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AN EXTENSION OF S–NOETHERIAN RINGS AND MODULES

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Abstract. For any commutative ring A we introduce a generalization of S–noetherian rings using a hereditary torsion theory σ instead of a multiplicatively closed subset $S \subseteq A$. It is proved that totally noetherian w.r.t. σ is a local property, and if A is a totally noetherian ring w.r.t σ, then σ is of finite type.

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1. Introduction

In [6], the authors study the problem of determining the structure of the polynomial ring $D[X]$, over an integral domain D with field of fractions K, through the structure of the Euclidean domain $K[X]$. In particular, an ideal $\mathfrak{a} \subseteq D[X]$ is said to be **almost principal** whenever there exist a polynomial $F \in \mathfrak{a}$, of positive degree, and an element $0 \neq s \in D$ such that $\mathfrak{a}s \subseteq FD[X] \subseteq \mathfrak{a}$. The integral domain D is an **almost principal domain** whenever every ideal $\mathfrak{a} \subseteq D[X]$, which extends properly to $K[X]$, is almost principal. Noetherian and integrally closed domains are examples of almost principal domains.

Later, in [2], the authors extend this notion to non–necessarily integral domains in defining, for a given multiplicatively closed subset $S \subseteq A$ of a ring A, an ideal $\mathfrak{a} \subseteq A$ to be S–**finite** if there exist a finitely generated ideal $\mathfrak{a}' \subseteq \mathfrak{a}$ and an element $s \in S$ such that $\mathfrak{a}s \subseteq \mathfrak{a}'$, and define a ring A to be S–**noetherian** whenever every ideal $\mathfrak{a} \subseteq A$ is S–finite. Many authors have worked on S–noetherian rings and related notions, and have shown relevant results about their structure. See for instance [1,4,7,11,12,13,14].

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The main aim of this paper is to give a new approach to S–noetherian rings and modules, and their applications, using the more abstract notion of hereditary torsion theory. From this new point of view, several results appear more evident and appear inlaid in a more general theory, which clarifies the original approach.

The background we use will be the hereditary torsion theories on a commutative (and unitary) ring A, see [5,15], we denote by **Mod**– A the category of A–modules. Thus, a hereditary torsion theory σ in **Mod**– A is given by one of the following objects:

(1) a **torsion class** $\mathcal{T}_\sigma$, a class of modules which is closed under submodules, homomorphic images, direct sums and group extensions,
(2) a **torsionfree class** $\mathcal{F}_\sigma$, a class of modules which is closed under submodules, essential extensions, direct products and group extensions,
(3) a **Gabriel filter** of ideals $\mathcal{L}(\sigma)$, a non–empty filter of ideals satisfying that every $\mathfrak{b} \subseteq A$, for which there exists an ideal $\mathfrak{a} \in \mathcal{L}(\sigma)$ such that $(\mathfrak{b} : a) \in \mathcal{L}(\sigma)$, for every $a \in \mathfrak{a}$, belongs to $\mathcal{L}(\sigma)$,
(4) a **left exact kernel functor** $\sigma : \mathbf{Mod}\text{–}\, A \longrightarrow \mathbf{Mod}\text{–}\, A$.

The relationships between these notions are the following. If σ is the left exact kernel functor, then

$$\begin{aligned} \mathcal{T}_\sigma &= \{M \in \mathbf{Mod}\text{–}\, A \mid \ \sigma M = M\}, \\ \mathcal{F}_\sigma &= \{M \in \mathbf{Mod}\text{–}\, A \mid \ \sigma M = 0\}, \\ \mathcal{L}(\sigma) &= \{\mathfrak{a} \subseteq A \mid \ A/\mathfrak{a} \in \mathcal{T}_\sigma\}. \end{aligned}$$

If $\mathcal{L}$ is the Gabriel filter of a hereditary torsion theory σ, and $\mathcal{T}$ is the torsion class, for any A–module M we have:

$$\sigma M \quad = \{m \in M \mid \ (0 : m) \in \mathcal{L}\} \quad = \sum\{N \subseteq M \mid \ N \in \mathcal{T}\}.$$

Example 1.1. (1) Let $\Sigma \subseteq A$ be a multiplicatively closed subset, there exists a hereditary torsion theory, σ_Σ, defined by

$$\mathcal{L}(\sigma_\Sigma) = \{\mathfrak{a} \subseteq A \mid \ \mathfrak{a} \cap \Sigma \neq \varnothing\}.$$

Observe that σ_Σ has a filter basis constituted by principal ideals. A hereditary torsion theory σ such that $\mathcal{L}(\sigma)$ has a filter basis of principal ideals is called a **principal hereditary torsion theory**. We can show there is a correspondence between principal hereditary torsion theories in **Mod**– A, and saturated multiplicatively closed subsets in A.

(2) For any ring A the set $\mathcal{L} = \{\mathfrak{a} \subseteq A \mid \ \mathrm{Ann}(\mathfrak{a}) = 0\}$ is a Gabriel filter, it defines the hereditary torsion theory λ.

The paper is organized in sections. In Section 2 we introduce totally σ–noetherian rings and modules and show that necessarily the hereditary torsion theory σ is of finite type whenever the ring A is totally σ–noetherian. In Section 3 we study how prime ideals and prime submodules appears naturally when studying totally σ–finitely generated modules and obtain a relative version of Cohen's theorem. The natural notion of maximal condition in relation with totally σ–noetherian modules is studied in Section 4. Section 5 is devoted to study some extensions of totally σ–noetherian rings. In particular, we find that it is necessary to impose some extra conditions to σ in order to assure that $A[X]$ is totally σ–noetherian whenever A is. The local behaviour of totally σ–noetherian modules is studied in Section 6 in which we can reduce to consider only prime ideals in $\mathcal{K}(\sigma)$. In the last section, Section 7, we study the particular case of totally σ–principal ideal rings.

Through this paper we try to study σ–noetherian rings and modules and, in a parallel way, totally σ-noetherian rings and modules. The first one (σ–noetherian) has a categorical behaviour, but not the second one (totally σ–noetherian). For that reason, the study of the last one is more difficult and it is not in the literature. On the other hand, it produces results as Proposition 6.2 which shows that to be totally σ–noetherian, as opposed to σ-noetherian, is a local property.

2. Totally σ–noetherian rings and modules

For any σ–torsion finitely generated A–module M, say $M = m_1A + \cdots + m_tA$, since $(0 : m_i) \in \mathcal{L}(\sigma)$, for any $i = 1, \ldots, t$, if $\mathfrak{h} := \cap_{i=1}^{t}(0 : m_i) \in \mathcal{L}(\sigma)$, it satisfies $M\mathfrak{h} = 0$. In general, this result does not hold for σ–torsion non–finitely generated A–modules. Therefore, we shall define an A–module M to be **totally σ–torsion** whenever there exists $\mathfrak{h} \in \mathcal{L}(\sigma)$ such that $M\mathfrak{h} = 0$. This notion of totally torsion appears, for instance, in [8, page 462].

For any ideal $\mathfrak{a} \subseteq A$ we have two different notions of finitely generated ideal relative to σ:

(1) $\mathfrak{a} \subseteq A$ is **σ–finitely generated** whenever there exists a finitely generated ideal $\mathfrak{a}' \subseteq \mathfrak{a}$ such that $\mathfrak{a}/\mathfrak{a}'$ is σ–torsion.
(2) $\mathfrak{a} \subseteq A$ is **totally σ–finitely generated** whenever there exists a finitely generated ideal $\mathfrak{a}' \subseteq \mathfrak{a}$ such that $\mathfrak{a}/\mathfrak{a}'$ is totally σ–torsion.

In the same way, for any ring A we have two different notions of noetherian ring relative to σ:

(1) A is **σ–noetherian** if every ideal is σ–finitely generated.
(2) A is **totally σ–noetherian** whenever every ideal is totally σ–finitely generated.

Example 2.1. (1) Every finitely generated ideal is totally σ–finitely generated and every totally σ–finitely generated ideal is σ–finitely generated.

(2) If $S \subseteq A$ is a multiplicatively closed subset, an ideal $\mathfrak{a} \subseteq A$ is S–finite if, and only if, it is totally σ_S–finitely generated; and the ring A is S–noetherian if, and only if, A is totally σ_S–noetherian.

These two notions of torsion, and the notions derived from them, are completely different in their behaviour and their categorical properties. For instance, due to the definition, for any A–module M there exists a maximum submodule belonging to $\mathcal{T}_\sigma$, the submodule: σM, and it satisfies $M/\sigma M \in \mathcal{F}_\sigma$. In the totally σ–torsion case we can not assure the existence of a maximal totally σ–torsion submodule. The existence of a maximum σ–torsion submodule allows us to build new concepts relative to σ as lattices, closure operators and localizations; concepts that we have not in the totally σ–torsion case. Nevertheless, the totally σ–torsion case allows us to study arithmetic properties of rings and modules which are hidden with the use of σ–torsion, and these properties are those which we are interested in studying.

As we pointed out before, the σ–torsion allows, for any A–module M, to define a lattice structure on

$$C(M,\sigma) = \{N \subseteq M \mid\ M/N \in \mathcal{F}_\sigma\},$$

and a closure operator $\mathrm{Cl}_\sigma^M(-) : \mathcal{L}(M) \longrightarrow C(M,\sigma) \subseteq \mathcal{L}(M)$, from $\mathcal{L}(M)$, the lattice of all submodules of M, defined by the equation $\mathrm{Cl}_\sigma^M(N)/N = \sigma(M/N)$. The elements in $C(M,\sigma)$ are called the σ–**closed** submodules of M, and the lattice operations in $C(M,\sigma)$, for any $N_1, N_2 \in C(M,\sigma)$, are defined by

$$\begin{aligned} N_1 \wedge N_2 &= N_1 \cap N_2, \\ N_1 \vee N_2 &= \mathrm{Cl}_\sigma^M(N_1 + N_2). \end{aligned}$$

Dually, the submodules $N \subseteq M$ such that $M/N \in \mathcal{T}_\sigma$ are called σ–**dense** submodules. The set of all σ–dense submodules of M is represented by $\mathcal{L}(M,\sigma)$.

In the following, we assume A is a ring, **Mod**– A is the category of A–modules and σ is a hereditary torsion theory on **Mod**– A. Modules are represented by Latin letters: $M, N, N_1, \ldots$, and ideals by Gothics letters: $\mathfrak{a}, \mathfrak{b}, \mathfrak{b}_1, \ldots$ Different hereditary torsion theories will be represented by Greek letters: $\sigma, \tau, \sigma_1, \ldots$, and induced hereditary torsion theories by adorned Greek letters: $\sigma', \overline{\tau}, \ldots$

The notions of (totally) σ–finitely generated and (totally) σ–noetherian can be extended to A–modules in an easy way. Properties on the behaviour of totally σ–finitely generated and σ–noetherian modules are collected in the following result.

Proposition 2.2. (1) *Every homomorphic image of a totally σ–finitely generated A–module also is.*

(2) *For every submodule $N \subseteq M$, we have: M is totally σ–noetherian if, and only if, N and M/N are totally σ–noetherian.*

(3) *Finite direct sums of totally σ–noetherian modules also are.*

The first restriction we have found in studying totally σ–noetherian rings is that the hereditary torsion theory σ must be of an special type: it is a **finite type** hereditary torsion theory. This is an extension of principal hereditary torsion theories, and means that $\mathcal{L}(\sigma)$ has a filter basis constituted by finitely generated ideals.

Proposition 2.3. *If A is a totally σ–noetherian ring then σ is of finite type.*

Proof. For any $\mathfrak{a} \in \mathcal{L}(\sigma)$ there exist $\mathfrak{a}' \subseteq \mathfrak{a}$, finitely generated, and $\mathfrak{h} \in \mathcal{L}(\sigma)$ such that $\mathfrak{a}\mathfrak{h} \subseteq \mathfrak{a}'$. Since $\mathcal{L}(\sigma)$ is closed under product of ideals, we have $\mathfrak{a}' \in \mathcal{L}(\sigma)$. □

Directly from the definition we have that every totally σ–torsion module is totally σ–noetherian, and an A–module M is totally σ–finitely generated if, and only if, it contains a finitely generated submodule $N \subseteq M$ such that M/N is totally σ–torsion. Our aim is to explore more conditions equivalent to totally σ–noetherian.

Remember, an A–module M is σ–**noetherian** if, and only if, for every submodule $N \subseteq M$ there exists a finitely generated submodule $H \subseteq N$ such that $\mathrm{Cl}_\sigma^M(H) = \mathrm{Cl}_\sigma^M(N)$; and M is **totally σ–noetherian** if, and only if, for every submodule $N \subseteq M$ there exist a finitely generated submodule $H \subseteq N$ and $\mathfrak{h} \in \mathcal{L}(\sigma)$ such that $N\mathfrak{h} \subseteq H$. In this sense, since every totally σ–noetherian module is σ–noetherian, the question is what properties are necessary to add to σ–noetherianness to get totally σ–noetherian.

The next proposition is based on [2, Proposition 2].

Proposition 2.4. *Let σ be a finite type hereditary torsion theory in **Mod**– A, and let M be an A–module. The following statements are equivalent:*

(a) *M is totally σ–noetherian.*

(b) *M is σ–noetherian, and for every $N \subseteq M$, finitely generated, there exists $\mathfrak{h} \in \mathcal{L}(\sigma)$, finitely generated, such that $\mathrm{Cl}_\sigma^M(N) = (N : \mathfrak{h})$.*

Proof. (a) ⇒ (b) If $\mathrm{Cl}_\sigma^M(N)$ is totally σ–finitely generated; for reader convenience we prove that $\mathrm{Cl}_\sigma^M(N) = (N : \mathfrak{h})$ for some $\mathfrak{h} \in \mathcal{L}(\sigma)$. By the hypothesis, there exist $H \subseteq \mathrm{Cl}_\sigma^M(N)$, finitely generated, and $\mathfrak{h} \in \mathcal{L}(\sigma)$, finitely generated, such that

$\mathrm{Cl}^M_\sigma(N)\mathfrak{h} \subseteq H \subseteq \mathrm{Cl}^M_\sigma(N)$. Since M is totally σ–noetherian, there exists $\mathfrak{h}' \in \mathcal{L}(\sigma)$, finitely generated, such that $H\mathfrak{h}' \subseteq N$. Therefore,

$$N\mathfrak{h}\mathfrak{h}' \subseteq \mathrm{Cl}^M_\sigma(N)\mathfrak{h}\mathfrak{h}' \subseteq H\mathfrak{h}' \subseteq N.$$

In particular, $\mathrm{Cl}^M_\sigma(N) \subseteq (N : \mathfrak{h}\mathfrak{h}')$. On the other hand, $(N : \mathfrak{h}\mathfrak{h}')\mathfrak{h}\mathfrak{h}' \subseteq N$, hence $\mathrm{Cl}^M_\sigma(N) = (N : \mathfrak{h}\mathfrak{h}')$.

(b) $\Rightarrow$ (a) For any submodule $H \subseteq M$, there is $N \subseteq H$, finitely generated, such that $\mathrm{Cl}^M_\sigma(H) = \mathrm{Cl}^M_\sigma(N)$, and there exists $\mathfrak{h} \in \mathcal{L}(\sigma)$, finitely generated such that $\mathrm{Cl}^M_\sigma(N) = (N : \mathfrak{h})$. Therefore we have:

$$H\mathfrak{h} \subseteq \mathrm{Cl}^M_\sigma(H)\mathfrak{h} = \mathrm{Cl}^M_\sigma(N)\mathfrak{h} \subseteq N \subseteq H. \qquad \square$$

We know how to induce hereditary torsion theories through a ring map; here we study the particular case of a ring map $f : A \longrightarrow B$ such that every ideal of B is extended of an ideal of A, i.e., for any ideal $\mathfrak{b} \subseteq B$ there exists an ideal $\mathfrak{a} \subseteq A$ such that $f(\mathfrak{a})B = \mathfrak{b}$.

If σ is a hereditary torsion theory in **Mod**– A, then $f(\sigma)$ is a hereditary torsion theory in **Mod**– B and their Gabriel filter is

$$\mathcal{L}(f(\sigma)) = \{\mathfrak{b} \subseteq B \mid \ f^{-1}(\mathfrak{b}) \in \mathcal{L}(\sigma)\}.$$

It is clear that $f(\sigma)$ is of finite type whenever σ is.

In this situation we have:

Lemma 2.5. *Let σ be a finite type hereditary torsion theory, and let $f : A \longrightarrow B$ be a ring map such that every ideal of B is an extended ideal. In this case $\mathcal{L}(f(\sigma)) = \{f(\mathfrak{a})B \mid \ \mathfrak{a} \in \mathcal{L}(\sigma)\}$.*

If A is totally σ–noetherian, then B is totally $f(\sigma)$–noetherian.

Proof. For any ideal $\mathfrak{b} \subseteq B$, there exists $\mathfrak{a} \subseteq A$ such that $\mathfrak{b} = f(\mathfrak{a})B$. There exists $\mathfrak{h} \in \mathcal{L}(\sigma)$, finitely generated, such that $\mathfrak{a}\mathfrak{h} \subseteq \mathfrak{a}' \subseteq \mathfrak{a}$, for some finitely generated ideal $\mathfrak{a}' \subseteq \mathfrak{a}$. Therefore, $\mathfrak{b}f(\mathfrak{h})B = f(\mathfrak{a})f(\mathfrak{h})B \subseteq f(\mathfrak{a}')B \subseteq \mathfrak{b}$. $\square$

Examples of this situation are the following:

(1) B is the quotient of a ring A by an ideal $\mathfrak{a}$, i.e., $p : A \longrightarrow A/\mathfrak{a}$.
(2) B is the localized ring of A at a multiplicatively closed subset $\Sigma \subseteq A$, i.e., $q : A \longrightarrow A_\Sigma$.

3. Prime ideals

If σ is a hereditary torsion theory in **Mod**– A, it is well known that for any prime ideal $\mathfrak{p} \subseteq A$ we have either $\mathfrak{p} \in C(A, \sigma)$ or $\mathfrak{p} \in \mathcal{L}(\sigma)$, i.e., either $A/\mathfrak{p}$ is σ–torsionfree or $A/\mathfrak{p}$ is σ–torsion. In consequence, σ produces a partition of $\operatorname{Spec}(A)$ in two sets: $\operatorname{Spec}(A) = \mathcal{K}(\sigma) \cup \mathcal{Z}(\sigma)$, with $\mathcal{K}(\sigma) \subseteq C(A, \sigma)$, and $\mathcal{Z}(\sigma) \subseteq \mathcal{L}(\sigma)$. In addition, for every $\mathfrak{p} \in \mathcal{K}(\sigma)$ we have $\sigma \leq \sigma_{A \setminus \mathfrak{p}}$, and $\sigma = \wedge\{\sigma_{A\setminus\mathfrak{p}} \mid \mathfrak{p} \in \mathcal{K}(\sigma)\}$ whenever σ is of finite type.

Is any maximal, among the non totally σ–finitely generated submodules, a prime submodule? We know that it holds in the case of finitely generated modules. We now prove it for totally σ–finitely generated modules.

Proposition 3.1. *Let σ be a finite type hereditary torsion theory, and let M be a totally σ–finitely generated A–module. Any $N \subseteq M$, maximal among the submodules of M which are not totally σ–finitely generated, is a prime submodule.*

Proof. Given a maximal submodule $N \subseteq M$. If $N \subseteq M$ is not prime, there exist $m \in M \setminus N$ and $a \in A \setminus (N : M)$ such that $ma \in N$.

Since $a \notin (N : M)$, we have $Ma \nsubseteq N$, and $N \subsetneqq N + Ma$ is totally σ–finitely generated. On the other hand, $N \subsetneqq (N : a)$ is totally σ–finitely generated, since $m \in (N : a) \setminus N$. Therefore, there exist finitely generated submodules $F = (f_1, \ldots, f_r) \subseteq N + Ma$ and $G = (g_1, \ldots, g_s) \subseteq (N : a)$, and $\mathfrak{h} = (h_1, \ldots, h_t) \in \mathcal{L}(\sigma)$, finitely generated, such that $(N + Ma)\mathfrak{h} \subseteq F \subseteq N + Ma$ and $(N : a)\mathfrak{h} \subseteq G \subseteq (N : a)$. Say $f_i = n_i + m_i a$ for $i = 1, \ldots, r$, $n_i \in N$ and $m_i \in M$.

For any $n \in N$ and $h_i \in \{h_1, \ldots, h_t\}$, since $nh_i \in F$, there exists a linear combination $nh_i = \sum_l f_l c_{i,l} = \sum_l n_l c_{i,l} + \sum_l m_l c_{i,l} a$, and $(\sum_l m_l c_{i,l})a = nh_i - \sum_l n_l c_{i,l} \in N$. In consequence $\sum_l m_l c_{i,l} \in (N : a)$.

For any $h_j \in \{h_1, \ldots, h_t\}$ we have $(\sum_l m_l c_{i,l})h_j \in G$, and there exists a linear combination

$$\left(\sum_l m_l c_{i,l}\right) h_j = \sum_k g_k d_{i,j,k}.$$

Therefore, we have

$$\begin{aligned} nh_ih_j = \left(\sum_l n_l c_{i,l} + \sum_l m_l c_{i,l} a\right) h_j &= \sum_l n_l c_{i,l} h_j + \sum_l m_l c_{i,l} h_j a \\ &= \sum_l n_l c_{i,l} h_j + \sum_k g_k d_{i,j,k} a, \end{aligned}$$

which means that $n\mathfrak{h}\mathfrak{h}$ is contained in the submodule generated by $\{n_1, \ldots, n_r\} \cup \{g_1 a, \ldots, g_s a\} \subseteq N$. Thus reaching a contradiction. □

Corollary 3.2. *Let σ be a finite type hereditary torsion theory in **Mod**– A, and let M be a totally σ–finitely generated module. If $N \subseteq M$ is maximal among all non–totally σ–finitely generated submodules of M, then $(N : M)$ is a prime ideal.*

The next one is a result of the finite type hereditary torsion theories.

Lemma 3.3. *Let σ be a finite type hereditary torsion theory in **Mod**– A. For every totally σ–finitely generated A–module M, and any $L \in C(M, \sigma)$, $L \subsetneqq M$, there exists a maximal element $N \in C(M, \sigma)$ such that $L \subseteq N$. In addition, if $\Gamma = \{N \subseteq M \mid \ L \subseteq N \in C(M, \sigma),\ N \neq M\}$, every maximal element in Γ is a prime submodule.*

Proof. For any chain $\{N_i \mid \ i \in I\}$ in Γ we define $N = \cup_{i \in I} N_i$. If $\mathrm{Cl}^M_\sigma(N) \neq M$, then $\mathrm{Cl}^M_\sigma(N)$ is an upper bound of the chain in Γ. If $\mathrm{Cl}^M_\sigma(N) = M$, since there exist $m_1, \ldots, m_t \in M$ and $\mathfrak{h} \in \mathcal{L}(\sigma)$, finitely generated, such that $M\mathfrak{h} \subseteq (m_1, \ldots, m_t)A \subseteq M$, then there exists $\mathfrak{b} \in \mathcal{L}(\sigma)$, finitely generated, such that $(m_1, \ldots, m_t)\mathfrak{b} \subseteq \cup_i N_i = N$; therefore, there exists an index i such that $(m_1, \ldots, m_t)\mathfrak{b} \subseteq N_i$. In consequence, $N_i = \mathrm{Cl}^M_\sigma(N_i) = M$, which is a contradiction because $N_i \in \Gamma$. □

The following result is based on [10, Theorem 1], see also [2, Proposition 4].

Theorem 3.4. *Given a finite type hereditary torsion theory σ in **Mod**– A. For any totally σ–finitely generated module M, the following statements are equivalent:*

(a) *M is totally σ–noetherian.*
(b) *For every prime ideal $\mathfrak{p} \in \mathcal{K}(\sigma)$ the submodule $M\mathfrak{p} \subseteq M$ is totally σ–finitely generated.*

Proof. Clearly, if $\mathfrak{p} \in \mathcal{Z}(\sigma)$, then $M\mathfrak{p} \subseteq M$ is σ–dense, whence $M\mathfrak{p}$ is totally σ–finitely generated, because M is.

(a) ⇒ (b) is obvious.

(b) ⇒ (a) Since M is totally σ–finitely generated, there exist $\mathfrak{h} \in \mathcal{L}(\sigma)$, finitely generated, and $a_1, \ldots, a_t \in M$ such that $M\mathfrak{h} \subseteq (a_1, \ldots, a_t) \subseteq M$. If M is not totally σ–noetherian, the set

$$\Gamma = \{N \subseteq M \mid \ N \text{ is not totally } \sigma\text{–finitely generated}\}$$

is not empty. Since any chain in Γ has a upper bound in Γ, by Zorn's lemma, there exists $N \in \Gamma$ maximal. By Corollary 3.2, the ideal $\mathfrak{p} = (N : M)$ is prime.

If $\mathfrak{p} \in \mathcal{Z}(\sigma)$, then N is totally σ–finitely generated as M is, which is a contradiction. Therefore, $\mathfrak{p} \in \mathcal{K}(\sigma)$, and, by the hypothesis, since $M\mathfrak{p}$ is totally σ–finitely generated, there exist $\mathfrak{h}' \in \mathcal{L}(\sigma)$, finitely generated, and $b_1, \ldots, b_s \in M\mathfrak{p}$, such that $M\mathfrak{p}\mathfrak{h}' \subseteq (b_1, \ldots, b_s) \subseteq M\mathfrak{p}$.

Since $\mathfrak{p}$ is prime and $\mathfrak{p} = (N : M) \subseteq (N : (a_1, \ldots, a_t)) \subseteq (N : M\mathfrak{h}) = ((N : M) : \mathfrak{h}) = (\mathfrak{p} : \mathfrak{h}) = \mathfrak{p}$, we have $\mathfrak{p} = (N : a_1) \cap \ldots \cap (N : a_t)$, and there exists an index i such that $\mathfrak{p} = (N : a_i)$; this means that $a_i \notin N$, whence $N \subsetneqq N + Aa_i$, and $N + Aa_i$ is totally σ–finitely generated. In consequence, there exist $\mathfrak{h}'' \in \mathcal{L}(\sigma)$, $n_1, \ldots, n_r \in N$, and $x_1, \ldots, x_r \in A$ such that $N\mathfrak{h}'' \subseteq (n_1 + x_1 a_i, \ldots, n_r + x_r a_i) \subseteq N + Aa_i$, whence $N\mathfrak{h}'' \subseteq (n_1, \ldots, n_r) + a_i\mathfrak{p}$. In conclusion, $N\mathfrak{h}''\mathfrak{h}' \subseteq (n_1, \ldots, n_r)\mathfrak{h}' + (b_1, \ldots, b_s) \subseteq N + M\mathfrak{p} \subseteq N$, and N must be totally σ–finitely generated, which is a contradiction. □

As a direct consequence we have:

Corollary 3.5. *[Cohen–like theorem] Given a finite type hereditary torsion theory σ in **Mod**– A, the following statements are equivalent:*

(a) *A is totally σ–noetherian.*

(b) *Every prime ideal in $\mathcal{K}(\sigma)$ is totally σ–finitely generated.*

Proof. It is a direct consequence of Theorem 3.4. □

When we particularize to the hereditary torsion theory $\sigma = 0$, i.e., when $\mathcal{L}(\sigma) = \{A\}$, we have that A is a noetherian ring if, and only if, every prime ideal is finitely generated, which is Cohen's Theorem. On the other hand, if $\sigma = \sigma_S$, for some multiplicatively closed subset $S \subseteq A$, then A is S–noetherian if, and only if, every prime ideal, in $\mathcal{K}(\sigma_S)$, is totally S–finite, see [2].

4. Maximal conditions

Let M be an A–module, an increasing chain of submodules $\{N_i \mid i \in I\}$ (as usual, this induces a relation in I: for any $i, j \in I$ we have $i \leq j$ whenever $N_i \subseteq N_j$) is **totally σ–stable** whenever there exist an index $j \in I$ and $\mathfrak{h} \in \mathcal{L}(\sigma)$ such that $(\cup_i N_i)\mathfrak{h} \subseteq N_j$. The chain $\{N_i \mid i \in I\}$ is a **countable chain** whenever the index set I is countable.

Proposition 4.1. *For any hereditary torsion theory σ in **Mod**– A and any A–module M, the following statements are equivalent:*

(a) *M is totally σ–noetherian.*

(b) *Every increasing chain $\{N_i \mid i \in I\}$ of submodules of M is totally σ–stable.*

(c) *Every increasing countable chain $\{N_n \mid n \in \mathbb{N}\}$ of submodules of M is totally σ–stable.*

Proof. (a) $\Rightarrow$ (b) Given $\{N_i \mid i \in I\}$, an increasing chain of submodules of M, we define $N = \cup_i N_i$. By the hypothesis, there exist $\mathfrak{h} \subseteq \mathcal{L}(\sigma)$ and $x_1, \ldots, x_t \in N$ such that $N\mathfrak{h} \subseteq (x_1, \ldots, x_t) \subseteq N$. Therefore, since there exists an index j such that $x_1, \ldots, x_t \in N_j$, and we have $(\cup_i N_i)\mathfrak{h} = N\mathfrak{h} \subseteq N_j$.

(b) $\Rightarrow$ (c) is obvious.

(c) $\Rightarrow$ (a) If M is not totally σ–noetherian there is a submodule $N \subseteq M$ which is not totally σ–finitely generated. Always we may take a non–zero finitely generated submodule $N_0 \subseteq N$, there exists a non–zero totally σ–finitely generated submodule of N, and since N is not totally σ–finitely generated, for any $\mathfrak{h} \in \mathcal{L}(\sigma)$ there exists $x \in N \setminus N_0$ such that $x\mathfrak{h} \nsubseteq N_0$. If we define $N_1 = N_0 + xA$, it is totally σ–finitely generated; hence we may build $N_2 \subseteq N$ totally σ–finitely generated such that $N_2\mathfrak{h} \nsubseteq N_1$ for every $\mathfrak{h} \in \mathcal{L}(\sigma)$, and so on.

In this way we have a countable chain $\{N_n \mid n \in \mathbb{N}\}$ such that $N_{n+1}\mathfrak{h} \nsubseteq N_n$ for any $\mathfrak{h} \in \mathcal{L}(\sigma)$, and any $n \in \mathbb{N}$. By the hypothesis there exist an ideal $\mathfrak{h} \in \mathcal{L}(\sigma)$ and an index $m \in \mathbb{N}$ such that $(\cup_n N_n)\mathfrak{h} \subseteq N_m$, which is a contradiction. $\square$

For any A–module M, we do the following definitions:

(1) Let $\mathcal{N} \subseteq \mathcal{L}(M)$ be a family of submodules of M. An element $N \in \mathcal{N}$ is σ–**maximal** if there exists $\mathfrak{h} \in \mathcal{L}(\sigma)$ such that for every $H \in \mathcal{N}$ satisfying $N \subseteq H$ we have $H\mathfrak{h} \subseteq N$.

(2) The A–module M satisfies the **σ-MAX condition** if every nonempty family of submodules of M has σ–maximal elements.

(3) A family $\mathcal{N}$ of submodules of M is σ–**upper closed** if for every submodule $H \subseteq M$ such that there exist $N \in \mathcal{N}$ and $\mathfrak{h} \in \mathcal{L}(\sigma)$ satisfying $N \subseteq H$ and $H\mathfrak{h} \subseteq N$, or equivalently $N \subseteq H \subseteq (N : \mathfrak{h})$, we have $H \in \mathcal{N}$.

Proposition 4.2. *For any A–module M, the following statements are equivalent:*

(a) *M is totally σ–noetherian.*

(b) *Every nonempty σ–upper closed family of submodules of M has maximal elements.*

(c) *Every nonempty family of submodules of M has σ–maximal elements.*

Proof. (a) $\Rightarrow$ (b) Let $\mathcal{N}$ be a nonempty σ–upper closed family of submodules of M. For any increasing chain $\{N_i \mid i \in I\}$ in $\mathcal{N}$ we define $N = \cup_i N_i$. By the hypothesis there exist an index $j \in J$ and $\mathfrak{h} \in \mathcal{L}(\sigma)$ such that $(\cup_i N_i)\mathfrak{h} \subseteq N_m$. Hence $N\mathfrak{h} \subseteq N_j$, and $N \in \mathcal{N}$. In consequence, by Zorn's lemma, $\mathcal{N}$ contains maximal elements.

(b) $\Rightarrow$ (c) Let $\mathcal{N}$ be a nonempty family of submodules of M. We define a new family

$$\overline{\mathcal{N}} = \{H \subseteq M \mid \text{ there exist } N \in \mathcal{N} \text{ and } \mathfrak{h} \in \mathcal{L}(\sigma) \text{ such that } H\mathfrak{h} \subseteq N\},$$

the σ–**upper closure** of $\mathcal{N}$. We claim $\overline{\mathcal{N}}$ is σ–upper closed. Indeed, if $L \subseteq M$, $H \in \overline{\mathcal{N}}$ and $\mathfrak{h} \in \mathcal{L}(\sigma)$ satisfy $L \subseteq (H : \mathfrak{h})$, by the hypothesis there exist $N \in \mathcal{N}$ and $\mathfrak{h}' \in \mathcal{L}(\sigma)$ such that $H \subseteq (N : \mathfrak{h}')$, hence we have $L \subseteq (H : \mathfrak{h}) \subseteq ((N : \mathfrak{h}') : \mathfrak{h}) = (N : \mathfrak{h}'\mathfrak{h})$, and $L \in \overline{\mathcal{N}}$.

By the hypothesis, there exists a maximal element, say H, in $\overline{\mathcal{N}}$, and there exist $N \in \mathcal{N}$ and $\mathfrak{h} \in \mathcal{L}(\sigma)$ such that $H \subseteq (N : \mathfrak{h})$. Since $(N : \mathfrak{h}) \in \overline{\mathcal{N}}$, we have $H = (N : \mathfrak{h})$. We claim N is σ–maximal in $\mathcal{N}$. Indeed, if $N \subseteq L$ for some $L \in \mathcal{N}$, then $H = (N : \mathfrak{h}) \subseteq (L : \mathfrak{h})$, by the maximality of H we have $(L : \mathfrak{h}) = (N : \mathfrak{h})$, hence $L\mathfrak{h} \subseteq N$.

(c) $\Rightarrow$ (a) For any $\{N_i \mid i \in I\}$ increasing chain of submodules of M. We consider the family $\mathcal{N} = \{N_i \mid i \in I\}$. By the hypothesis $\mathcal{N}$ has σ–maximal elements. If $N_j \in \mathcal{N}$ is σ–maximal, there exists $\mathfrak{h} \in \mathcal{L}(\sigma)$ such that $N_i\mathfrak{h} \subseteq N_j$ for every $i \geq j$; hence M is totally σ–noetherian. □

Observe that if M is an A–module, for any submodule $N \subseteq M$ we may consider the family $\mathcal{N} = \{N\}$, and their σ–**upper closure**

$$\overline{\{N\}} = \{H \subseteq M \mid \text{ there exists } \mathfrak{h} \in \mathcal{L}(\sigma) \text{ such that } N \subseteq H \subseteq (N : \mathfrak{h})\},$$

hence for every $H \in \overline{\{N\}}$ we have $N \subseteq H \subseteq \mathrm{Cl}_\sigma^M(N)$. In addition, we have:

Proposition 4.3. *If $\mathcal{N} = \{N\}$, then we have; $\overline{\mathcal{N}}$ has only one maximal element if, and only if, $\mathrm{Cl}_\sigma^M(N) \in \overline{\mathcal{N}}$.*

Proof. Given $H \in \overline{\mathcal{N}}$ be the only maximal element, if there exists $x \in \mathrm{Cl}_\sigma^M(N) \setminus H$, there are $\mathfrak{h} \in \mathcal{L}(\sigma)$ such that $H\mathfrak{h} \subseteq N$ and $x\mathfrak{h} \subseteq N$, hence $(H + (x))\mathfrak{h} \subseteq N$, and $H + (x) \in \overline{\mathcal{N}}$, which is a contradiction. □

Proposition 4.4. *Given an A–module M, and a totally σ–torsion submodule $T \subseteq M$, the following statements are equivalent:*

(a) *M is totally σ–noetherian.*

(b) *M/T is totally σ–noetherian.*

Proof. (a) $\Rightarrow$ (b) Given $\{N_i/T \mid i \in I\}$, an increasing chain of submodules of M/T, there exist $j \in I$ and $\mathfrak{h} \in \mathcal{L}(\sigma)$ such that $N_i\mathfrak{h} \subseteq N_j$, for every $i \geq j$. Therefore,

$$\frac{N_i}{T}\mathfrak{h} = \frac{N_i\mathfrak{h} + T}{T} \subseteq \frac{N_j}{T}.$$

(b) $\Rightarrow$ (a) If $\{N_i \mid \ i \in I\}$ is an increasing chain of submodules of M, then $\{(N_i+T)/T \mid \ i \in I\}$ is an increasing chain of submodules of M/T, and there exist $j \in I$, $\mathfrak{h} \in \mathcal{L}(\sigma)$ such that $\frac{N_i+T}{T}\mathfrak{h} \subseteq \frac{N_j+T}{T}$, for every $i \geq j$. On the other hand, there exists $\mathfrak{h}' \in \mathcal{L}(\sigma)$ such that $T\mathfrak{h}' = 0$. Therefore,

$$N_i\mathfrak{h}\mathfrak{h}' = (N_i\mathfrak{h} + T)\mathfrak{h}' \subseteq (N_j + T)\mathfrak{h}' = N_j\mathfrak{h}' \subseteq N_j. \qquad \square$$

5. Ring extensions

Let σ be a hereditary torsion theory in **Mod**– A, and $f : A \longrightarrow B$ be a ring map. To indicate that M is an A–module we may write M_A; thus for any B–module M we write M_B whenever we are considering the B–module structure on M, and write M_A for the A–module structure.

Proposition 5.1. *The set $\mathcal{L}(f(\sigma)) = \{\mathfrak{b} \subseteq B \mid \ f^{-1}(\mathfrak{b}) \in \mathcal{L}(\sigma)\}$ is a Gabriel filter in B, and it defines a hereditary torsion theory in **Mod**– B, being*

(1) $\mathcal{T}_{f(\sigma)} = \{M_B \mid \ M_A \in \mathcal{T}_\sigma\}$ *and*
(2) $\mathcal{F}_{f(\sigma)} = \{M_B \mid \ M_A \in \mathcal{F}_\sigma\}$.

We name $f(\sigma)$ the hereditary torsion theory **induced** by σ through the ring map f.

Proposition 5.2. *For any finite type hereditary torsion theory σ in **Mod**– A, and any ring map $f : A \longrightarrow B$, the induced hereditary torsion theory $f(\sigma)$ is of finite type.*

Proof. For any $\mathfrak{b} \in \mathcal{L}(f(\sigma))$, there exists $\mathfrak{a} \in \mathcal{L}(\sigma)$, finitely generated, such that $\mathfrak{a} \subseteq f^{-1}(\mathfrak{b})$. Therefore, $f(\mathfrak{a})B \in \mathcal{L}(f(\sigma))$ is finitely generated, and $f(\mathfrak{a}) \subseteq \mathfrak{b}$, hence $f(\sigma)$ is finitely generated. $\square$

Theorem 5.3. *[Eakin–Nagata–like theorem] Let σ be a finite type hereditary torsion theory in **Mod**– A, and $f : A \hookrightarrow B$ be a ring extension such that B is a totally σ–finitely generated A–module, and $\mathfrak{p}B$ is a totally $f(\sigma)$–finitely generated B–module for every prime ideal $\mathfrak{p} \in \mathcal{K}(\sigma)$, then A is totally σ–noetherian.*

Proof. It is sufficient to prove that B is a totally σ–noetherian A–module because A is a submodule of B, or equivalently that for every prime ideal $\mathfrak{p} \in \mathcal{K}(\sigma)$ we have that $\mathfrak{p}B \subseteq B$ is totally σ–finitely generated, see Theorem 3.4. There exists $\mathfrak{a} \in \mathcal{L}(\sigma)$ such that $\mathfrak{a}\mathfrak{p}B \subseteq (p_1, \ldots, p_t)B \subseteq \mathfrak{p}B$, and there exists $\mathfrak{a}' \in \mathcal{L}(\sigma)$ such that $\mathfrak{a}'B \subseteq (b_1, \ldots, b_s)A \subseteq A$, hence

$$\mathfrak{a}'\mathfrak{a}\mathfrak{p}B \subseteq \mathfrak{a}'(p_1, \ldots, p_t)B \subseteq (p_1, \ldots, p_t)(b_1, \ldots, b_s)A.$$

□

Corollary 5.4. *[Eakin–Nagata–like theorem] Let σ be a finite type hereditary torsion theory in **Mod**– A, $f : A \hookrightarrow B$ be a ring extension such that B is a totally σ–finitely generated A–module and totally $f(\sigma)$–noetherian B–module, then A is a totally σ–noetherian.*

Proposition 5.5. *For any finite type hereditary torsion theory σ in **Mod**– A, and any ring extension $f : A \hookrightarrow B$ such that $\mathfrak{a}B \cap A = \mathfrak{a}$ for every ideal $\mathfrak{a} \subseteq A$ (e.g., B is faithfully flat). If B is totally $f(\sigma)$–noetherian, then A is totally σ–noetherian.*

Proof. Given an ideal $\mathfrak{a} \subseteq A$, since $\mathfrak{a}B \subseteq B$ is totally $f(\sigma)$–finitely generated, there exists $\mathfrak{c} \in \mathcal{L}(\sigma)$ such that $\mathfrak{c}\mathfrak{a}B \subseteq (b_1, \ldots, b_t)B$, for some $b_1, \ldots, b_t \in \mathfrak{a}$. Therefore, $\mathfrak{c}\mathfrak{a} = \mathfrak{c}(\mathfrak{a}B \cap A) \subseteq (b_1, \ldots, b_t)B \cap A = (b_1, \ldots, b_t)A$. □

This means that faithfully flat extensions are a good test for checking the totally noetherian property.

In order to consider polynomial extensions, we introduce a new kind of finite type hereditary torsion theories. A finite type hereditary torsion theory σ in **Mod**– A is **almost jansian** or **anti–Archimedean** if for every ideal $\mathfrak{a} \in \mathcal{L}(\sigma)$ we have $\cap_{n=1}^{\infty} \mathfrak{a}^n \in \mathcal{L}(\sigma)$.

Examples 5.6. (1) An example of almost jansian hereditary torsion theories are the jansian torsion theories. A hereditary torsion theory σ is **jansian** whenever $\mathcal{L}(\sigma)$ has a filter basis constituted by an ideal $\mathfrak{a}$; in this case $\mathfrak{a}$ must be idempotent. If in addition, σ is of finite type then $\mathfrak{a}$ is finitely generated, hence generated by an idempotent element, say $e \in A$, and the localization of A at σ is just the ring eA.

(2) A multiplicatively closed subset $\Sigma \subseteq A$ is **anti–Archimedean** whenever, for each $a \in \Sigma$, we have $\cap_{n=1}^{\infty} a^n A \cap \Sigma \neq \varnothing$; hence if, and only if, σ_Σ is almost jansian.

(3) An integral domain D is **anti–Archimedean** if $\cap_{n=1}^{\infty} a^n D \neq 0$ for each $a \in D \setminus \{0\}$, hence we can rewrite D is anti–Archimedean if, and only if, $\sigma_{D\setminus\{0\}}$ is almost jansian.

(4) For every prime ideal $\mathfrak{p} \subseteq A$, the hereditary torsion theory $\sigma_{A\setminus\mathfrak{p}}$ is of finite type, and it is almost jansian if, and only if, for any ideal $\mathfrak{a} \subseteq A$ if $\mathfrak{a} \nsubseteq \mathfrak{p}$, then $\cap_n \mathfrak{a}^n \nsubseteq \mathfrak{p}$ if, and only if, for any $a \in A \setminus \mathfrak{p}$ we have $\cap_n a^n A \nsubseteq \mathfrak{p}$. Let us call such a $\mathfrak{p}$ an **almost jansian prime ideal** of A.

On the other hand, the integral domain $A/\mathfrak{p}$ is anti–Archimedean if, and only if, for any $a \in A \setminus \mathfrak{p}$ we have $\cap_n(a^nA + \mathfrak{p}) \neq \mathfrak{p}$ if, and only if, $\mathfrak{p}$ is not closed in the a–adic topology in A.

In this case we have the inclusions: $\mathfrak{p} \subseteq \mathfrak{p} + \cap_n a^nA \subseteq \cap_n(\mathfrak{p} + a^nA)$; therefore if $\sigma_{A\setminus\mathfrak{p}}$ is almost jansian then $A/\mathfrak{p}$ is anti–Archimedean, i.e., $\mathfrak{p}$ is not closed in the a–adic topology in A.

The converse not necessarily holds as the following example shows. Let $A = \mathbb{Z}$, and $\mathfrak{p} = 3\mathbb{Z}$. If we take $2 \in \mathbb{Z} \setminus 3\mathbb{Z}$ we have $\cap_n 2^n\mathbb{Z} = 0 \subseteq \mathfrak{p} = 3\mathbb{Z}$; hence and $\sigma_{\mathbb{Z}\setminus 3\mathbb{Z}}$ is not almost jansian. On the other hand, $\mathbb{Z}/3\mathbb{Z}_3 = \mathbb{F}_3$ is a finite field, hence it is anti–Archimedean.

In general, for any ring A and any maximal ideal $\mathfrak{m} \subseteq A$ we have that $A/\mathfrak{m}$ is an anti–Archimedean domain, but $\sigma_{A\setminus\mathfrak{m}}$ is not almost jansian.

(5) For every strongly prime ideal $\mathfrak{p} \subseteq A$, see [9], the hereditary torsion theory $\sigma_{A\setminus\mathfrak{p}}$ is almost jansian.

(6) Since the intersection of finitely many finite type hereditary torsion theories is of finite type, if $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_t\}$ are almost jansian prime ideals of A, then $\wedge_{i=1}^t \sigma_{A\setminus\mathfrak{p}_i}$ is almost jansian.

Theorem 5.7. *[Hilbert–like basis theorem] Let σ be a finite type almost jansian hereditary torsion theory in **Mod**– A, and σ' the induced hereditary torsion theory in **Mod**– $A[X]$. If A is totally σ–noetherian, then $A[X]$ is totally σ'–noetherian.*

Proof. Given $\mathfrak{b} \subseteq A[X]$ be an ideal, we define $\mathfrak{a} = \{\mathrm{lc}(F) \mid F \in \mathfrak{b}\}$, being $\mathrm{lc}(F)$ the leading coefficient of the polynomial F, and, for convenience, $\mathrm{lc}(0) = 0$. Thus $\mathfrak{a} \subseteq A$ is an ideal, and there exist $\mathfrak{h} \in \mathcal{L}(\sigma)$, finitely generated, and $a_1, \ldots, a_t \in \mathfrak{a}$ such that $\mathfrak{a}\mathfrak{h} \subseteq (a_1, \ldots, a_t)A \subseteq \mathfrak{a}$. Let $F_1, \ldots, F_t \in \mathfrak{b}$ such that $\mathrm{lc}(F_i) = a_i$ for any $i \in \{1, \ldots, t\}$, and $d = \max\{\deg(F_i) \mid i \in \{1, \ldots, t\}\}$.

For any $n \in \mathbb{N} \setminus \{0\}$, we define $\mathcal{H}_n = \{F \in \mathfrak{b} \mid \deg(F) < n\}$, thus $\mathcal{H}_n$ is an A–module isomorphic to a submodule of the free A–module A^n, hence it is totally σ–finitely generated. There exist $\mathfrak{h}_n \in \mathcal{L}(\sigma)$, finitely generated, and $H_1, \ldots, H_s \in \mathcal{H}_n$ such that $\mathcal{H}_n\mathfrak{h}_n \subseteq (H_1, \ldots, H_s)A \subseteq \mathcal{H}_n$.

When we take $\mathcal{H}_d$, we may assume that $\mathfrak{h}_n$ and $\mathfrak{h}$ are equal. Let us suppose that $\mathfrak{h} = (h_1, \ldots, h_r)A$.

For any $F \in \mathfrak{b}$, if $f = \deg(F) < d$, then $F \in \mathcal{H}_d$. If $f = \deg(F) \geq d$, we have $\mathrm{lc}(F) \in \mathfrak{a}$, and $\mathrm{lc}(F)\mathfrak{h} \subseteq (a_1, \ldots, a_t)A$. For any $j \in \{1, \ldots, r\}$ there exists an A–linear combination $\mathrm{lc}(F)h_j = \sum_{i=1}^t a_i c_{i,j}$. Hence there exist natural numbers $e_1, \ldots, e_t$ such that $Fh_j - \sum_{i=1}^t F_i X^{e_i} c_{i,j} = G_j$ is a polynomial in $\mathfrak{b}$ of degree less than f, i.e., $G_j \in \mathcal{H}_f$. Since $Fh_j = \sum_{i=1}^t F_i X^{e_i} c_{i,j} + G_j$, we have $F\mathfrak{h} \subseteq$

$(F_1,\ldots,F_t)A[X]+(G_1,\ldots,G_r)A$. We resume this fact saying that for any $0\neq F\in\mathfrak{b}$ of degree $f\geq d$, there exists a finite subset $\mathcal{G}\subseteq\mathcal{H}_f$ such that $F\mathfrak{h}\subseteq(F_1,\ldots,F_t)A[X]+\mathcal{G}A$; if $f<d$, then $\mathcal{G}=\{H_1,\ldots,H_s\}\subseteq\mathcal{H}_d$.

Starting from a polynomial $F\in\mathfrak{b}$ of degree $f\geq d$ there exists a finite subset $\mathcal{G}_1\subseteq\mathcal{H}_f$ such that $F\mathfrak{h}\subseteq(F_1,\ldots,F_t)A[X]+\mathcal{G}_1A$. For any $G\in\mathcal{G}_1$ there exists $\mathcal{G}'\subseteq\mathcal{H}_{f-1}$ such that $G\mathfrak{h}\subseteq(F_1,\ldots,F_t)A[X]+\mathcal{G}'A$, hence there exists a finite subset $\mathcal{G}_2\subseteq\mathcal{H}_{f-1}$ such that $F\mathfrak{h}^2\subseteq(F_1,\ldots,F_t)A[X]+\mathcal{G}_2A$. And, iterating this process, there exist $k\in\mathbb{N}$ and a finite subset $\mathcal{G}\subseteq\mathcal{H}_d$ such that $F\mathfrak{h}^k\subseteq(F_1,\ldots,F_t)A[X]+\mathcal{H}_dA$.

In consequence, $\mathfrak{b}(\cap_{k\in\mathbb{N}}\mathfrak{h}^k)\subseteq(F_1,\ldots,F_t,H_1,\ldots,H_s)A[X]\subseteq\mathfrak{b}$, where we have $H_1,\ldots,H_s\in\mathcal{H}_d$, and since σ is almost jansian $\cap_k\mathfrak{h}^k\in\mathcal{L}(\sigma)$, $\mathfrak{b}$ is totally σ'–finitely generated. □

The following consequence also holds.

Corollary 5.8. *Let σ be a finite type almost jansian hereditary torsion theory in **Mod**– A, and σ' be the induced hereditary torsion theory in **Mod**– $A[X_1,\ldots,X_n]$. If A is totally σ–noetherian, then $A[X_1,\ldots,X_n]$ is totally σ'–noetherian.*

For this result it is convenient to point out that if σ is almost jansian then σ' is also.

Since totally σ–noetherianness is preserved by localization at multiplicatively closed subsets, see Lemma 2.5, we also have the corollary.

Corollary 5.9. *Let A be a ring and σ be an almost jansian finite type hereditary torsion theory such that A is totally σ–noetherian, if σ' is the induced hereditary torsion theory in $A[X,X^{-1}]$, then $A[X,X^{-1}]$ is totally σ'–noetherian.*

Proof. We have ring maps $A\stackrel{f}{\longrightarrow}A[X]\stackrel{g}{\longrightarrow}A[X,X^{-1}]$. If we consider $\Sigma=\{X^t\mid t\in\mathbb{N}\}\subseteq A[X]$, then $A[X,X^{-1}]=A[X]_\Sigma$. Since σ is almost jansian, A is totally σ–noetherian, then $A[X]$ is totally $f(\sigma)$–noetherian, and $A[X,X^{-1}]$ is totally $gf(\sigma)$–noetherian, by Lemma 2.5. □

In the following theorem we'll use the characterization of totally σ–noetherian rings, see Cohen's Theorem 3.5, and totally σ–principal ideal integral domains, see Kaplansky's Theorem 7.1, see below.

Theorem 5.10. *[Hilbert–like basis theorem] Let A be an integral domain and σ be a finite type almost jansian hereditary torsion theory in **Mod**– A, and σ' the induced hereditary torsion theory in **Mod**– $A[\![X]\!]$. If A is a totally σ–principal ideal ring, then $A[\![X]\!]$ is totally σ'–noetherian.*

Proof. For any prime ideal $\mathfrak{q} \subseteq A[\![X]\!]$ in $\mathcal{K}(\sigma')$ we define an ideal of A as follows: $\mathfrak{q}_0 = \left\{ \left(\frac{F}{X^n}\right)(0) \mid F \in \mathfrak{q} \cap (X^n) \right\}$. By the hypothesis, there exist $a \in \mathfrak{q}_0$ and $\mathfrak{h} \in \mathcal{L}(\sigma)$ such that $\mathfrak{q}_0\mathfrak{h} \subseteq aA$.

If $X \in \mathfrak{q}$, then $\mathfrak{q} = \mathfrak{q}_0 A[\![X]\!] + (X^n)$, hence $\mathfrak{q}$ is totally σ–finitely generated. Indeed, since $a \in \mathfrak{q}$, we have: $\mathfrak{q}\mathfrak{h} = (\mathfrak{q}_0, X)A[\![X]\!] \subseteq (a, X)A[\![X]\!]$.

If $X \notin \mathfrak{q}$, let $H \in \mathfrak{q}$; if $H = XG$, then $G \in \mathfrak{q}$, and we may assume $H(0) \neq 0$. Let $F \in \mathfrak{q}$ such that $F(0) = a$; if $b = H(0)$, we have $b\mathfrak{h} \subseteq aA$, and, for any $h \in \mathfrak{h}$, there exists $c_h \in A$ such that $bh = ac_h$; hence $Hh - Fc_h = XH_2$, for some $H_2 \in \mathfrak{q}$; in consequence, $H\mathfrak{h} \subseteq (F, X)A[\![X]\!]$. By induction assume that $H\mathfrak{h}^t \subseteq (F, X^t)A[\![X]\!]$, for some $t \geq 1$; hence that exists a finitely generated ideal $\mathfrak{b} \subseteq A[\![X]\!]$ such that

$$H\mathfrak{h}^t \subseteq (F) + X^t\mathfrak{b}. \qquad (*)$$

In addition we have $\mathfrak{b}\mathfrak{h} \subseteq (F, X)A[\![X]\!]$, hence, multiplying by (*) we obtain $H\mathfrak{h}^{t+1} \subseteq (F, X^{t+1})A[\![X]\!]$. In consequence,

$$H\left(\cap_t \mathfrak{h}^t\right) \subseteq \cap_t H\mathfrak{h}^t \subseteq \cap_t (F, X^t)A[\![X]\!] = (F),$$

the last identity is consequence of [3, Lemma 2], hence $\mathfrak{q}$ is totally σ'–finitely generated, and $A[\![X]\!]$ is totally σ'–noetherian. □

Observe that Theorem 5.10 could be extended to consider A to be a totally σ–noetherian integral domain whenever $A[\![X]\!]$ satisfies the following condition:

- each finitely generated ideal of $A[\![X]\!]$ is closed in the X–adic topology.

Therefore, for any finite type almost jansian hereditary torsion theory σ on an integral domain A the power series ring $A[\![X]\!]$ is totally σ'–noetherian whenever A is.

6. Study through prime ideals

For any prime ideal $\mathfrak{p} \subseteq A$, we consider $\sigma_{A\setminus\mathfrak{p}}$, the hereditary torsion theory cogenerated by $A/\mathfrak{p}$, or equivalently, the hereditary torsion theory generated by the multiplicatively closed subset $A \setminus \mathfrak{p}$. For every torsion theory σ we consider the following sets of ideals:

(1) $\mathcal{L}(\sigma)$, the Gabriel filter of σ.
(2) $\mathcal{Z}(\sigma) = \mathcal{L}(\sigma) \cap \operatorname{Spec}(A)$. In particular, if $\mathfrak{p} \subseteq \mathfrak{q}$ are prime ideals and $\mathfrak{p} \in \mathcal{Z}(\sigma)$, then $\mathfrak{q} \in \mathcal{Z}(\sigma)$.
(3) $C(A, \sigma) = \{\mathfrak{a} \mid A/\mathfrak{a} \in \mathcal{F}_\sigma\}$.
(4) $\mathcal{K}(\sigma) = C(A, \sigma) \cap \operatorname{Spec}(A)$; it is the complement of $\mathcal{Z}(\sigma)$ in $\operatorname{Spec}(A)$. In particular, if $\mathfrak{p} \subseteq \mathfrak{q}$ are prime ideals and $\mathfrak{q} \in \mathcal{K}(\sigma)$, then $\mathfrak{p} \in \mathcal{K}(\sigma)$.

(5) $\mathcal{C}(\sigma) = \operatorname{Max}\mathcal{K}(\sigma)$.

If σ is of finite type, then $\sigma = \wedge\{\sigma_{A\setminus\mathfrak{p}} \mid \mathfrak{p} \in \mathcal{K}(\sigma)\}$. On the other hand, $\sigma = \wedge\{\sigma_{A\setminus\mathfrak{p}} \mid \mathfrak{p} \in \mathcal{C}(\sigma)\}$ whenever A is σ–noetherian, because $\sigma_{A\setminus\mathfrak{q}} \leq \sigma_{A\setminus\mathfrak{p}}$ if $\mathfrak{p} \subseteq \mathfrak{q}$, for any prime ideals $\mathfrak{p}, \mathfrak{q}$.

An A–module M is **totally $\mathfrak{p}$–noetherian** whenever M is totally $\sigma_{A\setminus\mathfrak{p}}$–noetherian.

Lemma 6.1. *Let A be a local (non necessarily noetherian) ring with maximal ideal $\mathfrak{m}$, and M be an A–module. The following statements are equivalent:*

(a) *M is noetherian.*

(b) *M is totally $\sigma_{A\setminus\mathfrak{m}}$–noetherian.*

Proof. It is immediate because every element in $A \setminus \mathfrak{m}$ is invertible. □

Proposition 6.2. *Let σ a finite type hereditary torsion theory in A, and M be an A–module. The following statements are equivalent:*

(a) *M is totally σ–noetherian.*

(b) *M is totally $\sigma_{A\setminus\mathfrak{p}}$–noetherian for every $\mathfrak{p} \in \mathcal{C}(\sigma) = \operatorname{Max}\mathcal{K}(\sigma)$.*

Proof. (a) $\Rightarrow$ (b) is immediate because $\sigma \leq \sigma_{A\setminus\mathfrak{p}}$.

(b) $\Rightarrow$ (a) Given $N \subseteq M$, for every $\mathfrak{m} \in \mathcal{C}(\sigma)$ there exist $s_{\mathfrak{m}} \in A \setminus \mathfrak{m}$ and $H(\mathfrak{m}) \subseteq N$, finitely generated, such that $Ns_{\mathfrak{m}} \subseteq H(\mathfrak{m}) \subseteq N$. Since $\mathfrak{b} = \sum_{\mathfrak{m}} s_{\mathfrak{m}}A$ belongs to $\mathcal{L}(\sigma) = \cap_{\mathfrak{m}}\mathcal{L}(\sigma_{A\setminus\mathfrak{m}})$, there exists $\mathfrak{c} \in \mathcal{L}(\sigma)$, finitely generated, such that $\mathfrak{c} \subseteq \mathfrak{b}$. In consequence, there are finitely many elements $s_{\mathfrak{m}_1}, \ldots, s_{\mathfrak{m}_t}$ such that $\mathfrak{c} \subseteq \sum_{i=1}^{t} s_{\mathfrak{m}_i}A \subseteq \mathfrak{b}$, and we have $N\mathfrak{c} \subseteq \sum_{i=1}^{t} H(\mathfrak{m}_i) \subseteq N$. Hence, N is totally σ–finitely generated, and M is totally σ–noetherian. □

When we take the hereditary torsion theory $\sigma = 0$, i.e., when $\mathcal{L}(\sigma) = \{A\}$, we have that M is noetherian if, and only if, M is totally $\sigma_{A\setminus\mathfrak{m}}$–noetherian for every maximal ideal $\mathfrak{m} \in \operatorname{Supp}(M)$, which is [2, Proposition 12].

7. Principal ideal rings

If σ be a hereditary torsion theory in **Mod**– A, and $\mathfrak{a} \subseteq A$ an ideal, then we have:

(1) $\mathfrak{a}$ is σ–**principal** if there exists $a \in \mathfrak{a}$ such that $\operatorname{Cl}_{\sigma}^{A}(aA) = \operatorname{Cl}_{\sigma}^{A}(\mathfrak{a})$.

(2) A is a σ–**principal ideal ring**, σ-PIR, whenever every ideal is σ–principal.

(3) $\mathfrak{a}$ is **totally** σ–**principal** if there exist $a \in \mathfrak{a}$ and $\mathfrak{h} \subseteq \mathcal{L}(\sigma)$ such that $\mathfrak{a}\mathfrak{h} \subseteq aA \subseteq \mathfrak{a}$.

(4) A is a **totally σ–principal ideal ring**, totally σ–PIR, whenever every ideal is totally σ–principal.

Proposition 7.1. *[Kaplansky–like Theorem] For any finite type hereditary torsion theory σ in **Mod**– A, the following statements are equivalent:*

(a) *A is totally σ–PIR.*

(b) *Every prime ideal $\mathfrak{p} \in \mathcal{K}(\sigma)$ is totally σ–principal.*

See also [2, Proposition 16].

Proof. (a) $\Rightarrow$ (b) This direction is evident.

(b) $\Rightarrow$ (a) Let $\Gamma = \{\mathfrak{a} \subseteq A \mid \mathfrak{a} \text{ is not totally } \sigma\text{–principal}\}$. Let us assume $\Gamma \neq \varnothing$, $\{\mathfrak{a}_i \mid i \in I\} \subseteq \Gamma$ be a chain in Γ, and take $\mathfrak{a} = \cup_i \mathfrak{a}_i$. If $\mathfrak{a}$ is totally σ–principal, there are $\mathfrak{h} \in \mathcal{L}(\sigma)$, finitely generated, and $a \in \mathfrak{a}$ such that $\mathfrak{a}\mathfrak{h} \subseteq aA \subseteq \mathfrak{a}$, and there exists an index i such that $a \in \mathfrak{a}_i$, hence $\mathfrak{a}_i\mathfrak{h} \subseteq \mathfrak{a}\mathfrak{h} \subseteq aA \subseteq \mathfrak{a}_i$, which is a contradiction. In conclusion, every chain in Γ has a upper bound in Γ, and by Zorn's lemma there exist maximal elements in Γ.

For any $\mathfrak{a} \in \Gamma$ maximal; we claim $\mathfrak{a} = \mathrm{Cl}^A_\sigma(\mathfrak{a})$. If $\mathfrak{a} \neq \mathrm{Cl}^A_\sigma(\mathfrak{a})$ then $\mathrm{Cl}^A_\sigma(\mathfrak{a})$ is totally σ–principal, and there exist $\mathfrak{h}_1 \in \mathcal{L}(\sigma)$, $x \in \mathrm{Cl}^A_\sigma(\mathfrak{a})$ such that $\mathrm{Cl}^A_\sigma(\mathfrak{a})\mathfrak{h}_1 \subseteq xA \subseteq \mathrm{Cl}^A_\sigma(\mathfrak{a})$. On the other hand, there exists $\mathfrak{h}_2 \in \mathcal{L}(\sigma)$ such that $x\mathfrak{h}_2 \subseteq \mathfrak{a}$; therefore $\mathfrak{a}\mathfrak{h}_1\mathfrak{h}_2 \subseteq xA\mathfrak{h}_2 \subseteq \mathfrak{a}$, and $\mathfrak{a}$ is totally σ–principal, which is a contradiction.

We claim that any $\mathfrak{a} \in \Gamma$, maximal, is prime. Let $a, b \in A \setminus \mathfrak{a}$ such that $ab \in \mathfrak{a}$. Since $\mathfrak{a} + aA$ is totally σ–principal, there exist $x \in \mathfrak{a} + aA$ and $\mathfrak{h}_1 \in \mathcal{L}(\sigma)$, finitely generated, such that $(\mathfrak{a}+aA)\mathfrak{h}_1 \subseteq xA$. Since $bx \in \mathfrak{a}$, we have $\mathfrak{a} \subsetneqq (\mathfrak{a} : x)$. Therefore, $(\mathfrak{a} : x)$ is totally σ–principal, and there exist $y \in (\mathfrak{a} : x)$ and $\mathfrak{h}_2 \in \mathcal{L}(\sigma)$ such that $(\mathfrak{a} : x)\mathfrak{h}_2 \subseteq yA$.

If $x \in \mathfrak{a}$, then $(\mathfrak{a}+aA)\mathfrak{h}_2 \subseteq xA \subseteq \mathfrak{a}$, and $a \in \mathrm{Cl}^A_\sigma(\mathfrak{a}) = \mathfrak{a}$, which is a contradiction. Hence, $x \notin \mathfrak{a}$; since $\mathfrak{a}\mathfrak{h}_1 \subseteq xA$, there exists an ideal $\mathfrak{k} \subseteq A$ such that $\mathfrak{a}\mathfrak{h}_1 = x\mathfrak{k}$; hence $\mathfrak{k} \subseteq (\mathfrak{a} : x)$. Therefore,

$$\mathfrak{a}\mathfrak{h}_1\mathfrak{h}_2 = x\mathfrak{k}\mathfrak{h}_2 \subseteq xyA,$$

and $\mathfrak{a}$ is totally σ–principal, which is a contradiction. In conclusion, any maximal element of Γ belongs to $\mathcal{K}(\sigma)$, hence it is totally σ–principal, which is a contradiction. □

Corollary 7.2. *For any a finite type hereditary torsion theory σ in **Mod**– A, the following statements are equivalent:*

(a) *A is totally σ–PIR.*

(b) *A is σ-PIR and every prime ideal $\mathfrak{p}$ is totally σ–finitely generated.*

Proof. (a) $\Rightarrow$ (b) is evident.

(b) $\Rightarrow$ (a) Given $\mathfrak{p} \in \mathcal{K}(\sigma)$, by the hypothesis, there exist $\mathfrak{h}_1 \in \mathcal{L}(\sigma)$, finitely generated, and $p_1, \ldots, p_t \in \mathfrak{p}$ such that $\mathfrak{p}\mathfrak{h}_1 \subseteq (p_1, \ldots, p_t) \subseteq \mathfrak{p}$. On the other hand, there exist $p \in (p_1, \ldots, p_t)$ and $\mathfrak{h}_2 \in \mathcal{L}(\sigma)$, finitely generated, such that $(p_1, \ldots, p_t)\mathfrak{h}_2 \subseteq pA \subseteq (p_1, \ldots, p_t)$. Observe that $\mathrm{Cl}_\sigma^A(pA) = \mathfrak{p} = \mathrm{Cl}_\sigma^A(p_1, \ldots, p_t)$. Therefore, we have

$$\mathfrak{p}\mathfrak{h}_1\mathfrak{h}_2 \subseteq (p_1, \ldots, p_t)\mathfrak{h}_2 \subseteq pA \subseteq \mathfrak{p},$$

and $\mathfrak{p}$ is totally σ–principal. □

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