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DEFORMATIONS AND EXTENSIONS OF BIHOM-ALTERNATIVE ALGEBRAS

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Abstract. The aim of this paper is to deal with BiHom-alternative algebras which are a generalization of alternative and Hom-alternative algebras, their structure is defined with two commuting multiplicative linear maps. We study cohomology and one-parameter formal deformation theory of left BiHom-alternative algebras. Moreover, we study central and T_θ-extensions of BiHom-alternative algebras and their relationship with cohomology. Finally, we investigate generalized derivations and give some relevant results.

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1. Introduction

Algebras of Hom-type have been recently investigated by many authors. They appeared in the physics literature, during the nineties, in quantum deformations of some algebras of vector fields, such as the Witt and Virasoro algebras. Hartwig, Larsson and Silvestrov introduced and studied, in [21], the classes of quasi-Lie, quasi-Hom-Lie and Hom-Lie algebras. The Hom-analogues of associative, alternative, Jordan and Novikov algebras were introduced in [25,26,28]. A categorical approach to Hom-algebras was first investigated in [4] and a generalization has been introduced in [20]. A construction of a Hom-category including a group action brought to concepts of BiHom-type algebras. Hence BiHom-associative and BiHom-Lie algebras were introduced in [20]. In [22], BiHom-pre-Lie algebras and BiHom-Leibniz algebras were defined and studied via Rota-Baxter operators. The BiHom-versions of alternative, pre-alternative, quadri-alternative and Malcev algebras were introduced in [7] and [8]. The authors gave several constructions connecting such structures.

Deformation theory arose essentially from geometry and physics. It is a tool to study a mathematical object by deforming it into a family of the same kind of objects depending on a certain parameter. In the sixties, M. Gerstenhaber introduced algebraic formal deformations for associative algebras in a series of papers (see [13,14,15,16]). He used formal power series and showed that the theory is intimately connected to the cohomology of the algebra. The same approach was extended to several algebraic structures (see [1,2,5,6,23,24]). Other approaches to study deformations exist. For details see [12,17,18,19,27].

In this paper, we introduce representations, cohomology and formal deformation theory of left BiHom-alternative algebras. This concept generalizes the classical and Hom-type cases studied by M. Elhamdadi and A. Makhlouf in [10,11]. In Section 1, we review the basic definitions and properties related to left BiHom-alternative algebras and recall their representations. In Section 2, we define first, second and third coboundary operators and corresponding cohomology groups of left BiHom-alternative algebras. Section 3 is dedicated to develop a one-parameter formal deformation theory for left BiHom-alternative algebras. Section 4 is devoted to the study of central and T_θ-extensions of left BiHom-alternative algebras. In the last section, we provide some results involving generalized derivations on BiHom-alternative algebras.

Throughout this paper $\mathbb{K}$ is a field of characteristic 0 and all vector spaces are over $\mathbb{K}$. We refer to a BiHom-algebra as quadruple (A, μ, α, β) where A is a vector space, μ is a multiplication and α, β are two linear maps. It is said to be regular if α, β are invertible. The BiHom-associator of a BiHom-algebra (A, μ, α, β) is the trilinear map $as_{\alpha,\beta}$ defined for all $x, y, z \in A$ by $as_{\alpha,\beta}(x, y, z) = (xy)\beta(z) - \alpha(x)(yz)$. When there is no ambiguity, we denote for simplicity the multiplication and composition by concatenation.

2. Preliminaries and basics

In this section, we give some basics and properties on left BiHom-alternative algebras. We refer for the definitions to [7].

Definition 2.1. A left BiHom-alternative algebra (resp. right BiHom-alternative algebra) is a quadruple (A, μ, α, β) consisting of a $\mathbb{K}$-vector space A, a bilinear map $\mu : A \times A \longrightarrow A$ and two homomorphisms $\alpha, \beta : A \longrightarrow A$ such that $\alpha\beta = \beta\alpha$, $\alpha\mu = \mu\alpha^{\otimes^2}$ and $\beta\mu = \mu\beta^{\otimes^2}$, satisfying the left BiHom-alternative identity, i.e. for all $x, y, z \in A$, one has

$$as_{\alpha,\beta}(\beta(x), \alpha(y), z) + as_{\alpha,\beta}(\beta(y), \alpha(x), z) = 0, \tag{2.1}$$

respectively, the right BiHom-alternative identity, i.e. for all $x, y, z \in A$, one has

$$as_{\alpha,\beta}(x, \beta(y), \alpha(z)) + as_{\alpha,\beta}(x, \beta(z), \alpha(y)) = 0. \tag{2.2}$$

A BiHom-alternative algebra is one which is both a left and a right BiHom-alternative algebra.

Observe that when $\alpha = \beta = Id$ the left BiHom-alternative identity (2.1) reduces to the left alternative identity and when $\alpha = \beta$ we recover left Hom-alternative identity.

Lemma 2.2. *A quadruple (A, μ, α, β) is a left BiHom-alternative algebra (resp. right BiHom-alternative algebra) if and only if*

$$as_{\alpha,\beta}(\beta(x), \alpha(x), y) = 0$$

respectively

$$as_{\alpha,\beta}(x, \beta(y), \alpha(y)) = 0$$

for any $x, y \in A$. (Since the characteristic of the field $\mathbb{K}$ is not 2.)

Definition 2.3. Let (A, μ, α, β) and $(A^{'}, \mu^{'}, \alpha^{'}, \beta^{'})$ be two BiHom-alternative algebras. A homomorphism $f : A \longrightarrow A^{'}$ is said to be a BiHom-alternative algebra morphism if the following hold

$$f \circ \mu = \mu^{'} \circ (f \otimes f), \; f \circ \alpha = \alpha^{'} \circ f \text{ and } f \circ \beta = \beta^{'} \circ f.$$

Definition 2.4. Let (A, μ, α, β) be a BiHom-alternative algebra and V be a vector space. Let $\ell, r : A \to gl(V)$ and $\phi, \psi \in gl(V)$ be two commuting linear maps. Then (V, ℓ, r, ϕ, ψ) is called a bimodule, or a representation, of (A, μ, α, β), if for any $x, y \in A, \; v \in V$,

$$\phi\ell(x) = \ell(\alpha(x))\phi, \; \phi r(x) = r(\alpha(x))\phi, \tag{2.3}$$

$$\psi\ell(x) = \ell(\beta(x))\psi, \; \psi r(x) = r(\beta(x))\psi \tag{2.4}$$

$$\ell(\beta(x)\alpha(x))\psi(v) = \ell(\alpha\beta(x))\ell(\alpha(x))v, \tag{2.5}$$

$$r(\beta(x)\alpha(x))\phi(v) = r(\alpha\beta(x))r(\beta(x))v, \tag{2.6}$$

$$r(\beta(y))\ell(\beta(x))\phi(v) - \ell(\alpha\beta(x))r(y)\phi(v) = r(\alpha(x)y)\phi\psi(v) - r(\beta(y))r(\alpha(x))\psi(v), \tag{2.7}$$

$$\ell(\alpha(y))r(\alpha(x))\psi(v) - r(\alpha\beta(x))\ell(y)\psi(v) = \ell(y\beta(x))\phi\psi(v) - \ell(\alpha(y))\ell(\beta(x))\phi(v). \tag{2.8}$$

The following observation is straightforward.

Proposition 2.5. *A tuple (V, ℓ, r, ϕ, ψ) is a bimodule of a BiHom-alternative algebra (A, μ, α, β) if and only if the direct sum $(A \oplus V, *, \alpha + \phi, \beta + \psi)$ is turned into* *a BiHom-alternative algebra (the semidirect product) where*

$$(x_1 + v_1) * (x_2 + v_2) = \mu(x_1, x_2) + \ell(x_1)v_2 + r(x_2)v_1, \tag{2.9}$$

$$(\alpha + \phi)(x + v) = \alpha(x) + \phi(v), \quad (\beta + \psi)(x + v) = \beta(x) + \psi(v). \tag{2.10}$$

Remark 2.6. Let (A, μ, α, β) be a left BiHom-alternative algebra, define the operator multiplication $L, R : A \to gl(A)$ by $L(x)(y) = R(y)(x) = xy$ for x, y in A. Then (A, L, R, α, β) is a representation of A called the adjoint representation.

3. Cohomology of left BiHom-alternative algebras

In this section, we construct a cochain complex that defines a BiHom-type cohomology of a left BiHom-alternative algebra in an A-bimodule V.

Let $V = (V, \ell, r, \phi, \psi)$ be a representation of a left BiHom-alternative algebra $A = (A, \mu, \alpha, \beta)$.

Definition 3.1. A left BiHom-alternative n-cochain on A with values in V is an n-linear map f from $A \times \cdots \times A$ (n-times) to V satisfying:

$$\phi(f(x_1, \dots, x_n)) = f(\alpha(x_1), \dots, \alpha(x_n)), \tag{3.11}$$

$$\psi(f(x_1, \dots, x_n)) = f(\beta(x_1), \dots, \beta(x_n)). \tag{3.12}$$

The set of left BiHom-alternative n-cochains on A with values in V is denoted by $C^n(A; V)$.

For $n = 0$, we have $C^0(A; V) = V$.

Definition 3.2. The coboundary operator $\delta^1 : C^1(A; V) \to C^2(A; V)$ is defined by:

$$\delta^1 f(x, y) = \ell(x)f(y) + r(y)f(x) - f(xy). \tag{3.13}$$

We notice that the first differential of a left BiHom-alternative algebra is similar to the first differential map of Hochschild cohomology of a BiHom-associative algebra (1-cocycles are derivations) given in [9].

Definition 3.3. The coboundary operator $\delta^2 : C^2(A; V) \to C^3(A; V)$ is defined as follows:

$$\begin{aligned}
\delta^2 f(x, y, z) &= r(\beta(z))f(\beta(x), \alpha(y)) - \ell(\alpha\beta(x))f(\alpha(y), z) \\
&+ r(\beta(z))f(\beta(y), \alpha(x)) - \ell(\alpha\beta(y))f(\alpha(x), z) \\
&+ f(\beta(x)\alpha(y), \beta(z)) - f(\alpha\beta(x), \alpha(y)z) \\
&+ f(\beta(y)\alpha(x), \beta(z)) - f(\alpha\beta(y), \alpha(x)z).
\end{aligned}$$

Remark 3.4. We have $\delta^2 f = \delta^2 f \circ \tau_{12}$, where $\tau_{12}(x,y,z) = (y,x,z)$, for all $x,y,z \in A$.

Lemma 3.5. *The operators δ_1 and δ_2 are well defined, that is*

$$\delta^1 f \circ \alpha^{\otimes 2} = \phi \circ \delta^1 f, \quad \delta^1 f \circ \beta^{\otimes 2} = \psi \circ \delta^1 f,$$
$$\delta^2 f \circ \alpha^{\otimes 3} = \phi \circ \delta^2 f, \quad \delta^2 f \circ \beta^{\otimes 3} = \psi \circ \delta^2 f.$$

Proof. For any $x,y,z \in A$, we have

$$\begin{aligned}
\delta^1 f\alpha(x,y) &= \delta^1 f(\alpha(x),\alpha(y)) \\
&= \ell(\alpha(x))f(\alpha(y)) + r(\alpha(y))f(\alpha(x)) - f(\alpha(x)\alpha(y)) \\
&= \ell(\alpha(x))\phi(f(y)) + r(\alpha(y))\phi(f(x)) - \phi(f(xy)) \\
&= \phi(\ell(x)f(y)) + \phi(r(y)f(x)) - \phi(f(xy)) \\
&= \phi \circ \delta^1 f(x,y)
\end{aligned}$$

and

$$\begin{aligned}
&\delta^2 f \circ \alpha(x,y,z) = \delta^2 f(\alpha(x),\alpha(y),\alpha(z)) \\
=&r(\alpha\beta(z))f(\alpha\beta(x),\alpha^2(y)) - \ell(\alpha^2\beta(x))f(\alpha^2(y),\alpha(z)) \\
&+r(\alpha\beta(z))f(\alpha\beta(y),\alpha^2(x)) - \ell(\alpha^2\beta(y))f(\alpha^2(x),\alpha(z)) \\
&+f(\alpha\beta(x)\alpha^2(y),\alpha\beta(z)) - f(\alpha^2\beta(x),\alpha^2(y)z) \\
&+f(\alpha\beta(y)\alpha^2(x),\alpha\beta(z)) - f(\alpha^2\beta(y),\alpha^2(x)\alpha(z)) \\
=&r(\alpha\beta(z))\phi f(\beta(x),\alpha(y)) - \ell(\alpha^2\beta(x))\phi f(\alpha(y),z) \\
&+r(\alpha\beta(z))\phi f(\beta(y),\alpha(x)) - \ell(\alpha^2\beta(y))\phi f(\alpha(x),z) \\
&+\phi f(\beta(x)\alpha(y),\beta(z)) - \phi f(\alpha\beta(x),\alpha(y)z) \\
&+\phi f(\beta(y)\alpha(x),\beta(z)) - \phi f(\alpha\beta(y),\alpha(x)z) \\
=&\phi \circ \delta^2 f(x,y,z).
\end{aligned}$$

The other identities can be shown using the same computation. $\square$

Proposition 3.6. *The composite $\delta^2 \circ \delta^1$ is zero.*

Proof. Let $x,\ y,\ z \in A$ and $f \in C^1(A,V)$. Then

$$\begin{aligned}
\delta^2\circ\delta^1 f(x,y,z) &= r(\beta(z))\delta^1 f(\beta(x),\alpha(y)) - \ell(\alpha\beta(x))\delta^1 f(\alpha(y),z)\\
&+r(\beta(z))\delta^1 f(\beta(y),\alpha(x)) - \ell(\alpha\beta(y))\delta^1 f(\alpha(x),z) + \delta^1 f(\beta(x)\alpha(y),\beta(z))\\
&-\delta^1 f(\alpha\beta(x),\alpha(y)z) + \delta^1 f(\beta(y)\alpha(x),\beta(z)) - \delta^1 f(\alpha\beta(y),\alpha(x),z)\\
&=r(\beta(z))\ell(\beta(x))f(\alpha(y)) + r(\beta(z))r(\alpha(y))f(\beta(x)) - r(\beta(z))f(\beta(x)\alpha(y))\\
&-\ell(\alpha\beta(x))\ell(\alpha(y))f(z) - \ell(\alpha\beta(x))r(z)f(\alpha(y)) + \ell(\alpha\beta(x))f(\alpha(y)z)\\
&+r(\beta(z))\ell(\beta(y))f(\alpha(x)) + r(\beta(z))r(\alpha(x))f(\beta(y)) - r(\beta(z))f(\beta(y)\alpha(x))\\
&-\ell(\alpha\beta(y))\ell(\alpha(x))f(z) - \ell(\alpha\beta(y))r(z)f(\alpha(x)) + \ell(\alpha\beta(y))f(\alpha(x)z)\\
&+\ell(\beta(x)\alpha(y))f(\beta(z)) + r(\beta(z))f(\beta(x)\alpha(y)) - f((\beta(x)\alpha(y))\beta(z))\\
&-\ell(\alpha\beta(x))f(\alpha(y)z) - r(\alpha(y)z)f(\alpha\beta(x)) + f(\alpha\beta(x)(\alpha(y)z))\\
&+\ell(\beta(y)\alpha(x))f(\beta(z)) + r(\beta(z))f(\beta(y)\alpha(x)) - f((\beta(y)\alpha(x))\beta(z))\\
&-\ell(\alpha\beta(y))f(\alpha(x)z) - r(\alpha(x)z)f(\alpha\beta(y)) + f(\alpha\beta(y)(\alpha(x)z))\\
&=r(\beta(z))\ell(\beta(x))f(\alpha(y)) + r(\beta(z))r(\alpha(x))f(\beta(y)) \qquad (3.14)\\
&-\ell(\alpha\beta(x))r(z)f(\alpha(y)) - r(\alpha(x)z)f(\alpha\beta(y)) \qquad (3.15)\\
&+r(\beta(z))\ell(\beta(y))f(\alpha(x)) + r(\beta(z))r(\alpha(y))f(\beta(x)) \qquad (3.16)\\
&-\ell(\alpha\beta(y))r(z)f(\alpha(x)) - r(\alpha(y)z)f(\alpha\beta(x)) \qquad (3.17)\\
&+\ell(\beta(x)\alpha(y))f(\beta(z)) - \ell(\alpha\beta(x))\ell(\alpha(y))f(z) \qquad (3.18)\\
&+\ell(\beta(y)\alpha(x))f(\beta(z)) - \ell(\alpha\beta(y))\ell(\alpha(x))f(z) \qquad (3.19)\\
&-f(as_{\alpha,\beta}(\beta(x),\alpha(y),z) + as_{\alpha,\beta}(\beta(y),\alpha(x),z). \qquad (3.20)
\end{aligned}$$

We have (3.14)+(3.15)=0 and (3.16)+(3.17)=0, using (2.7). Moreover, (3.18)+(3.19)=0, using (2.5). In addition, (3.20)=0, since A is left BiHom-alternative.

Then, we get $\delta^2\circ\delta^1 = 0$. □

Definition 3.7. The coboundary operator $\delta^3 : C^3(A;V) \to C^4(A;V)$ is defined by

$$\begin{aligned}
\delta^3 f(x_1,x_2,x_3,x_4) &= \ell(\alpha(x_1))f(\beta(x_2),\beta(x_3),\beta(x_4)) - \ell(\alpha(x_1))f(\beta(x_3),\beta(x_2),\beta(x_4))\\
&+r(\beta(x_4))f(\alpha(x_1),\alpha(x_2),\alpha(x_3)) - r(\beta(x_4))f(\alpha(x_2),\alpha(x_1),\alpha(x_3))\\
&-f(\alpha(x_1)\beta(x_2),x_3,x_4) - f(\alpha(x_2)\beta(x_3),x_1,x_4) + f(x_1,\alpha(x_2)\beta(x_3),x_4)\\
&+f(x_3,a(x_1)\beta(x_2),x_4) - f(x_1,x_2,\alpha(x_3)\beta(x_4)) + f(x_2,x_1,\alpha(x_3)\beta(x_4)).
\end{aligned}$$

Lemma 3.8. *The operator δ^3 is well defined, that is*

$$\delta^3 f\circ\alpha^{\otimes 4} = \phi\circ\delta^3 f \ \textit{and}\ \delta^3 f\circ\beta^{\otimes 4} = \psi\circ\delta^3 f.$$

Proof. We can check these identities by a direct computation. □

Proposition 3.9. *The composite $\delta^3 \circ \delta^2$ is zero.*

Proof. Let $x_1, x_2, x_3, x_4 \in A$ and $f \in C^2(A;V)$. Then, by substituting f with $\delta^2 f$ in the previous formula and rearranging the terms we get

$$\begin{aligned}
&\delta^3(\delta^2 f)(x_1,x_2,x_3,x_4) =\\
&\ell(\alpha(x_1))\delta^2 f(\beta(x_2),\beta(x_3),\beta(x_4)) - \ell(\alpha(x_1))\delta^2 f(\beta(x_3),\beta(x_2),\beta(x_4))\\
&+ r(\beta(x_4))\delta^2 f(\alpha(x_1),\alpha(x_2),\alpha(x_3)) - r(\beta(x_4))\delta^2 f(\alpha(x_2),\alpha(x_1),\alpha(x_3))\\
&- \delta^2 f(\alpha(x_1)\beta(x_2),x_3,x_4) - \delta^2 f(\alpha(x_2)\beta(x_3),x_1,x_4)\\
&+ \delta^2 f(x_1,\alpha(x_2)\beta(x_3),x_4) + \delta^2 f(x_3,\alpha(x_1)\beta(x_2),x_4)\\
&- \delta^2 f(x_1,x_2,\alpha(x_3)\beta(x_4)) + \delta^2 f(x_2,x_1,\alpha(x_3)\beta(x_4)).\\
&= \ell(\alpha(x_1))(\delta^2 f(\beta(x_2),\beta(x_3),\beta(x_4)) - \delta^2 f(\beta(x_3),\beta(x_2),\beta(x_4)))\\
&+ r(\beta(x_4))(\delta^2 f(\alpha(x_1),\alpha(x_2),\alpha(x_3) - \delta^2 f(\alpha(x_2),\alpha(x_1),\alpha(x_3))\\
&+ \delta^2 f(x_3,\alpha(x_1)\beta(x_2),x_4) - -\delta^2 f(\alpha(x_1)\beta(x_2),x_3,x_4)\\
&+ \delta^2 f(x_1,\alpha(x_2)\beta(x_3),x_4) - \delta^2 f(\alpha(x_2)\beta(x_3),x_1,x_4)\\
&+ \delta^2 f(x_2,x_1,\alpha(x_3)\beta(x_4)). - \delta^2 f(x_1,x_2,\alpha(x_3)\beta(x_4))\\
&= 0 \quad (\text{since } \delta^2 f = \delta^2 f \circ \tau_{12}).
\end{aligned}$$

□

One can complete the complex by considering $\delta^p = 0$, for $p > 3$. For $n = 1,2,3$, the map $f \in C^n(A;V)$ is called an n-BiHom-cocycle (or simply an n-cocycle), if $\delta^n f = 0$. We denote by $Z^n(A;V)$ the subspace spanned by n-BiHom-cocycles and by $B^n(A;V) = \delta^{n-1}(C^{n-1}(A;V))$. Since $\delta^2 \circ \delta^1 = 0$ and $\delta^3 \circ \delta^2 = 0$, $B^2(A;V)$ is a subspace of $Z^2(A;V)$ and $B^3(A;V)$ is a subspace of $Z^3(A;V)$. Hence, we can define cohomology spaces of (A,μ,α,β) as

$$H^2(A;V) = \frac{Z^2(A;V)}{B^2(A;V)}, \qquad H^3(A;V) = \frac{Z^3(A;V)}{B^3(A;V)}.$$

4. Deformations of left BiHom-alternative algebras

We develop, in this section, a deformation theory for left BiHom-alternative algebras by analogy with Gerstenhaber's deformation theory [13,14,15,16]. Heuristically, a formal deformation of an algebra A is a one-parameter family of multiplications (of the same sort) which reduces, when the parameter is zero, to the original multiplication of A.

Let (A, μ, α, β) be a left BiHom-alternative algebra over $\mathbb{K}$. Let $\mathbb{K}[[t]]$ be the ring of formal power series with coefficients in $\mathbb{K}$. Suppose that $A[[t]]$ is the set of formal power series with coefficients in A. Then any $\mathbb{K}$-bilinear map $f : A \times A \to A$ (resp. a $\mathbb{K}$-linear map $g : A \to A$), can be extended naturally to a $\mathbb{K}[[t]]$-bilinear map $f : A[[t]] \times A[[t]] \to A[[t]]$ (resp. $g : A[[t]] \to A[[t]]$) by setting

$$f\Big(\sum_{i\geq 0} x_i t^i, \sum_{j\geq 0} y_j t^j\Big) = \sum_{i,j\geq 0} f(x_i, y_j) t^{i+j} \quad (\text{resp. } g\Big(\sum_{i\geq 0} x_i t^i\Big) = \sum_{i\geq 0} g(x_i) t^i).$$

Definition 4.1. Let $A = (A, \mu, \alpha, \beta)$ be a left BiHom-alternative algebra. A one-parameter formal deformation of A is a formal power series $d_t : A[[t]] \times A[[t]] \to A[[t]]$ of the form

$$d_t(x, y) = \sum_{i\geq 0} d_i(x, y) t^i = d_0(x, y) + d_1(x, y)t + d_2(x, y)t^2 + \dots,$$

where each d_i is a $\mathbb{K}$-bilinear map $d_i : A \times A \to A$ (extended to be $\mathbb{K}[[t]]$-bilinear) and $d_0(x, y) = \mu(x, y)$, such that the following identities are satisfied:

$$d_t(\alpha(x), \alpha(y)) = \alpha \circ d_t(x, y), \tag{4.21}$$

$$d_t(\beta(x), \beta(y)) = \beta \circ d_t(x, y), \tag{4.22}$$

$$\begin{aligned} &d_t(d_t(\beta(x), \alpha(y)), \beta(z)) - d_t(\alpha\beta(x), d_t(\alpha(y), z)) \\ &+ d_t(d_t(\beta(y), \alpha(x)), \beta(z)) - d_t(\alpha\beta(y), d_t(\alpha(x), z)) = 0. \end{aligned} \tag{4.23}$$

Conditions (4.21)-(4.23) are called deformation equations of a left BiHom-alternative algebra.

Note that $A[[t]]$ is a module over $\mathbb{K}[[t]]$ and d_t defines the $\mathbb{K}[[t]]$-bilinear multiplication on $A[[t]]$ such that $A_t = (A[[t]], d_t, \alpha, \beta)$ is a left BiHom-alternative algebra.

4.1. Deformation equations and obstructions. Now we investigate deformation equations (4.21)-(4.23). Conditions (4.21) and (4.22) are equivalent to the following equations:

$$d_i(\alpha(x), \alpha(y)) = \alpha \circ d_i(x, y), \tag{4.24}$$

$$d_i(\beta(x), \beta(y)) = \beta \circ d_i(x, y), \tag{4.25}$$

respectively, for $i = 0, 1, 2, \cdots$. Expanding the left hand side of equation (4.23) and collecting the coefficients of t^k yield to

$$\begin{aligned} \sum_{i+j=k, i,j\geq 0} &\big(d_i(d_j(\beta(x), \alpha(y)), \beta(z)) - d_i(\alpha\beta(x), d_j(\alpha(y), z)) \\ &+ d_i(d_j(\beta(y), \alpha(x)), \beta(z)) - d_i(\alpha\beta(y), d_j(\alpha(x), z))\big) = 0, \quad \text{for } k = 0, 1, 2, \cdots . \end{aligned}$$

This equation gives the necessary and sufficient conditions of the left BiHom-alternativity of d_t. It may be written

$$\sum_{i=0}^{k}\big(d_i(d_{k-i}(\beta(x),\alpha(y)),\beta(z)) - d_i(\alpha\beta(x), d_{k-i}(\alpha(y),z)) + d_i(d_{k-i}(\beta(y),\alpha(x)),\beta(z)) - d_i(\alpha\beta(y), d_{k-i}(\alpha(x),z))\big) = 0. \tag{4.26}$$

The first equation ($k = 0$) is the left BiHom-alternative condition with respect to μ_0.
For $k = 1$, we obtain the following equation:

$$\mu(d_1(\beta(x),\alpha(y)),\beta(z)) - \mu(\alpha\beta(x), d_1(\alpha(y),z)) + \mu(d_1(\beta(y),\alpha(x)),\beta(z)) - \mu(\alpha\beta(y), d_1(\alpha(x),z)) + d_1(\mu(\beta(x),\alpha(y)),\beta(z)) - d_1(\alpha\beta(x),\mu(\alpha(y),z)) + d_1(\mu(\beta(y),\alpha(x)),\beta(z)) - d_1(\alpha\beta(y),\mu(\alpha(x),z)) = 0.$$

This means that d_1 must be a 2-cocycle with respect to the left BiHom-alternative algebra cohomology defined above, that is $d_1 \in Z^2(A, A)$.
More generally, suppose that d_p be the first non-zero coefficient after μ_0 in the deformation d_t. This d_p is called the infinitesimal of d_t.

Theorem 4.2. *The map d_p is a 2-cocycle of a left BiHom-alternative algebra cohomology of A with coefficient in itself.*

Proof. In Eq. (4.26) make the following substitution $k = p$ and $d_1 = \cdots = d_{p-1} = 0$. $\square$

Definition 4.3. Let (A, μ, α, β) be a left BiHom-alternative algebra and d_1 be an element in $Z^2(A, A)$. The 2-cocycle d_1 is called integrable if there exists a family $(d_i)_{i\geq 0}$ such that $d_t = \sum_{i\geq 0} d_i t^i$ defines a formal deformation $A_t = (A[[t]], d_t, \alpha, \beta)$ of A.

The integrability of d_p implies an infinite sequence of relations which may be interpreted as the vanishing of the obstruction to the integration of d_p.

For an arbitrary k, with $k > 1$, the k^{th} equation of the system (4.26) may be written

$$\begin{aligned}\delta^2 d_k\,(x,y,z) &= \sum_{i=1}^{k-1} d_i\,(d_{k-i}\,(\beta(x),\alpha(y))\,\beta(z)) - d_i\,(\alpha\beta(x), d_{k-i}\,(\alpha(y),z)) \\ &+ \quad d_i\,(d_{k-i}\,(\beta(y),\alpha(x))\,,\beta(z)) - d_i\,(\alpha\beta(y), d_{k-i}\,(\alpha(x),z))\,.\end{aligned}$$

Suppose that the truncated deformation of order $m-1$

$$d_t = d_0 + td_1 + t^2 d_2 + \cdots + t^{m-1} d_{m-1}$$

satisfies the deformation equation. This deformation is extended to a deformation of order m, $d_t = d_0 + td_1 + t^2 d_2 + \cdots + t^{m-1} d_{m-1} + t^m d_m$, satisfying the deformation equation if

$$\begin{aligned}\delta^2 d_m (x,y,z) &= \sum_{i=1}^{m-1} d_i \left(d_{m-i}\left(\beta(x),\alpha(y)\right)\beta(z)\right) - d_i\left(\alpha\beta(x), d_{m-i}\left(\alpha(y),z\right)\right)\\ &+ \quad d_i\left(d_{m-i}\left(\beta(y),\alpha(x)\right),\beta(z)\right) - d_i\left(\alpha\beta(y), d_{m-i}\left(\alpha(x),z\right)\right).\end{aligned}$$

The right-hand side of this equation is called the *obstruction* to finding d_m extending the deformation.

We define a diamond operation on 2-cochains by

$$\begin{aligned}&d_i \diamond d_j (x,y,z) = d_i\left(d_j\left(\beta(x),\alpha(y)\right)\beta(z)\right) - d_i\left(\alpha\beta(x), d_j\left(\alpha(y),z\right)\right)\\ &+ d_i\left(d_j\left(\beta(y),\alpha(x)\right),\beta(z)\right) - d_i\left(\alpha\beta(y), d_j\left(\alpha(x),z\right)\right).\end{aligned}$$

Then the obstruction may be written

$$\sum_{i=1}^{m-1} d_i \diamond d_{m-i} \text{ or } \sum_{i+j=m \quad i,j\neq m} d_i \diamond d_j.$$

A straightforward computation gives the following result.

Theorem 4.4. *The obstructions are left BiHom-alternative 3-cocycles.*

We provide the following observations

(1) The cohomology class of the element $\sum_{i+j=m,\quad i,j\neq m} d_i \diamond d_j$ is the first obstruction to the integrability of d_m.

Let us consider now how to extend an infinitesimal deformation to a deformation of order 2. Suppose $m = 2$ and $d_t = d_0 + td_1 + t^2 d_2$. The deformation equation of the truncated deformation of order 2 is equivalent to the finite system

$$\begin{cases} d_0 \diamond d_0 &= 0 \quad (d_0 \text{ is left BiHom-alternative})\\ \delta^2 d_1 &= 0 \quad \left(d_1 \in Z^2(A,A)\right)\\ d_1 \diamond d_1 &= \delta^2 d_2\end{cases}$$

Then $d_1 \diamond d_1$ is the first obstruction to integrate d_1 and $d_1 \diamond d_1 \in Z^3(A,A)$.

The elements $d_1 \diamond d_1$ which are coboundaries permit to extend the deformation of order one to a deformation of order 2. But the elements of $H^3(A,A)$ gives the obstruction to the integration of d_1.

(2) If d_m is integrable, then the cohomological class of $\sum\limits_{i+j=m,\quad i,j\neq m} d_i \diamond d_j$ must be 0. In the previous example, d_1 is integrable implies $d_1 \diamond d_1 = \delta^2 d_2$ which means that the cohomology class of $d_1 \diamond d_1$ vanishes.

Corollary 4.5. *If $H^3(A,A) = 0$, then all obstructions vanish and every $d_m \in Z^2(A,A)$ is integrable.*

4.2. Equivalent and trivial deformations. In this paragraph, we characterize the equivalent and trivial deformations of left BiHom-alternative algebras.

Definition 4.6. Let (A,μ,α,β) be a left BiHom-alternative algebra. Suppose that $d_t(x,y) = \sum_{i\geq 0} d_i(x,y)t^i$ and $d'_t(x,y) = \sum_{i\geq 0} d'_i(x,y)t^i$ are two one-parameter formal deformations of (A,μ,α,β). They are called equivalent, denoted by $d_t \sim d'_t$, if there is a formal isomorphism of $\mathbb{K}[[t]]$-modules

$$\phi_t(x) = \sum_{i\geq 0} \phi_i(x)t^i : (A[[t]], d_t, \alpha, \beta) \longrightarrow (A[[t]], d'_t, \alpha, \beta),$$

where each $\phi_i : A \to A$ is a $\mathbb{K}$-linear map (extended to be $\mathbb{K}[[t]]$-linear) and $\phi_0 = Id_A$, satisfying

$$\begin{aligned} &\phi_t \circ \alpha = \alpha \circ \phi_t,\ \ \phi_t \circ \beta = \beta \circ \phi_t,\\ &\phi_t \circ d_t(x,y) = d'_t(\phi_t(x), \phi_t(y)). \end{aligned}$$

When $d_1 = d_2 = \cdots = 0$, $d_t = d_0$ is said to be the null deformation. A 1-parameter formal deformation d_t is called trivial if $d_t \sim d_0$. A left BiHom-alternative algebra (A,μ,α,β) is called analytically rigid, if every 1-parameter formal deformation d_t of A is trivial.

Theorem 4.7. *Let $d_t(x,y) = \sum_{i>0} d_i(x,y)t^i$ and $d'_t(x,y) = \sum_{i\geq 0} d'_i(x,y)t^i$ be equivalent 1-parameter formal deformations of a left BiHom-alternative algebra (A,μ,α,β). Then d_1 and d'_1 belong to the same cohomology class in $H^2(A,A)$.*

Proof. Suppose that $\phi_t(x) = \sum_{i\geq 0}\phi_i(x)t^i$ is the formal $\mathbb{K}[[t]]$-module isomorphism such that $\phi_t \circ \alpha = \alpha \circ \phi_t, \phi_t \circ \beta = \beta \circ \phi_t$ and

$$\sum_{i\geq 0} \phi_i\Big(\sum_{j>0} d_j(x,y)t^j\Big)t^i = \sum_{i\geq 0} d'_i\Big(\sum_{k\geq 0}\phi_k(x)t^k, \sum_{k\geq 0}\phi_l(y)t^l\Big)t^i.$$

It follows that

$$\sum_{i+j=n} \phi_i\big(d_j(x,y)\big)t^{i+j} = \sum_{i+k+l=n} d'_i\big(\phi_k(x), \phi_l(y)\big)t^{i+k+l}.$$

In particular,

$$\sum_{i+j=1} \phi_i\big(d_j(x,y)\big)t^{i+j} = \sum_{i+k+l=1} d'_i\big(\phi_k(x),\phi_l(y)\big)t^{i+k+l}.$$

Since $\phi_0 = id$, we obtain

$$d_1(x,y) + \phi_1(\mu(x,y)) = \mu(\phi_1(x),y) + \mu(x,\phi_1(y)) + d'_1(x,y).$$

Then $d_1 - d'_1 \in B^2(A,A)$. Therefore, d_1 and d'_1 are cohomologous. □

Theorem 4.8. *Let (A,μ,α,β) be a left BiHom-alternative algebra such that $H^2(A,A) = 0$. Then (A,μ,α,β) is analytically rigid.*

Proof. Let d_t be a one-parameter formal deformation of (A,μ,α,β). Suppose that $d_t = d_0 + \sum_{i\geq n} d_i t^i$. Then

$$\delta^2 d_n = d_1 \diamond d_{n-1} + d_2 \diamond d_{n-2} + \cdots + d_{n-1} \diamond d_0 = 0,$$

that is $d_n \in Z^2(A,A) = B^2(A,A)$. It follows that there exists $f_n \in C^1(A,A)$ such that $d_n = \delta^1 f_n$.

Let $\phi_t = id_A - t^n f_n : (A[[t]], d'_t, \alpha, \beta) \to (A[[t]], d_t, \alpha, \beta)$, where $d'_t(x,y) = \phi_t^{-1} d_t(\phi_t(x), \phi_t(y))$. Here, ϕ_t is an isomorphism since

$$\phi_t \circ \sum_{i\geq 0} f_n^i t^{in} = \sum_{i\geq 0} f_n^i t^{in} \circ \phi_t = id_{A[[t]]}.$$

Moreover, we have $\phi_t \circ \alpha = \alpha \circ \phi_t$ and $\phi_t \circ \beta = \beta \circ \phi_t$.

It is straightforward to prove that d'_t is a one-parameter formal deformation of (A,μ,α,β) and $d_t \sim d'_t$. Assume that $d'_t(x,y) = \sum_{i\geq 0} d'_i(x,y)t^i$ and use the fact that $\phi_t \circ d'_t(x,y) = d_t(\phi_t(x), \phi_t(y))$. Then

$$(id_A - f_n t^n)\Big(\sum_{i\geq 0} d'_i(x,y)t^i\Big) = \Big(d_0 + \sum_{i\geq n} d_i t^i\Big)(x - f_n(x)t^n, y - f_n(y)t^n),$$

i.e.,

$$\begin{aligned} & d'_0(x,y) + \sum_{i\geq 1} d'_i(x,y)t^i - \sum_{i\geq 0} f_n \circ d'_i(x,y)t^{i+n} \\ = \;& d_0(x,y) - (d_0(f_n(x),y) + d_0(x,f_n(y)))t^n + d_0(f_n(x),f_n(y))t^{2n} \\ & + \sum_{i\geq n} d_i(x,y)t^i - \sum_{i\geq n}(d_i(f_n(x),y) + d_i(x,f_n(y))t^{i+n} + \sum_{i\geq n} d_i(f_n(x),f_n(y))t^{i+2n}. \end{aligned}$$

Then we have $d'_1 = d'_2 = ... = d'_{n-1} = 0$. Since $d_0 = d'_0$, we have

$$d'_n(x,y) - f_n \circ d_0(x,y) = -(d_0(f_n(x),y) + d_0(x,f_n(y))) + d_n(x,y).$$

Hence $d'_n = d_n - \delta^1 f_n = 0$ and $d'_t(x,y) = d'_0 + \sum_{i\geq n+1} d'_i(x,y)t^i$. By induction, this procedure ends with $d_t \sim d_0$, so, (A,μ,α,β) is analytically rigid. □

5. Extensions of BiHom-alternative algebras

5.1. Dual representations of regular BiHom-alternative algebras. In this section, we construct the dual representation of a representation of a BiHom-alternative algebra without any additional condition. This is nontrivial and to our knowledge, people needed to add a very strong condition to obtain a representation on the dual space in former studies which restricts its development.

Let (V, ℓ, r, ϕ, ψ) be a regular representation of a regular BiHom-alternative algebra (A, μ, α, β). Define $\ell^*, r^* : A \longrightarrow \mathrm{gl}(V^*)$ as usual by

$$\langle \ell^*(x)(\xi), u\rangle = \langle \xi, \ell(x)(u)\rangle \quad \text{and } \langle r^*(x)(\xi), u\rangle = \langle \xi, r(x)(u)\rangle$$

for all $x \in A, u \in V, \xi \in V^*$. However, in general (r^*, ℓ^*) is not a representation of A. Define $r^\star, \ell^\star : A \longrightarrow \mathrm{gl}(V^*)$ by

$$\ell^\star(x)(\xi) := \ell^*(\alpha^{-1}\beta^2(x))\big((\phi^{-1}\psi^{-1})^*(\xi)\big) \tag{5.27}$$

and

$$r^\star(x)(\xi) := r^*(\alpha^2\beta^{-1}(x))\big((\phi^{-1}\psi^{-1})^*(\xi)\big) \tag{5.28}$$

for all $x \in A, \xi \in V^*$. More precisely, we have

$$\langle \ell^\star(x)(\xi), u\rangle = \langle \xi, \ell(\alpha^{-2}\beta(x))(\phi^{-1}\psi^{-1}(u))\rangle \tag{5.29}$$

$$\langle r^\star(x)(\xi), u\rangle = \langle \xi, r(\alpha\beta^{-2}(x))(\phi^{-1}\psi^{-1}(u))\rangle \tag{5.30}$$

for all $x \in A, u \in V, \xi \in V^*$.

Theorem 5.1. *Let (V, ℓ, r, ϕ, ψ) be a representation of a BiHom-alternative algebra (A, μ, α, β). Then $(V^*, r^\star, \ell^\star, (\phi^{-1})^*, (\psi^{-1})^*)$ is a representation of (A, μ, α, ψ) where the linear maps $\ell^\star, r^\star : A \longrightarrow gl(V^*)$ are defined above by* (5.27) *and* (5.28).

Proof. For all $x \in A, \xi \in V^*$, we have

$$\begin{aligned}\ell^\star(\alpha(x))((\phi^{-1})^*(\xi)) &= \ell^*(\beta^2(x))(\phi^{-2}\psi^{-1})^*(\xi) = (\phi^{-1})^*(\ell^*(\alpha^{-1}\beta^2(x))(\phi^{-1}\psi^{-1})^*(\xi))\\ &= (\phi^{-1})^*(\ell^\star(x)(\xi))\end{aligned}$$

and

$$\begin{aligned}\ell^\star(\beta(x))((\psi^{-1})^*(\xi)) &= \ell^*(\alpha^{-1}\beta^3(x))(\phi^{-1}\psi^{-2})^*(\xi) = (\psi^{-1})^*(\ell^*(\alpha^{-1}\beta^2(x))(\phi^{-1}\psi^{-1})^*(\xi))\\ &= (\psi^{-1})^*(\ell^\star(x)(\xi)).\end{aligned}$$

Similarly, we can check that

$$r^\star(\alpha(x))((\phi^{-1})^*(\xi)) = (\phi^{-1})^*(r^\star(x)(\xi)),$$

$$r^\star(\beta(x))((\psi^{-1})^*(\xi)) = (\psi^{-1})^*(r^\star(x)(\xi)).$$

On the other hand, by Eqs (2.5), (2.6) and (5.29), for all $x \in A, \xi \in V^*$ and $u \in V$, we have

$$\begin{aligned}
&\langle \ell^\star(\beta(x)\alpha(x))(\phi^{-1})^*(\xi) - \ell^\star(\alpha\beta(x))\ell^\star(\beta(x))(\xi), u\rangle \\
&= \langle \ell^\star(\beta(x)\alpha(x))(\phi^{-1})^*(\xi), u\rangle - \langle \ell^\star(\alpha\beta(x))\ell^\star(\beta(x))(\xi), u\rangle \\
&= \langle (\phi^{-1})^*(\xi), \ell(\alpha^{-2}\beta^2(x)\alpha^{-1}\beta(x))(\phi^{-1}\psi^{-1}(u))\rangle \\
&- \langle \xi, \ell(\alpha^{-2}\beta^2(x))\ell(\alpha^{-2}\beta(x))(\phi^{-2}\psi^{-2}(u))\rangle \\
&= \langle \xi, \ell(\alpha^{-3}\beta^2(x)\alpha^{-2}\beta(x))(\phi^{-2}\psi^{-1}(u)) - \ell(\alpha^{-2}\beta^2(x))\ell(\alpha^{-2}\beta(x))(\phi^{-2}\psi^{-2}(u))\rangle = 0.
\end{aligned}$$

Using Eqs (2.6), (2.6) and (5.30), for all $x \in A, \xi \in V^*$ and $u \in V$, we have

$$\begin{aligned}
&\langle r^\star(\beta(x)\alpha(x))(\psi^{-1})^*(\xi) - r^\star(\alpha\beta(x))r^\star(\alpha(x))(\xi), u\rangle \\
&= \langle r^\star(\beta(x)\alpha(x))(\psi^{-1})^*(\xi), u\rangle - \langle r^\star(\alpha\beta(x))r^\star(\alpha(x))(\xi), u\rangle \\
&= \langle (\psi^{-1})^*(\xi), r(\alpha\beta^{-1}(x)\alpha^2\beta^{-2}(x))(\phi^{-1}\psi^{-1}(u))\rangle \\
&- \langle \xi, r(\alpha^2\beta^{-2}(x))r(\alpha\beta^{-2}(x))(\phi^{-2}\psi^{-2}(u))\rangle \\
&= \langle \xi, r(\alpha\beta^{-2}(x)\alpha^2\beta^{-3}(x))(\phi^{-1}\psi^{-2}(u)) - r(\alpha^2\beta^{-2}(x))r(\alpha\beta^{-2}(x))(\phi^{-2}\psi^{-2}(u))\rangle = 0.
\end{aligned}$$

Similarly, we obtain the other identities. Therefore, $(V^*, r^\star, \ell^\star, (\phi^{-1})^*, (\psi^{-1})^*)$ is a representation of (A, μ, α, β). □

Lemma 5.2. *Let (V, ℓ, r, ϕ, ψ) be a representation of a BiHom-alternative algebra (A, μ, α, β). Then we have*

$$(\ell^\star)^\star = \ell \circ \alpha^{-3}\beta^3 \text{ and } (r^\star)^\star = r \circ \alpha^3\beta^{-3}.$$

Proof. Let $x \in A, u \in V, \xi \in V^*$, we have

$$\begin{aligned}
\langle \xi, (\ell^\star)^\star(x)(u)\rangle &= \langle \xi, (\ell^\star)^*(\alpha^{-1}\beta^2(x))(\phi\psi(u))\rangle = \langle \ell^\star(\alpha^{-1}\beta^2(x))(\xi), \phi\psi(u)\rangle \\
&= \langle \xi, \ell(\alpha^{-3}\beta^3(x))(u)\rangle,
\end{aligned}$$

which implies that $(\ell^\star)^\star = \ell \circ \alpha^{-3}\beta^3$. Similarly, we can check $(r^\star)^\star = r \circ \alpha^3\beta^{-3}$. □

Corollary 5.3. *Let (A, μ, α, β) be a BiHom-alternative algebra. Then $R^\star, L^\star : A \longrightarrow gl(A^*)$ defined by*

$$L^\star(x)(\xi) = L^*(\alpha^{-1}\beta^2(x))(\alpha^{-1}\beta^{-1})^*(\xi) \text{ and } R^\star(x)(\xi) = R^*(\alpha^2\beta^{-1}(x))(\alpha^{-1}\beta^{-1})^*(\xi) \tag{5.31}$$

for all $x \in A, \xi \in A^*$, is a representation of the BiHom-alternative algebra (A, μ, α, β) on A^* with respect to $\big((\alpha^{-1})^*, (\beta^{-1})^*\big)$. It is called the coadjoint representation.

Using the coadjoint representation $(R^\star, L^\star)$, we obtain a semidirect product BiHom-alternative algebra structure on $A \oplus A^*$.

Corollary 5.4. *Let (A,μ,α,β) be a BiHom-alternative algebra. Then there is a natural BiHom-alternative algebra $(A\oplus A^*,\mu_{A\oplus A^*},\alpha\oplus(\alpha^{-1})^*,\beta\oplus(\beta^{-1})^*)$, where the BiHom-alternative product $\mu_{A\oplus A^*}$ is given by*

$$\mu_{A\oplus A^*}(x+\xi,y+\eta)=\mu(x,y)+R^\star(\eta)+L^\star(x)(y)(\xi) \tag{5.32}$$

for all $x,y\in A,\xi,\eta\in A^*$.

5.2. Central extensions and T_θ-extensions of BiHom-alternative algebras. An extension $\tilde{A}$ with twist maps $(\tilde{\alpha},\tilde{\beta})$ of a BiHom-alternative algebra $(A,\cdot,\alpha,\beta)$ by a representation(V,ℓ,r,ϕ,ψ) is an exact sequence

$$0\longrightarrow V\xrightarrow{\ i\ }\tilde{A}\xrightarrow{\ \pi\ }A\longrightarrow 0$$

satisfying $\tilde{\alpha}\circ i=i\circ\phi,\ \tilde{\beta}\circ i=i\circ\psi,\ \alpha\circ\pi=\pi\circ\tilde{\alpha},\ \beta\circ\pi=\pi\circ\tilde{\beta}$.

A central extension of a BiHom-alternative algebra (A,μ,α,β) is an extension in which the annihilator of $\tilde{A}$ contains $i(V)$.

Example 5.5. Let $(A,\cdot,\alpha,\beta)$ be a BiHom-alternative algebra, V be a vector space and $\omega:A\times A\to V$ be a bilinear map. We define the following multiplication on the vector space direct sum $A\oplus V$ by

$$(x+u)\bullet(y+v)=x\cdot y+\omega(x,y),\ \forall\ x,y\in A,\ u,v\in V. \tag{5.33}$$

Define the linear maps $\overline{\alpha},\overline{\beta}:A\oplus V\to A\oplus V$ by

$$\overline{\alpha}(x+u)=\alpha(x)+u,\quad \overline{\beta}(x+u)=\beta(x)+u,\ \forall x\in A,\ u\in V.$$

It is clear that $\overline{\alpha}\overline{\beta}=\overline{\beta}\overline{\alpha}$. Then $(A\oplus V,\overline{\alpha},\overline{\beta})$ is a BiHom-alternative algebra if and only if

$$\begin{aligned}
&\overline{\alpha}\big((x+u)\bullet(y+v)\big)=\overline{\alpha}(x+u)\bullet\overline{\alpha}(y+v),\\
&\overline{\beta}\big((x+u)\bullet(y+v)\big)=\overline{\beta}(x+u)\bullet\overline{\beta}(y+v),\\
&\big(\overline{\beta}(x+u)\bullet\overline{\alpha}(x+u)\big)\bullet\overline{\beta}(y+v)-\overline{\alpha}\overline{\beta}(x+u)\bullet\big(\overline{\alpha}(x+u)\bullet(y+v)\big)=0,\\
&\big((x+u)\bullet\overline{\beta}(y+v)\big)\bullet\overline{\alpha}\overline{\beta}(y+v)-\overline{\alpha}(x+u)\bullet\big(\overline{\beta}(y+v)\bullet\overline{\alpha}(y+v)\big)=0,
\end{aligned}$$

which are equivalent to

$$\omega(\alpha(x),\alpha(y))=\omega(\beta(x),\beta(y))=\omega(x,y), \tag{5.34}$$

$$\omega(\beta(x)\alpha(x),\beta(y))=\omega(\alpha\beta(x),\alpha(x)y), \tag{5.35}$$

$$\omega(x\beta(y),\alpha\beta(y))=\omega(\alpha(x),\beta(y)\alpha(y)). \tag{5.36}$$

In this case, we have an exact sequence

$$0\longrightarrow V\xrightarrow{\ i\ }A\oplus V\xrightarrow{\ \pi\ }A\longrightarrow 0$$

such that $ker(\pi) = \{0\} \oplus V \subset Ann(A \oplus V)$. Then the extension is central. It is called the central extension of A by V via ω.
Note that if we consider V as a trivial bimodule of A, then $A \oplus V$ is a central extension of A by V via ω if and only if $\omega \in Z^2(A;V)$.

The notion of T_θ-Extensions was introduced first in [3]. In the sequel we show that this concept can be extended to BiHom-alternative algebras.

Proposition 5.6. *Let $(A, \cdot, \alpha, \beta)$ be a BiHom-alternative algebra and (V, ℓ, r, ϕ, ψ) be an A-bimodule. Let $\theta : A \times A \to V$ be a bilinear map. Define on the direct sum $A \oplus V$ the product*

$$(x + u) \circ (y + v) = xy + \ell(x)v + r(y)u + \theta(x, y). \tag{5.37}$$

Then $(A \oplus V, \circ, \alpha + \phi, \beta + \psi)$ is a BiHom-alternative algebra if and only if the following conditions hold

$$\phi\theta(x, y) = \theta(\alpha(x), \alpha(y)),\ \ \psi\theta(x, y) = \theta(\beta(x), \beta(y)), \tag{5.38}$$

$$\begin{aligned} &\theta(\beta(x)\alpha(y), \beta(z)) + r(\beta(z))\theta(\beta(x), \alpha(y)) - \theta(\alpha\beta(x), \alpha(y)z) - \ell(\alpha\beta(x))\theta(\alpha(y), z) \\ +&\theta(\beta(y)\alpha(x), \beta(z)) + r(\beta(z))\theta(\beta(y), \alpha(x)) - \theta(\alpha\beta(y), \alpha(x)z) - \ell(\alpha\beta(y))\theta(\alpha(x), z) = 0, \end{aligned} \tag{5.39}$$

$$\begin{aligned} &\theta(x\beta(y), \alpha\beta(z)) + r(\alpha\beta(z))\theta(x, \beta(y)) - \theta(\alpha(x), \beta(y)\alpha(z)) - \ell(\alpha(x))\theta(\beta(y), \alpha(z)) \\ +&\theta(x\beta(z), \alpha\beta(y)) + r(\alpha\beta(y))\theta(x, \beta(z)) - \theta(\alpha(x), \beta(z)\alpha(y)) - \ell(\alpha(x))\theta(\beta(z), \alpha(y)) = 0. \end{aligned} \tag{5.40}$$

This means that $\theta \in Z^2(A;V)$. In this situation, $A \oplus V$ is called the T_θ-extension of A by V via θ.
As an application of the previous result, let us consider the dual representation. Let $(V^*, r^\star, \ell^\star, (\phi^{-1})^*, (\psi^{-1})^*)$ be the dual representation of (V, ℓ, r, ϕ, ψ). Define a bilinear map $\theta : A \times A \to V^*$ and consider on the vector space $A \oplus V^*$ the product

$$(x + f) \circ (y + g) = xy + r^\star(x)g + \ell^\star(y)f + \theta(x, y). \tag{5.41}$$

Then $(A\oplus V^*, \circ, \alpha+(\phi^{-1})^*, \beta+(\psi^{-1})^*)$ is a BiHom-alternative algebra if and only if the following conditions hold

$$(\phi^{-1})^*\theta(x,y)=\theta(\alpha(x),\alpha(y)),\ (\psi^{-1})^*\theta(x,y)=\theta(\beta(x),\beta(y)), \tag{5.42}$$

$$\begin{aligned}&\theta(\beta(x)\alpha(y),\beta(z))+\ell^\star(\beta(z))\theta(\beta(x),\alpha(y))-\theta(\alpha\beta(x),\alpha(y)z)-r^\star(\alpha\beta(x))\theta(\alpha(y),z)\\ &+\theta(\beta(y)\alpha(x),\beta(z))+\ell^\star(\beta(z))\theta(\beta(y),\alpha(x))-\theta(\alpha\beta(y),\alpha(x)z)-r^\star(\alpha\beta(y))\theta(\alpha(x),z)=0,\end{aligned} \tag{5.43}$$

$$\begin{aligned}&\theta(x\beta(y),\alpha\beta(z))+\ell^\star(\alpha\beta(z))\theta(x,\beta(y))-\theta(\alpha(x),\beta(y)\alpha(z))-r^\star(\alpha(x))\theta(\beta(y),\alpha(z))\\ &+\theta(x\beta(z),\alpha\beta(y))+\ell^\star(\alpha\beta(y))\theta(x,\beta(z))-\theta(\alpha(x),\beta(z)\alpha(y))-r^\star(\alpha(x))\theta(\beta(z),\alpha(y))=0.\end{aligned} \tag{5.44}$$

Under these considerations, $A\oplus V^*$ is called the T^*_θ-extension of A by V^* via θ. Note that the T^*-extension corresponds to the case of $\theta=0$ in the above construction.

6. Generalized derivations of BiHom-alternative algebras

This section is devoted to investigate generalized derivations of BiHom-alternative algebras. In the sequel A denotes a BiHom-alternative algebra (A,μ,α,β).

For any integer k and l, denote by α^k the k-times composition of α and by β^l the l-times composition of β, i.e.,

$$\alpha^k=\underbrace{\alpha\circ\cdots\circ\alpha}_{k-times},\quad \beta^l=\underbrace{\beta\circ\cdots\circ\beta}_{l-times}.$$

Let U be the subspace of $End(A)$ defined by

$$U:=\{u\in End(A)\mid u\circ\alpha=\alpha\circ u,\ u\circ\beta=\beta\circ u\}$$

and let $\widetilde{\alpha},\widetilde{\beta}:U\to U$ be two linear maps defined as follow

$$\widetilde{\alpha}(u)=\alpha\circ u,\ \widetilde{\beta}(u)=\beta\circ u,\ \forall u\in U.$$

The space $(U,[\cdot,\cdot],\widetilde{\alpha},\widetilde{\beta})$ is a BiHom-Lie algebra where $[u,v]=u\circ v-v\circ u$ for any $u,v\in U$.

Definition 6.1. A linear map $D:A\to A$ is said to be an $\alpha^k\beta^l$-derivation on A, for $k,l\in\mathbb{N}$, if it satisfies the following conditions:

$$[D,\alpha]=0,\ [D,\beta]=0, \tag{6.45}$$

$$D(xy)=D(x)\alpha^k\beta^l(y)+\alpha^k\beta^l(x)D(y),\ \forall x,y\in A. \tag{6.46}$$

We denote the set of all $\alpha^k\beta^l$-derivations by $Der_{\alpha^k\beta^l}(A)$ and

$$Der(A)=\bigoplus_{k,l\geq 0} Der_{\alpha^k\beta^l}(A).$$

We can easily show that $Der(A)$ is equipped with a Lie algebra structure. In fact, for $D \in Der_{\alpha^k\beta^l}(A)$ and $D' \in Der_{\alpha^s\beta^t}(A)$, we have $[D, D'] \in Der_{\alpha^{k+s}\beta^{l+t}}(A)$, where $[D, D']$ is the standard commutator defined by $[D, D'] = DD' - D'D$.

It is well known that the BiHom-commutator algebra A^- of A is a BiHom-Malcev algebra and the BiHom-anti-commutator algebra A^+ of A is a BiHom-Jordan algebra. We state the following result.

Proposition 6.2. *Let D be an $\alpha^k\beta^l$-derivation on A. Then D is still an $\alpha^k\beta^l$-derivation on its associated BiHom-Jordan algebra A^+ and on its associated BiHom-Malcev algebra A^-.*

Definition 6.3. A linear map $D \in End(A)$ is said to be an $\alpha^k\beta^l$-quasi-derivation of A if there exists a linear map $D' \in End(A)$ such that

$$[D, \alpha] = 0,\ [D^{'}, \alpha] = 0, [D, \beta] = 0,\ [D^{'}, \beta] = 0,$$
$$D^{'}(xy) = D(x)\alpha^k\beta^l(y) + \alpha^k\beta^l(x)D(y),\ \forall\ x, y \in A,$$

and we say that D associates with D'.

Definition 6.4. A linear map $D \in End(A)$ is said to be an $\alpha^k\beta^l$-generalized derivation of A if there exist linear maps $D', D'' \in End(A)$ such that

$$[D, \alpha] = 0,\ [D^{'}, \alpha] = 0,\ [D^{''}, \alpha] = 0, [D, \beta] = 0,\ [D^{'}, \beta] = 0,\ [D^{''}, \beta] = 0,$$
$$D^{''}(xy) = D(x)\alpha^k\beta^l(y) + \alpha^k\beta^l(x)D^{'}(y),\ \forall x, y \in A,$$

and we also say that D associates with D' and $D^{''}$.

Definition 6.5. An $\alpha^k\beta^l$-generalized derivation D of A associated with D' and $D^{''}$ is said to be symmetric if (for any $x, y \in A$)

$$D^{''}(xy) = D(x)\alpha^k\beta^l(y) + \alpha^k\beta^l(x)D^{'}(y) = D'(x)\alpha^k\beta^l(y) + \alpha^k\beta^l(x)D(y).$$

The sets of generalized derivations, quasi-derivations and symmetric $\alpha^k\beta^l$-generalized derivations will be denoted by $GDer(A)$, $QDer(A)$ and $SGDer(A)$ respectively.

It is easy to see that

$$Der(A) \subset QDer(A) \subset SGDer(A) \subset GDer(A).$$

Definition 6.6. A linear map $\theta \in End(A)$ is said to be an $\alpha^k\beta^l$-centroid of A if

$$[\theta, \alpha] = 0,\ [\theta, \beta] = 0, \tag{6.47}$$
$$\theta(xy) = \theta(x)\alpha^k\beta^l(y) = \alpha^k\beta^l(x)\theta(y),\ \forall\ x, y \in A. \tag{6.48}$$

The set of $\alpha^k\beta^l$-centroids of A is denoted by $C(A)$.

Definition 6.7. A linear map $\theta \in End(A)$ is said to be an $\alpha^k\beta^l$-quasi-centroid of A if

$$[\theta, \alpha] = 0,\ [\theta, \beta] = 0, \tag{6.49}$$

$$\theta(x)\alpha^k\beta^l(y) = \alpha^k\beta^l(x)\theta(y),\ \forall\ x, y \in A. \tag{6.50}$$

The set of $\alpha^k\beta^l$-quasi-centroids of A is denoted by $QC(A)$. We have $C(A) \subset QC(A)$.

Proposition 6.8. *Let $D \in Der(A)$ and $\theta \in C(A)$. Then*

$$[D, \theta] \in C(A).$$

Proof. Assume that $D \in Der_{\alpha^k\beta^l},\ \theta \in C_{\alpha^s\beta^t}(A)$. For arbitrary $x, y \in A$, we have

$$\begin{aligned} D\theta(x)\alpha^{k+s}\beta^{l+t}(y) &= D(\theta(x)\alpha^s\beta^t(y)) - \alpha^k\beta^l(\theta(x))D(\alpha^s\beta^t(y)) \\ &= D(\theta(x)\alpha^s\beta^t(y)) - \theta(\alpha^k\beta^l(x))D(\alpha^s\beta^t(y)) \\ &= D\theta(xy) - \alpha^{k+s}\beta^{l+t}(x)\theta D(y). \end{aligned} \tag{6.51}$$

and

$$\begin{aligned} \theta D(x)\alpha^{k+s}\beta^{l+t}(y) &= \theta(D(x)\alpha^k\beta^l(y)) \\ &= \theta D(xy) - \theta(\alpha^k\beta^l(x)D(y)) \\ &= \theta D(xy) - (\alpha^{k+s}\beta^{l+t}(x)\theta D(y)). \end{aligned} \tag{6.52}$$

By making the difference of equations (6.51) and (6.52), we get

$$[D, \theta](xy) = [D, \theta](x)\alpha^{k+s}\beta^{l+t}(y).$$

$\square$

Proposition 6.9. *$C(A) \subseteq QDer(A)$.*

Proof. Let $\theta \in C_{\alpha^k\beta^l}(A)$ and $x, y \in A$, then we have

$$\begin{aligned} \theta(x)\alpha^k\beta^l(y) + \alpha^k\beta^l(x)\theta(y) &= \theta(x)\alpha^k\beta^l(y) + \theta(x)\alpha^k\beta^l(y) \\ &= 2\theta(xy) = D^{'}(xy). \end{aligned}$$

Then $\theta \in QDer_{\alpha^k\beta^l}(A)$. $\square$

Proposition 6.10. *$[QC(A), QC(A)] \subseteq QDer(A)$.*

Proof. Assume that $\theta \in QC_{\alpha^k\beta^l}(A)$ and $\theta' \in QC_{\alpha^s\beta^t}(A)$. Then for all $x, y \in A$, we have

$$\theta(x)\alpha^k\beta^l(y) = \alpha^k\beta^l(x)\theta(y),$$
$$\theta'(x)\alpha^s\beta^t(y) = \alpha^s\beta^t(x)\theta'(y).$$

Hence we obtain

$$\begin{aligned}
[\theta,\theta'](x)\alpha^{k+s}\beta^{l+t}(y) &= (\theta\theta' - \theta'\theta)(x)\alpha^{k+s}\beta^{l+t}(y)\\
&= \theta\theta'(x)\alpha^{k+s}\beta^{l+t}(y) - \theta'\theta(x)\alpha^{k+s}\beta^{l+t}(y)\\
&= \alpha^{k+s}\beta^{l+t}(x)\theta'\theta(y) - \alpha^{k+s}\beta^{l+t}(x)\theta\theta'(y)\\
&= -\alpha^{k+s}\beta^{l+t}(x)[\theta,\theta'](y),
\end{aligned}$$

which implies that $[\theta,\theta'](x)\alpha^{k+s}\beta^{l+t}(y) + \alpha^{k+s}\beta^{l+t}(x)[\theta,\theta'](y) = 0$.
Then $[\theta,\theta'] \in QDer_{\alpha^{k+s}\beta^{l+t}}(A)$. □

Proposition 6.11. *The spaces* $GDer(A)$, $QDer(A)$ *and* $C(A)$ *are BiHom-subalgebras of* $(U, [\cdot,\cdot], \widetilde{\alpha}, \widetilde{\beta})$.

Proof. For any generalized $\alpha^k\beta^l$-derivation D, it is obvious to see that

$$D \circ \widetilde{\alpha} = \widetilde{\alpha} \circ D \in GDer_{\alpha^{k+1}\beta^l} \text{ and } D \circ \widetilde{\beta} = \widetilde{\beta} \circ D \in GDer_{\alpha^k\beta^{l+1}}.$$

So that $\widetilde{\alpha}(GDer(A)) \subseteq GDer(A)$ and $\widetilde{\beta}(GDer(A)) \subseteq GDer(A)$.

Let $D_1 \in GDer_{\alpha^k\beta^l}(A)$ and $D_2 \in GDer_{\alpha^s\beta^t}(A)$, then, for $x, y \in A$, we have

$$D_1''(xy) = D_1(x)\alpha^k\beta^l(y) + \alpha^k\beta^l(x)D_1'(y), \tag{6.53}$$
$$D_2''(xy) = D_2(x)\alpha^s\beta^t(y) + \alpha^s\beta^t(x)D_2'(y). \tag{6.54}$$

Moreover

$$\begin{aligned}
[D_1, D_2](x)\alpha^{k+s}\beta^{l+t}(y) =& D_1D_2(x)\alpha^{k+s}\beta^{l+t}(y) - D_2D_1(x)\alpha^{k+s}\beta^{l+t}(y)\\
=& D_1''(D_2(x)\alpha^s\beta^t(y)) - \alpha^k\beta^l D_2(x)D_1'\alpha^s\beta^t(y)\\
&- D_2''(D_1(x)\alpha^k\beta^l(y)) + \alpha^s\beta^t D_1(x)D_2'\alpha^k\beta^l(y)\\
=& D_1''D_2''(xy) - D_1''(\alpha^s\beta^t(x)D_2'(y))\\
&- D_2''(\alpha^k\beta^l(x)D_1'(y)) + \alpha^{k+s}\beta^{l+t}(x)D_2'D_1'(y)\\
&- D_2''D_1''(xy) + D_2''(\alpha^k\beta^l(x)D_1'(y))\\
&+ D_1''(\alpha^s\beta^t(x)D_2'(y)) - \alpha^{k+s}\beta^{l+t}(x)D_1'D_2'(y)\\
=& [D_1'', D_2''](xy) - \alpha^{k+s}\beta^{l+t}(x)[D_1', D_2'](y).
\end{aligned}$$

Then $[D_1, D_2]$ is a generalized $\alpha^{k+s}\beta^{l+t}$-derivation on A. Therefore $GDer(A)$ is a BiHom-subalgebra of $(U, [\cdot,\cdot], \widetilde{\alpha}, \widetilde{\beta})$.

Similarly, we can show that $QDer(A)$ and $C(A)$ are also BiHom-subalgebras of $(U, [\cdot,\cdot], \widetilde{\alpha}, \widetilde{\beta})$. □

Proposition 6.12. *We have $SGDer(A) = QDer(A) + QC(A)$.*

Proof. Let $D \in SGDer(A)$ associated with D' and D''. Then for any $x, y \in A$, we have

$$D''(xy) = D(x)\alpha^k\beta^l(y) + \alpha^k\beta^l(x)D'(y) = D'(x)\alpha^k\beta^l(y) + \alpha^k\beta^l(x)D(y).$$

We remark that $D = \frac{D+D'}{2} + \frac{D-D'}{2}$. So we will prove that $\frac{D+D'}{2} \in QDer(A)$ and $\frac{D-D'}{2} \in CD(A)$. For this, take $x, y \in A$, then

$$\begin{aligned}
&\frac{D+D'}{2}(x)\alpha^k\beta^l(y) + \alpha^k\beta^l(x)\frac{D+D'}{2}(y)\\
&= \frac{1}{2}(D(x)\alpha^k\beta^l(y) + D'(x)\alpha^k\beta^l(y) + \alpha^k\beta^l(x)D(y) + \alpha^k\beta^l(x)D'(y))\\
&= D"(xy),
\end{aligned}$$

which implies that $\frac{D+D'}{2} \in QDer(A)$.

On the other hand, we have

$$\begin{aligned}
\frac{D-D'}{2}(x)\alpha^k\beta^l(y) =& \frac{1}{2}(D(x)\alpha^k\beta^l(y) - D'(x)\alpha^k\beta^l(y))\\
=& \frac{1}{2}(\alpha^k\beta^l(x)D(y) - \alpha^k\beta^l(x)D'(y))\\
=& \alpha^k\beta^l(x)\frac{D-D'}{2}(y),
\end{aligned}$$

which means that $\frac{D-D'}{2} \in CD(A)$. Hence $D \in QDer(A) + QC(A)$. Moreover, it is straightforward to prove $QDer(A) + QC(A) \subset SGD(A)$. Therefore $SGD(A) = QDer(A) + QC(A)$. □

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