambda + \partial)e - 2a_h(-\lambda)f + (a_e(-\lambda) + b_f(\lambda + \partial))h) \\ &= -2b_h(\lambda + \partial)(a_e(\partial)e + a_f(\partial)f + a_h(\partial)h) - 2a_h(-\lambda)(b_e(\partial)e + b_f(\partial)f + b_h(\partial)h) \\ &\quad + (a_e(-\lambda) + b_f(\lambda + \partial))(c_e(\partial)e + c_f(\partial)f + c_h(\partial)h). \end{aligned}$$

We get three new identities:

$$\begin{aligned} & 2(a_h(-\lambda)b_e(\lambda + \partial) - a_e(-\lambda)b_h(\lambda + \partial)) \\ & = -2b_h(\lambda + \partial)a_e(\partial) - 2a_h(-\lambda)b_e(\partial) + (a_e(-\lambda) + b_f(\lambda + \partial))c_e(\partial), \end{aligned} \quad (19)$$

$$\begin{aligned} & -2(a_h(-\lambda)b_f(\lambda + \partial) - a_f(-\lambda)b_h(\lambda + \partial)) \\ & = -2b_h(\lambda + \partial)a_f(\partial) - 2a_h(-\lambda)b_f(\partial) + (a_e(-\lambda) + b_f(\lambda + \partial))c_f(\partial), \end{aligned} \quad (20)$$

$$\begin{aligned} & a_e(-\lambda)b_f(\lambda + \partial) - a_f(-\lambda)b_e(\lambda + \partial) \\ & = -2b_h(\lambda + \partial)a_h(\partial) - 2a_h(-\lambda)b_h(\partial) + (a_e(-\lambda) + b_f(\lambda + \partial))c_h(\partial). \end{aligned} \quad (21)$$

Further, we write down,

$$\begin{aligned} [R(e)_\lambda R(h)] & = [(a_e(\partial)e + a_f(\partial)f + a_h(\partial)h)_\lambda(c_e(\partial)e + c_f(\partial)f + c_h(\partial)h)] \\ & = 2(a_h(-\lambda)c_e(\lambda + \partial) - a_e(-\lambda)c_h(\lambda + \partial))e \\ & \quad - 2(a_h(-\lambda)c_f(\lambda + \partial) - a_f(-\lambda)c_h(\lambda + \partial))f \\ & \quad + (a_e(-\lambda)c_f(\lambda + \partial) - a_f(-\lambda)c_e(\lambda + \partial))h; \end{aligned}$$

$$\begin{aligned} & R([R(e)_\lambda h] + [e_\lambda R(h)]) \\ & = R([(a_e(\partial)e + a_f(\partial)f + a_h(\partial)h)_\lambda h] + [e_\lambda(c_e(\partial)e + c_f(\partial)f + c_h(\partial)h)]) \\ & = R(-2(a_e(-\lambda) + c_h(\lambda + \partial))e + 2a_f(-\lambda)f + c_f(\lambda + \partial)h) \\ & = -2(a_e(-\lambda) + c_h(\lambda + \partial))(a_e(\partial)e + a_f(\partial)f + a_h(\partial)h) \\ & \quad + 2a_f(-\lambda)(b_e(\partial)e + b_f(\partial)f + b_h(\partial)h) + c_f(\lambda + \partial)(c_e(\partial)e + c_f(\partial)f + c_h(\partial)h). \end{aligned}$$

We get three new identities:

$$\begin{aligned} & 2(a_h(-\lambda)c_e(\lambda + \partial) - a_e(-\lambda)c_h(\lambda + \partial)) \\ & = -2(a_e(-\lambda) + c_h(\lambda + \partial))a_e(\partial) + 2a_f(-\lambda)b_e(\partial) + c_f(\lambda + \partial)c_e(\partial), \end{aligned} \quad (22)$$

$$\begin{aligned} & -2(a_h(-\lambda)c_f(\lambda + \partial) - a_f(-\lambda)c_h(\lambda + \partial)) \\ & = -2(a_e(-\lambda) + c_h(\lambda + \partial))a_f(\partial) + 2a_f(-\lambda)b_f(\partial) + c_f(\lambda + \partial)c_f(\partial), \end{aligned} \quad (23)$$

$$\begin{aligned} & a_e(-\lambda)c_f(\lambda + \partial) - a_f(-\lambda)c_e(\lambda + \partial) \\ & = -2(a_e(-\lambda) + c_h(\lambda + \partial))a_h(\partial) + 2a_f(-\lambda)b_h(\partial) + c_f(\lambda + \partial)c_h(\partial). \end{aligned} \quad (24)$$

Finally, we have,

$$\begin{aligned}
[R(f)_\lambda R(h)] &= [(b_e(\partial)e + b_f(\partial)f + b_h(\partial)h)_\lambda(c_e(\partial)e + c_f(\partial)f + c_h(\partial)h)] \\
&= 2(b_h(-\lambda)c_e(\lambda + \partial) - b_e(-\lambda)c_h(\lambda + \partial))e \\
&\quad - 2(b_h(-\lambda)c_f(\lambda + \partial) - b_f(-\lambda)c_h(\lambda + \partial))f \\
&\quad + (b_e(-\lambda)c_f(\lambda + \partial) - b_f(-\lambda)c_e(\lambda + \partial))h;
\end{aligned}$$

$$\begin{aligned}
R([R(f)_\lambda h] + [f_\lambda R(h)]) &= R([(b_e(\partial)e + b_f(\partial)f + b_h(\partial)h)_\lambda h] + [f_\lambda(c_e(\partial)e + c_f(\partial)f + c_h(\partial)h)]) \\
&= R(-2b_e(-\lambda)e + 2(b_f(-\lambda) + c_h(\lambda + \partial))f - c_e(\lambda + \partial)h) \\
&\quad = -2b_e(-\lambda)(a_e(\partial)e + a_f(\partial)f + a_h(\partial)h) \\
&\quad + 2(b_f(-\lambda) + c_h(\lambda + \partial))(b_e(\partial)e + b_f(\partial)f + b_h(\partial)h) \\
&\quad - c_e(\lambda + \partial)(c_e(\partial)e + c_f(\partial)f + c_h(\partial)h).
\end{aligned}$$

The last three identities are

$$\begin{aligned}
&2(b_h(-\lambda)c_e(\lambda + \partial) - b_e(-\lambda)c_h(\lambda + \partial)) \\
&= -2b_e(-\lambda)a_e(\partial) + 2(b_f(-\lambda) + c_h(\lambda + \partial))b_e(\partial) - c_e(\lambda + \partial)c_e(\partial), \quad (25)
\end{aligned}$$

$$\begin{aligned}
&-2(b_h(-\lambda)c_f(\lambda + \partial) - b_f(-\lambda)c_h(\lambda + \partial)) \\
&= -2b_e(-\lambda)a_f(\partial) + 2(b_f(-\lambda) + c_h(\lambda + \partial))b_f(\partial) - c_e(\lambda + \partial)c_f(\partial), \quad (26)
\end{aligned}$$

$$\begin{aligned}
&b_e(-\lambda)c_f(\lambda + \partial) - b_f(-\lambda)c_e(\lambda + \partial) \\
&= -2b_e(-\lambda)a_h(\partial) + 2(b_f(-\lambda) + c_h(\lambda + \partial))b_h(\partial) - c_e(\lambda + \partial)c_h(\partial). \quad (27)
\end{aligned}$$

We can extend any automorphism of $\mathfrak{sl}_2(\mathbb{C})$ to an automorphism of the Lie conformal algebra $L = \text{Cur}(\mathfrak{sl}_2(\mathbb{C}))$. Hence, by Proposition 2.4 and by Corollary 2.8 we may assume that

$$a_e(0) = a_f(0) = a_h(0) = 0, \quad b_f(0) = 0, \quad c_f(0) = 0. \quad (28)$$

Let us derive consequences of these identities. Consider (19), (13), (17), (15), (18), (23), and (25) respectively with $\lambda + \partial = 0$:

$$2a_h(\partial)(b_e(\partial) + b_e(0)) = c_e(\partial)a_e(\partial), \quad (29)$$

$$2a_h(\partial)a_e(\partial) = c_e(\partial)a_f(\partial), \quad (30)$$

$$2b_h(\partial)b_f(\partial) = c_f(\partial)(b_e(\partial) - b_e(0)), \quad (31)$$

$$2a_h^2(\partial) = a_f(\partial)c_h(\partial), \quad (32)$$

$$2b_h(\partial)(b_h(\partial) - b_h(0)) = b_e(\partial)c_h(\partial) + b_e(0)(b_f(\partial) - c_h(\partial)), \quad (33)$$

$$0 = a_f(\partial)(a_e(\partial) - b_f(\partial) + 2c_h(0)), \quad (34)$$

$$c_e(0)(c_e(\partial) + 2b_h(\partial)) = 2b_e(\partial)(b_f(\partial) - a_e(\partial) + 2c_h(0)). \quad (35)$$

Take $\lambda = 0$ in (14), (16), and (20) to get

$$0 = a_f(\partial)(c_f(\partial) - 2a_h(\partial)), \quad (36)$$

$$b_e(0)(c_e(\partial) + 2b_h(\partial)) = b_e(\partial)(c_e(\partial) - 2b_h(\partial) + 4b_h(0)), \quad (37)$$

$$2b_h(\partial)a_f(\partial) = c_f(\partial)b_f(\partial). \quad (38)$$

Substituting $\lambda + \partial = 0$ in (22) and (26), we obtain

$$a_e^2(\partial) - a_f(\partial)b_e(\partial) + a_h(\partial)c_e(0) = 0, \quad (39)$$

$$2b_f^2(\partial) - 2a_f(\partial)b_e(\partial) - c_e(0)c_f(\partial) = 0. \quad (40)$$

CASE I: $a_f(\partial) = 0$. By (32), we get $a_h(\partial) = 0$. By (39), we have $a_e(\partial) = 0$. By (38) and (40), we obtain $b_f(\partial) = 0$. By (23) considered at $\lambda = 0$, we have $c_f(\partial) = 0$. We have the following system on $b_e(\partial), b_h(\partial), c_e(\partial), c_h(\partial)$ remaining from the equations (10), (16), (18), (25), (27):

$$c_e(\lambda + \partial)c_h(-\lambda) - c_e(-\lambda)c_h(\lambda + \partial) = 0, \quad (41)$$

$$\begin{aligned} & 2b_e(\lambda + \partial)b_h(-\lambda) - 2b_e(-\lambda)b_h(\lambda + \partial) \\ &= c_e(\partial)(b_e(-\lambda) - b_e(\lambda + \partial)) + 2b_e(\partial)(b_h(\lambda + \partial) - b_h(-\lambda)), \end{aligned} \quad (42)$$

$$c_h(\partial)(b_e(-\lambda) - b_e(\lambda + \partial)) + 2b_h(\partial)(b_h(\lambda + \partial) - b_h(-\lambda)) = 0, \quad (43)$$

$$2(b_h(-\lambda)c_e(\lambda + \partial) - b_e(-\lambda)c_h(\lambda + \partial)) = 2c_h(\lambda + \partial)b_e(\partial) - c_e(\lambda + \partial)c_e(\partial), \quad (44)$$

$$2c_h(\lambda + \partial)b_h(\partial) - c_e(\lambda + \partial)c_h(\partial) = 0. \quad (45)$$

CASE IA: $c_h(\partial) = 0$. Then $b_h(\partial) = b_h(0) \in \mathbb{C}$ by (43). By (44), we have $c_e(\partial) = c_e(0) \in \mathbb{C}$ and, moreover, $c_e(0) = 0$ or $c_e(0) = -2b_h(0)$. Excluding the case

when $R = R_0$, we get

$$R_1(e) = 0, \quad R_1(f) = b_e(\partial)e + \alpha h, \quad R_1(h) = -2\alpha e$$

for some $\alpha \in \mathbb{C}$ and nonconstant $b_e(\partial)$.

CASE IB: $c_h(\partial) \neq 0$. By (41), we have

$$\frac{c_e(-\lambda)}{c_h(-\lambda)} = \frac{c_e(\lambda + \partial)}{c_h(\lambda + \partial)},$$

it means that $c_e(\partial) = \alpha c_h(\partial)$ for some $\alpha \in \mathbb{C}$. Thus, by (45), we get $b_h(\partial) = (\alpha/2)c_h(\partial)$. By (44), $b_e(\partial) = (\alpha^2/2)c_h(\partial)$. Therefore, we get the RB-operator:

$$R_2(e) = 0, \quad R_2(f) = (\alpha/2)q(\partial)(\alpha e + h), \quad R_2(h) = q(\partial)(\alpha e + h).$$

CASE II: $a_f(\partial) \neq 0$ and $b_e(\partial) = 0$. By (29), we have $a_e(\partial)c_e(\partial) = 0$. Thus, by (30), $a_f(\partial)c_e(\partial) = 0$. Since $a_f(\partial) \neq 0$, we conclude that $c_e(\partial) = 0$.

By (39) and (40), we get $a_e(\partial) = b_f(\partial) = 0$. The equality (38) implies that $b_h(\partial) = 0$. Now we may conjugate R with an automorphism φ such that $\varphi(e) = f$, $\varphi(f) = e$, and $\varphi(h) = -h$. So, we move to the case I.

CASE III: $a_f(\partial) \neq 0$ and $b_e(\partial) \neq 0$. By (34) and by (36), we have

$$c_f(\partial) = 2a_h(\partial), \tag{46}$$

$$a_e(\partial) = b_f(\partial) - 2c_h(\partial). \tag{47}$$

The last equality implies $c_h(0) = (b_f(0) - a_e(0))/2 = 0$. So, $a_e(\partial) = b_f(\partial)$.

From (38), (46), (30) and the relation $a_e(\partial) = b_f(\partial)$, we conclude

$$2b_h(\partial)a_f(\partial) = c_f(\partial)b_f(\partial) = 2a_h(\partial)b_f(\partial) = 2a_h(\partial)a_e(\partial) = c_e(\partial)a_f(\partial),$$

i. e.,

$$c_e(\partial) = 2b_h(\partial). \tag{48}$$

The equality (35) considered at $\partial = 0$ implies $c_e(0) = b_h(0) = 0$.

Therefore, by (39), we have $a_e^2(\partial) = b_f^2(\partial) = a_f(\partial)b_e(\partial) \neq 0$. Subtracting (31) from (29) we obtain, by (48),

$$a_h(\partial)b_e(0) = 0.$$

CASE IIIA: $a_h(\partial) = 0$. By (46) and (38), $c_f(\partial) = b_h(\partial) = 0$. Then by (30), we derive $c_e(\partial) = 0$. By (15) considered at $\lambda = 0$, we get $c_h(\partial)a_f(\partial) = 0$, so, $c_h(\partial) = 0$. By (21) computed at $\lambda + \partial = 0$, we have $b_e(0)a_f(\partial) = 0$, so, $b_e(0) = 0$. Since $a_e(\partial) \neq 0$, we apply (15) to get the equality $a_f(\partial) = \alpha a_e(\partial)$ with some nonzero $\alpha \in \mathbb{C}$. Thus, we obtain the RB-operator R such that

$$R_3(e) = q(\partial)(e + \alpha f), \quad R_3(f) = q(\partial)((1/\alpha)e + f), \quad R_3(h) = 0.$$

CASE IIIb: $a_h(\partial) \neq 0$. Then $b_e(0) = 0$ and $b_h(\partial) \neq 0$ by (48), since $c_e(\partial) \neq 0$ by (29). By (29) and (30), we get

$$\frac{b_h(\partial)}{a_h(\partial)} = \frac{b_f(\partial)}{a_f(\partial)} = \frac{b_e(\partial)}{a_e(\partial)} = \varphi(\partial),$$

where $\varphi(\partial)$ is a rational function on ∂ .

By (48), (38), and (21) with $\lambda = 0$, we get

$$\frac{2b_h(\partial)}{b_f(\partial)} = \frac{c_h(\partial)}{a_h(\partial)} = \frac{c_f(\partial)}{a_f(\partial)} = \frac{c_e(\partial)}{a_e(\partial)} = 2\psi(\partial),$$

for another rational function $\psi(\partial)$. By $c_e(\partial) = 2b_h(\partial)$, $c_f(\partial) = 2a_h(\partial)$, and $b_f(\partial) = a_e(\partial)$, we get the following form of the matrix $[R]$ of R in the basis e, f, h :

$$[R] = a_f(\partial) \begin{pmatrix} \varphi(\partial) & 1 & \psi(\partial) \\ \varphi^2(\partial) & \varphi(\partial) & \varphi(\partial)\psi(\partial) \\ 2\varphi(\partial)\psi(\partial) & 2\psi(\partial) & 2\psi^2(\partial) \end{pmatrix}.$$

Denote $a_f(\partial) = q(\partial)$. The right hand-side of (11) multiplied by $\varphi(\partial)$ and $2\psi(\partial)$ coincides with the right hand-side of (10) and (12) respectively. Thus, we get the equalities

$$\begin{aligned} q(-\lambda)q(\lambda + \partial)\psi(\lambda + \partial)\psi(-\lambda)(\varphi(\lambda + \partial)\psi(-\lambda) - \varphi(-\lambda)\psi(\lambda + \partial)) \\ = q(-\lambda)q(\lambda + \partial)\psi(\lambda + \partial)\psi(-\lambda)\varphi(\partial)(\psi(\lambda + \partial) - \psi(-\lambda)), \end{aligned}$$

$$\begin{aligned} q(-\lambda)q(\lambda + \partial)\psi(\lambda + \partial)\psi(-\lambda)(\varphi(-\lambda) - \varphi(\lambda + \partial)) \\ = 2q(-\lambda)q(\lambda + \partial)\psi(\lambda + \partial)\psi(-\lambda)\psi(\partial)(\psi(\lambda + \partial) - \psi(-\lambda)). \end{aligned}$$

Simplifying both relations, we obtain

$$\begin{aligned} \varphi(\lambda + \partial)\psi(-\lambda) - \varphi(-\lambda)\psi(\lambda + \partial) &= \varphi(\partial)(\psi(\lambda + \partial) - \psi(-\lambda)), \\ (\varphi(-\lambda) - \varphi(\lambda + \partial)) &= 2\psi(\partial)(\psi(\lambda + \partial) - \psi(-\lambda)). \end{aligned}$$

Thus, we get

$$\varphi(-\lambda)(\psi(-\lambda) - \psi(\lambda + \partial)) = (2\psi(\partial)\psi(-\lambda) + \varphi(\partial))(\psi(\lambda + \partial) - \psi(-\lambda)).$$

Suppose that $\psi(\partial)$ is not a constant. Then $2\psi(\partial)\psi(-\lambda) + \varphi(\partial) = -\varphi(-\lambda)$. Setting $-\lambda = \partial$, we get $\psi^2(\partial) = -\varphi(\partial)$. So, $\psi(\partial)(2\psi(\partial) - \psi(\partial)) = -\varphi(\partial)$. It means that both rational functions $\psi(\partial)$ and $\varphi(\partial)$ are constant, a contradiction.

Thus $\psi(\partial) = \beta \in \mathbb{C} \setminus \{0\}$, and it is easy to show that $\varphi(\partial) = \alpha \in \mathbb{C} \setminus \{0\}$. Summarizing, we get

$$R_4(e) = q(\partial)(\alpha e + f + \beta h), \quad R_4(\alpha e - f) = R_4(2\beta e - h) = 0, \quad \alpha, \beta \neq 0.$$

Theorem 3.2. *Up to conjugation with an automorphism of $\text{Cur}(\text{sl}_2(\mathbb{C}))$ and up to a scalar multiple, we have that a Rota–Baxter operator R of weight 0 on $\text{Cur}(\text{sl}_2(\mathbb{C}))$ is either a ∂ -linear extension of an RB-operator of weight 0 on $\text{sl}_2(\mathbb{C})$ or one of the following for some nonzero $q(\partial) \in \mathbb{C}[\partial]$:*

- (R1) $R(e) = 0, R(f) = q(\partial)e + \alpha h, R(h) = -2\alpha e, \alpha \in \mathbb{C};$
- (R2) $R(e) = R(f) = 0, R(h) = q(\partial)h.$

Proof. Above, we have obtained the RB-operators $R_1, R_2, R_3,$ and R_4 . The RB-operator R_1 coincides with (R1).

For R_2 , define an automorphism ψ of $\text{sl}_2(\mathbb{C})$ as follows:

$$\psi(e) = e, \quad \psi(f) = f - (\alpha/2)h - (\alpha^2/4)e, \quad \psi(h) = \alpha e + h.$$

Then $\psi^{-1}R_2\psi$ is exactly (R2).

For R_3 , $\xi^{-1}R_3\xi$ gives again (R2). Here

$$\xi(e) = \frac{\sqrt{\alpha}h - e + \alpha f}{2\sqrt{\alpha}}, \quad \xi(f) = \frac{\sqrt{\alpha}h + e - \alpha f}{2\sqrt{\alpha}}, \quad \xi(h) = \frac{e + \alpha f}{\sqrt{\alpha}}.$$

Finally, consider the RB-operator R_4 . If $\alpha + \beta^2 = 0$, then $\pi^{-1}R_4\pi$ equals to a particular case of (R1), where

$$\pi(e) = -\beta^2e + f + \beta h, \quad \pi(f) = e, \quad \pi(h) = 2\beta e - h.$$

Otherwise, $\theta^{-1}R_4\theta$ is exactly (R2). Here

$$\begin{aligned} \theta(e) &= \frac{\alpha e - f + (\beta + D)(2\beta e - h)}{2iD}, & \theta(f) &= \frac{\alpha e - f + (\beta - D)(2\beta e - h)}{2iD}, \\ \theta(h) &= \frac{\alpha e + f + \beta h}{D}, & D &= \sqrt{\alpha + \beta^2}. \end{aligned}$$

□

3.2. Connection with conformal CYBE. In [18], it was shown that every solution to the conformal classical Yang–Baxter equation on a Lie conformal algebra L endowed with a non-degenerate symmetric invariant conformal bilinear form gives rise to a Rota–Baxter operator on L .

Proposition 3.3 ([18, Corollary 3.3]). *Let L be a Lie conformal algebra of finite type, which is free as a $C[\partial]$ -module. Suppose that there exists a non-degenerate symmetric invariant conformal bilinear form on L , and $r \in L \otimes L$ is skew-symmetric. Then r is a solution to the CCYBE if and only if P_0^r is a Rota–Baxter operator of weight 0 on L for $P^r \in \text{Cend}(R)$ defined by the formula*

$$\langle r, u \otimes v \rangle_{(\lambda, \mu)} = \langle P_{\lambda - \partial}^r(u), v \rangle_{\mu}. \tag{49}$$

Let us clarify which Rota—Baxter operators on $L = \text{Cur}(\mathfrak{sl}_2(\mathbb{C}))$ we get from the solutions to the weak conformal classical Yang—Baxter equation on L .

Theorem 3.4 ([16]). *Let $L = \text{Cur}(\mathfrak{sl}_2(\mathbb{C}))$, $r = \sum A_{ql}(\partial_{\otimes 1}, \partial_{\otimes 2})q \otimes l \in L \otimes L$, $q, l \in \{e, f, h\}$, be a skew-symmetric solution to CCYBE. Then $A_{qq}(0, 0) = 0$ for any q and*

$$\begin{aligned} A_{fe}(0, 0) &= -A_{ef}(0, 0) = \beta, & A_{he}(0, 0) &= -A_{eh}(0, 0) = \alpha, \\ A_{hf}(0, 0) &= -A_{fh}(0, 0) = \gamma \end{aligned}$$

for some $\alpha, \beta, \gamma \in \mathbb{C}$. Moreover, $A_{ql}(x, -x) - A_{ql}(0, 0) = a_{ql}xf(x^2)$ for a unitary polynomial $f(x)$ and $a_{q,l} \in \mathbb{C}$, and up to the action of $\text{Aut}(L)$, we have three cases:

- (i) $a_{ee} = 1$, $a_{ql} = 0$ for $(q, l) \neq (e, e)$, and $\beta = \gamma = 0$;
- (ii) $a_{hh} = \lambda \in \mathbb{C} \setminus \{0\}$, $a_{ql} = 0$ for $(q, l) \neq (h, h)$, and $\alpha = \beta = \gamma = 0$;
- (iii) $a_{ql} = 0$ for all q, l .

Recall that given a semisimple finite-dimensional Lie algebra \mathfrak{g} over \mathbb{C} , we define the required form on $\text{Cur}(\mathfrak{g})$ as follows. Firstly, $\langle a, b \rangle_\lambda := \langle a, b \rangle$, i. e. the Killing form on \mathfrak{g} . Secondly, extend it by the rule

$$\langle \partial a, b \rangle_\lambda = -\lambda \langle a, b \rangle_\lambda = -\langle a, \partial b \rangle_\lambda, \quad a, b \in L.$$

Such form is invariant, which means that $\langle [a_\mu b], c \rangle_\lambda = \langle a, [b_{\lambda-\partial} c] \rangle_\mu$ for all $a, b, c \in \mathfrak{g}$. We additionally set for $a \otimes b, c \otimes d \in L \otimes L$,

$$\langle a \otimes b, c \otimes d \rangle_{(\lambda, \mu)} = \langle a, c \rangle_\lambda \langle b, d \rangle_\mu.$$

Let $r = \sum_{i,j=1}^3 \sum_{k,l \geq 0} \gamma_{kl}^{ij} \partial^k q_i \otimes \partial^l q_j$ be a skew-symmetric solution to CCYBE, here $q_1 = e, q_2 = f, q_3 = h$. Take $u, v \in \{e, f, h\}$, then

$$\begin{aligned} \langle r, u \otimes v \rangle_{(\lambda, \mu)} &= \sum_{i,j=1}^3 \sum_{k,l \geq 0} \gamma_{kl}^{ij} \langle \partial^k q_i, u \rangle_\lambda \langle \partial^l q_j, v \rangle_\mu \\ &= \sum_{i,j=1}^3 \sum_{k,l \geq 0} \gamma_{kl}^{ij} (-\lambda)^k \langle q_i, u \rangle_\lambda \langle \partial^l q_j, v \rangle_\mu. \end{aligned}$$

Since v is arbitrary and the Killing form is non-degenerate on $\mathfrak{sl}_2(\mathbb{C})$, we have by (49)

$$P_{\lambda-\partial}^r(u) = \sum_{i,j=1}^3 \sum_{k,l \geq 0} \gamma_{kl}^{ij} (-\lambda)^k \langle q_i, u \rangle_\lambda \partial^l q_j.$$

For $\lambda = \partial$, we get in terms of the polynomials $A_{ij}(\partial_{\otimes 1}, \partial_{\otimes 2})$,

$$P_0^r(u) = \sum_{i,j=1}^3 \sum_{k,l \geq 0} \gamma_{kl}^{ij} (-\partial)^k \partial^l \langle q_i, u \rangle_\lambda q_j = \sum_{i,j=1}^3 A_{ij}(-\partial, \partial) \langle q_i, u \rangle q_j.$$

By Theorem 3.4, we get either an RB-operator which is a ∂ -linear extension of an RB-operator of weight 0 on $\mathfrak{sl}_2(\mathbb{C})$ or both RB-operators (R1) and (R2) with odd polynomial $q(\partial)$.

4. Rota—Baxter operators of weight 1 on $\text{Cur}(\mathfrak{sl}_2(\mathbb{C}))$

Let R be an RB-operator of weight 1 on $\text{Cur}(\mathfrak{sl}_2(\mathbb{C}))$. We again use the formulas (9) and repeat the computations similar to the ones from the case of weight 0. Thus, we obtain the system:

$$\begin{aligned} c_e(\lambda + \partial)c_h(-\lambda) - c_e(-\lambda)c_h(\lambda + \partial) \\ = a_e(\partial)(c_e(\lambda + \partial) - c_e(-\lambda)) + b_e(\partial)(c_f(-\lambda) - c_f(\lambda + \partial)), \end{aligned} \quad (50)$$

$$\begin{aligned} c_f(-\lambda)c_h(\lambda + \partial) - c_f(\lambda + \partial)c_h(-\lambda) \\ = a_f(\partial)(c_e(\lambda + \partial) - c_e(-\lambda)) + b_f(\partial)(c_f(-\lambda) - c_f(\lambda + \partial)), \end{aligned} \quad (51)$$

$$\begin{aligned} c_e(-\lambda)c_f(\lambda + \partial) - c_e(\lambda + \partial)c_f(-\lambda) \\ = 2a_h(\partial)(c_e(\lambda + \partial) - c_e(-\lambda)) + 2b_h(\partial)(c_f(-\lambda) - c_f(\lambda + \partial)), \end{aligned} \quad (52)$$

$$\begin{aligned} 2a_e(\lambda + \partial)a_h(-\lambda) - 2a_e(-\lambda)a_h(\lambda + \partial) \\ = c_e(\partial)(a_f(\lambda + \partial) - a_f(-\lambda)) + 2a_e(\partial)(a_h(-\lambda) - a_h(\lambda + \partial)), \end{aligned} \quad (53)$$

$$\begin{aligned} 2a_f(-\lambda)a_h(\lambda + \partial) - 2a_f(\lambda + \partial)a_h(-\lambda) \\ = c_f(\partial)(a_f(\lambda + \partial) - a_f(-\lambda)) + 2a_f(\partial)(a_h(-\lambda) - a_h(\lambda + \partial)), \end{aligned} \quad (54)$$

$$\begin{aligned} a_e(-\lambda)a_f(\lambda + \partial) - a_e(\lambda + \partial)a_f(-\lambda) \\ = c_h(\partial)(a_f(\lambda + \partial) - a_f(-\lambda)) + 2a_h(\partial)(a_h(-\lambda) - a_h(\lambda + \partial)), \end{aligned} \quad (55)$$

$$\begin{aligned} 2b_e(\lambda + \partial)b_h(-\lambda) - 2b_e(-\lambda)b_h(\lambda + \partial) \\ = c_e(\partial)(b_e(-\lambda) - b_e(\lambda + \partial)) + 2b_e(\partial)(b_h(\lambda + \partial) - b_h(-\lambda)), \end{aligned} \quad (56)$$

$$\begin{aligned} & 2b_f(-\lambda)b_h(\lambda + \partial) - 2b_f(\lambda + \partial)b_h(-\lambda) \\ &= c_f(\partial)(b_e(-\lambda) - b_e(\lambda + \partial)) + 2b_f(\partial)(b_h(\lambda + \partial) - b_h(-\lambda)), \end{aligned} \quad (57)$$

$$\begin{aligned} & b_e(-\lambda)b_f(\lambda + \partial) - b_e(\lambda + \partial)b_f(-\lambda) \\ &= c_h(\partial)(b_e(-\lambda) - b_e(\lambda + \partial)) + 2b_h(\partial)(b_h(\lambda + \partial) - b_h(-\lambda)), \end{aligned} \quad (58)$$

$$\begin{aligned} & 2(a_h(-\lambda)b_e(\lambda + \partial) - a_e(-\lambda)b_h(\lambda + \partial)) \\ &= -2b_h(\lambda + \partial)a_e(\partial) - 2a_h(-\lambda)b_e(\partial) + (a_e(-\lambda) + b_f(\lambda + \partial) + 1)c_e(\partial), \end{aligned} \quad (59)$$

$$\begin{aligned} & -2(a_h(-\lambda)b_f(\lambda + \partial) - a_f(-\lambda)b_h(\lambda + \partial)) \\ &= -2b_h(\lambda + \partial)a_f(\partial) - 2a_h(-\lambda)b_f(\partial) + (a_e(-\lambda) + b_f(\lambda + \partial) + 1)c_f(\partial), \end{aligned} \quad (60)$$

$$\begin{aligned} & a_e(-\lambda)b_f(\lambda + \partial) - a_f(-\lambda)b_e(\lambda + \partial) \\ &= -2b_h(\lambda + \partial)a_h(\partial) - 2a_h(-\lambda)b_h(\partial) + (a_e(-\lambda) + b_f(\lambda + \partial) + 1)c_h(\partial), \end{aligned} \quad (61)$$

$$\begin{aligned} & 2(a_h(-\lambda)c_e(\lambda + \partial) - a_e(-\lambda)c_h(\lambda + \partial)) \\ &= -2(a_e(-\lambda) + c_h(\lambda + \partial) + 1)a_e(\partial) + 2a_f(-\lambda)b_e(\partial) + c_f(\lambda + \partial)c_e(\partial), \end{aligned} \quad (62)$$

$$\begin{aligned} & -2(a_h(-\lambda)c_f(\lambda + \partial) - a_f(-\lambda)c_h(\lambda + \partial)) \\ &= -2(a_e(-\lambda) + c_h(\lambda + \partial) + 1)a_f(\partial) + 2a_f(-\lambda)b_f(\partial) + c_f(\lambda + \partial)c_f(\partial), \end{aligned} \quad (63)$$

$$\begin{aligned} & a_e(-\lambda)c_f(\lambda + \partial) - a_f(-\lambda)c_e(\lambda + \partial) \\ &= -2(a_e(-\lambda) + c_h(\lambda + \partial) + 1)a_h(\partial) + 2a_f(-\lambda)b_h(\partial) + c_f(\lambda + \partial)c_h(\partial), \end{aligned} \quad (64)$$

$$\begin{aligned} & 2(b_h(-\lambda)c_e(\lambda + \partial) - b_e(-\lambda)c_h(\lambda + \partial)) \\ &= -2b_e(-\lambda)a_e(\partial) + 2(b_f(-\lambda) + c_h(\lambda + \partial) + 1)b_e(\partial) - c_e(\lambda + \partial)c_e(\partial), \end{aligned} \quad (65)$$

$$\begin{aligned} & -2(b_h(-\lambda)c_f(\lambda + \partial) - b_f(-\lambda)c_h(\lambda + \partial)) \\ &= -2b_e(-\lambda)a_f(\partial) + 2(b_f(-\lambda) + c_h(\lambda + \partial) + 1)b_f(\partial) - c_e(\lambda + \partial)c_f(\partial), \end{aligned} \quad (66)$$

$$\begin{aligned} & b_e(-\lambda)c_f(\lambda + \partial) - b_f(-\lambda)c_e(\lambda + \partial) \\ &= -2b_e(-\lambda)a_h(\partial) + 2(b_f(-\lambda) + c_h(\lambda + \partial) + 1)b_h(\partial) - c_e(\lambda + \partial)c_h(\partial). \end{aligned} \quad (67)$$

By Proposition 2.5 and by Corollary 2.8 we may assume that

$$a_f(0) = 0, \quad b_e(0) = b_f(0) = b_h(0) = 0, \quad c_e(0) = c_f(0) = 0. \quad (68)$$

Theorem 4.1. *Up to conjugation with an automorphism of $\text{Cur}(\mathfrak{sl}_2(\mathbb{C}))$ we have that a nontrivial Rota–Baxter operator R of weight 1 on $\text{Cur}(\mathfrak{sl}_2(\mathbb{C}))$ is either a ∂ -linear extension of an RB-operator of weight 1 on $\mathfrak{sl}_2(\mathbb{C})$ or the following one for some nonzero $q(\partial) \in \mathbb{C}[\partial]$:*

$$(Q1) \quad R(e) = -e, \quad R(f) = 0, \quad R(h) = q(\partial)h.$$

Proof. By (55), (56), (58), (60), (61), and (66) considered at $\lambda + \partial = 0$, we have

$$a_f(\partial)(c_h(\partial) - a_e(0)) = 2a_h(\partial)(a_h(\partial) - a_h(0)), \quad (69)$$

$$b_e(\partial)(c_e(\partial) - 2b_h(\partial)) = 0, \quad (70)$$

$$b_e(\partial)c_h(\partial) = 2b_h^2(\partial), \quad (71)$$

$$(a_e(\partial) + 1)c_f(\partial) = 2a_h(\partial)b_f(\partial), \quad (72)$$

$$(a_e(\partial) + 1)c_h(\partial) = 2a_h(\partial)b_h(\partial), \quad (73)$$

$$a_f(\partial)b_e(\partial) = b_f(\partial)(b_f(\partial) + 1). \quad (74)$$

Also, by (63)–(65) with $\lambda = 0$, we get

$$2(a_e(0) + 1 + c_h(\partial))a_f(\partial) = c_f(\partial)(c_f(\partial) + 2a_h(0)), \quad (75)$$

$$2(a_e(0) + 1 + c_h(\partial))a_h(\partial) = c_f(\partial)(c_h(\partial) - a_e(0)), \quad (76)$$

$$2b_e(\partial)(c_h(\partial) + 1) = c_e^2(\partial). \quad (77)$$

CASE I: $b_e(\partial) = 0$. By (71), (74), and (77), we have $b_h(\partial) = b_f(\partial) = c_e(\partial) = 0$. By (62), we conclude that $a_e(\partial)(a_e(\partial) + 1) = 0$.

CASE IA: $a_e(\partial) = 0$. Then the formulas (72) and (73) imply $c_f(\partial) = c_h(\partial) = 0$. Further, by (75) and (76), $a_f(\partial) = a_h(\partial) = 0$, i. e., $R = 0$.

CASE IB: $a_e(\partial) = -1$.

CASE IBA: $c_f(\partial) = 0$. The equalities (75) and (76) give $a_f(\partial)c_h(\partial) = a_h(\partial)c_h(\partial) = 0$. If $c_h(\partial) \neq 0$, then $a_f(\partial) = a_h(\partial) = 0$ and it is (Q1).

Suppose that $c_h(\partial) = 0$. Consider (54) and (55) with $\lambda + \partial = 0$:

$$a_f(\partial)(a_h(\partial) - 2a_h(0)) = 0, \quad a_f(\partial) = 2a_h(\partial)(a_h(\partial) - a_h(0)).$$

They imply that $a_h(\partial)$ and $a_f(\partial)$ are constant. Thus, R is defined by Proposition 2.5.

CASE IBB: $c_f(\partial) \neq 0$. From the equality (51), it follows that $c_h(\partial) = \alpha c_f(\partial)$ for some $\alpha \in \mathbb{C}$. By (76), we write down

$$2a_h(\partial)c_h(\partial) = 2\alpha a_h(\partial)c_f(\partial) = c_f(\partial)(1 + \alpha c_f(\partial)).$$

If $\alpha = 0$, then $c_f(\partial) = 0$, a contradiction. Hence, $a_h(\partial) = \frac{\alpha c_f(\partial) + 1}{2\alpha}$ and $a_h(0) = 1/(2\alpha)$. By (75), we derive that

$$a_f(\partial) = \frac{1}{2\alpha} \left(c_f(\partial) + \frac{2}{2\alpha} \right) = \frac{\alpha c_f(\partial) + 1}{2\alpha^2}.$$

Therefore, $0 = a_f(0) = 1/(2\alpha^2)$, a contradiction.

CASE II: $b_e(\partial) \neq 0$. Then, by (70), we deduce that $c_e(\partial) = 2b_h(\partial)$. Due to (71), we get $c_h(\partial) = c_e^2(\partial)/(2b_e(\partial))$. Finally, we apply this equality to (77),

$$c_e^2(\partial) = 2b_e(\partial)(c_h(\partial) + 1) = 2b_e(\partial) \left(\frac{c_e^2(\partial)}{2b_e(\partial)} + 1 \right) = c_e^2(\partial) + 2b_e(\partial),$$

a contradiction. □

We may generalize the RB-operator (Q1) from Theorem 4.1 for $L = \text{Cur}(\mathfrak{g})$, where \mathfrak{g} is a finite-dimensional semisimple Lie algebra.

Example 4.2. Let \mathfrak{g} be a finite-dimensional semisimple Lie algebra over \mathbb{C} with a root system Φ . Let a linear operator R act on $L = \text{Cur}(\mathfrak{g})$ as follows, $R(\mathfrak{h}) \subset \mathbb{C}[\partial]\mathfrak{h}$ for the Cartan subalgebra \mathfrak{h} of \mathfrak{g} , $R(e_\lambda) = -e_\lambda$, when $\lambda \in \Phi_+$, and $R(e_\lambda) = 0$ for all $\lambda \in \Phi_-$. Then R is a Rota—Baxter operator of weight 1 on L .

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