

BAER SUBMODULES OF MODULES OVER COMMUTATIVE RINGS

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ABSTRACT. Let R be a commutative ring and M be an R -module. A submodule N of M is called a d -submodule (resp., an fd -submodule) if $\text{Ann}_R(m) \subseteq \text{Ann}_R(m')$ (resp., $\text{Ann}_R(F) \subseteq \text{Ann}_R(m')$) for some $m \in N$ (resp., finite subset $F \subseteq N$) and $m' \in M$ implies that $m' \in N$. Many examples, characterizations, and properties of these submodules are given. Moreover, we use them to characterize modules satisfying Property T, reduced modules, and von Neumann regular modules.

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1. Introduction

Throughout this paper, all rings are assumed to be commutative with nonzero identity and all modules are nonzero unitary. Let R be a ring and M be an R -module. If N is a submodule of M and K is a nonempty subset of M , then the ideal $\{x \in R \mid xK \subseteq N\}$ of R is denoted by $(N :_R K)$ and the annihilator of N is the ideal $\text{Ann}_R(N) := (0 :_R N)$. For any ideal I of R and any submodule N of M , the submodule $\{m \in M \mid Im \subseteq N\}$ is denoted by $(N :_M I)$ and the annihilator of I in M is the submodule $\text{Ann}_M(I) := (0 :_M I)$. An R -module M is said to be faithful if $\text{Ann}_R(M)$ is the zero ideal of R . A proper submodule N of M is said to be a maximal submodule if it is not properly contained in any other proper submodule of M . A proper submodule N of M is a prime submodule if for any $x \in R$ and $m \in M$, $xm \in N$ implies either $m \in N$ or $x \in (N :_R M)$. For any submodule N of M , the (prime) radical of N in M , denoted by $\text{rad}_M(N)$, is defined to be the intersection of all prime submodules of M containing N . If there is no prime submodule containing N , we define $\text{rad}_M(N) = M$. A submodule N of M is called radical if $N = \text{rad}_M(N)$. We shall need the notion of the envelope of a submodule

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introduced by McCasland and Moore in [20]. For a submodule N of M , the envelope of N in M , denoted by $E_M(N)$, is defined to be the collection $E_M(N) := \{rm \mid r \in R \text{ and } m \in M \text{ such that } r^k m \in N \text{ for some positive integer } k\}$. Note that, in general, $E_M(N)$ is not an R -module. We use $\langle E_M(N) \rangle$ to denote the submodule of M generated by $E_M(N)$. Obviously $(N :_R M)M \subseteq N \subseteq E_M(N) \subseteq \text{rad}_M(N)$ for every submodule N of M , where the equalities do not hold in general. A submodule N of M satisfies *the radical formula* in M if $\langle E_M(N) \rangle = \text{rad}_M(N)$. Moreover, we say that an R -module M *satisfies the radical formula* if every submodule of M satisfies the radical formula in M (for details, see [15], [18], [20], [22] and [24]). We say that an R -module M is a *multiplication module* [4] if every submodule of M has the form IM for some ideal I of R , i.e., $N = (N :_R M)M$. It is well known that maximal submodules and prime submodules exist in multiplication modules (for details, see [7]). Recall from [1], an R -module M satisfies *Property T* (resp., *strong Property T*) if for every finitely generated submodule N (resp., finite subset F) of M with $N \subseteq T_R(M)$ (resp., $F \subseteq T_R(M)$), $\text{Ann}_R(N) \neq 0$ (resp., $\text{Ann}_R(F) \neq 0$), where $T_R(M) := \{m \in M \mid rm = 0 \text{ for some nonzero } r \in R\}$. According to [17], an R -module M is called *reduced* if for any $m \in M$ and $x \in R$, $xm = 0$ implies $Rm \cap xM = 0$. An R -module M is called *von Neumann regular* [12] if for each $m \in M$, there exists an element $r \in R$ such that $rM = r^2M$ and $Rm = rM$. It is well known that a finitely generated R -module M is von Neumann regular if and only if for any $m \in M$, there exists a weak idempotent $e \in R$ of M (i.e., $e - e^2 \in \text{Ann}_R(M)$) such that $Rm = eM$ (see [12, Lemma 5]).

The notion of d-ideals in a commutative ring was introduced by Speed [25] who called them Baer ideals. These ideals were also put to good use in 1972 by Evans [8] when characterizing commutative rings that are finite direct sums of integral domains. In [11], Jayaram introduced fd-ideals (as strongly Baer ideals) and 0-ideals in reduced rings and characterize quasi regular and von Neumann regular rings. In [16], Khabazian, Vedadi and Safaeeyan extended the concept of d-ideals to the category of modules and investigated the modules for which their submodules are d-submodules (see [17, Theorem 2.1]). In [23], Safaeeyan and Taherifar studied d-ideals and fd-ideals in general rings, and not just the reduced ones. In [2], the authors studied rings in which every ideal contained in the set of zero-divisors is a d-ideal. The authors of [13] recently extended the concepts of d-ideals and 0-ideals to submodules and they called them Baer submodules and *-submodules, respectively. They used them to characterize quasi regular modules and weak quasi regular modules.

Our objective in this study is to investigate the concepts of d-submodules, fd-submodules, and 0-submodules of a module over a commutative ring. In Section 2, we establish some examples of these submodules (see Example 2.2). Also we characterize modules which have an element with zero annihilator and satisfies the Property T (see Proposition 2.9). This result can be applied to produce examples of modules in which all maximal submodules are fd-submodules. We use these concepts to characterize reduced modules (see Proposition 2.14). Moreover, we observe that in any faithful reduced p.p. R -module, the sum of two d-submodules is a d-submodule (see Theorem 2.16), where an R -module M is called *p.p.* if for any $m \in M$, $\text{Ann}_R(m) = Re$ for some idempotent $e \in R$ [17]. We also characterize finitely generated von Neumann regular modules in terms of d-submodules (see Theorem 2.25).

2. Main results

Let M be an R -module and let S be a multiplicative subset of R , i.e., S satisfies: $1 \in S$ and if $s_1, s_2 \in S$, then $s_1 s_2 \in S$. If N is a submodule of M , we set $N_S(M) := \{x \in M \mid sx \in N \text{ for some } s \in S\}$. Then $N_S(M)$ is a submodule of M containing N , which is called the *component of N determined by S* , or simply the *S -component of N* . In particular, the S -component of the zero submodule 0 is $\ker(\phi)$, where $\phi : M \rightarrow S^{-1}M$, the localization of M by S , is a canonical map. Note that $N_S(M) = M$ if $(N :_R M) \cap S \neq \emptyset$. We shall begin with the following definitions:

Definition 2.1. Let M be an R -module and N be a submodule of M .

- (1) N is said to be a *d-submodule* (or *Baer submodule*)¹ if $\text{Ann}_R(m) \subseteq \text{Ann}_R(m')$ for some $m \in N$ and $m' \in M$ implies that $m' \in N$.
- (2) N is called an *fd-submodule* (or a *strongly Baer submodule*) if $\text{Ann}_R(F) \subseteq \text{Ann}_R(m')$ for some finite subset F of N and $m' \in M$ implies that $m' \in N$.
- (3) N is said to be a *0-submodule* if $N = 0_S(M)$ for some multiplicative subset S of R .

It can be easily seen that for a submodule N of an R -module M ,

$$N \text{ is a } 0\text{-submodule} \Rightarrow N \text{ is an fd-submodule} \Rightarrow N \text{ is a d-submodule.}$$

Example 2.2. Let M be an R -module.

¹As Ebrahim Ghashghaei kindly informed the authors, in [9, Definition 1] and [3, Definition 2.3], d-submodules are also referred to as *perpetual* submodules.

- (a) An ideal I of R is a d -ideal (resp., an fd -ideal or a 0 -ideal) if and only if I is a d -submodule (resp., an fd -submodule or a 0 -submodule) of the R -module R .
- (b) If N is a d -submodule (resp., an fd -submodule) of M , then $(N :_M I)$ is also a d -submodule (resp., an fd -submodule) for every ideal I of R . In particular, $\text{Ann}_M(I) = (0 :_M I)$ is an fd -submodule. Also, if S is a multiplicative subset of R , then $\text{Ann}_M(S) = \{0\}$ is trivially a 0 -submodule.
- (c) Any intersection of d -submodules (resp., fd -submodules) is a d -submodule (resp., an fd -submodule).
- (d) Every submodule N of M contains a 0 -submodule. In fact, put $S := 1 + (N :_R M)$. Clearly $0_S(M)$ is a 0 -submodule contained in N .
- (e) Let N be a d -submodule (resp., an fd -submodule) of M and S be any multiplicative subset of R such that $(N :_R M) \cap S = \emptyset$. Then $N_S(M)$, the S -component of N , is a d -submodule (resp., an fd -submodule).

Let M be an R -module and N be a submodule of M . In [14, Lemma 3.9], it was shown that the following statements are equivalent:

- (1) N is a d -submodule;
- (2) $\text{Ann}_M(\text{Ann}_R(n)) \subseteq N$ for each $n \in N$;
- (3) $N = \bigcup_{n \in N} \text{Ann}_M(\text{Ann}_R(n))$.

Let M be an R -module. By the trace of a submodule N in M , we mean $\text{Tr}(N, M) := \{\sum \text{Im}(f) \mid f \in \text{Hom}_R(N, M)\}$.

Proposition 2.3. *Let M be an R -module and N be a submodule of M . If N is a d -submodule of M , then $\text{Tr}(N, M) = N$.*

Proof. Suppose that N is a d -submodule of M . One can see that $N \subseteq \text{Tr}(N, M)$. Now, if $f \in \text{Hom}_R(N, M)$, then $\text{Ann}_R(m) \subseteq \text{Ann}_R(f(m))$ for each $m \in N$. It follows that $\text{Im}(f) \subseteq N$. Thus $\text{Tr}(N, M) = N$. \square

The converse of Proposition 2.3 is not true in general. In order to provide a counterexample, we need some terminology. Let R be an integral domain with quotient field K . Let $\mathbf{F}(R)$ be the set of nonzero fractional ideals of R . For an $I \in \mathbf{F}(R)$, define $I^{-1} = \{x \in K \mid xI \subseteq R\}$. The v -operation on R is a mapping on $\mathbf{F}(R)$ defined by $I \mapsto I_v = (I^{-1})^{-1}$. Recall that an ideal J of R is called a *Glaz-Vasconcelos ideal* (GV-ideal) if J is finitely generated and $J^{-1} = R$. We denote the set of GV-ideals by $\text{GV}(R)$. The w -operation on R is a mapping on $\mathbf{F}(R)$ defined by $I \mapsto I_w = \{x \in K \mid Jx \subseteq I \text{ for some } J \in \text{GV}(R)\}$. An $I \in \mathbf{F}(R)$ is said to be a

v -ideal (resp., w -ideal) if $I_v = I$ (resp., $I_w = I$). An integral domain R is called a DW -domain if every ideal of R is a w -ideal.

Example 2.4. Let R be an integral domain which is not DW (for a concrete example, see [10, Proposition 5.2]). Take a proper GV-ideal J of R . Since R is a w -ideal, it follows from [26, Theorem 6.1.14] that $\text{Ext}_R^1(R/J, R) = \{0\}$. Thus by [19, Example 3.9 (1)], $\text{Tr}(J, R) = J$. Now take $m' \in J_v \setminus J$ since $J \subsetneq J_v = R$. Then $\text{Ann}_R(m) = \text{Ann}_R(m') = \{0\}$ for any $0 \neq m \in J$, but $m' \notin J$. Therefore, J is not a d -ideal of R .

Let M be an R -module. We say that an ideal I of R is a d_M -ideal if for each $x, y \in R$, $\text{Ann}_M(x) \subseteq \text{Ann}_M(y)$ and $x \in I$ implies that $y \in I$. Also, an ideal I of R is said to be an fd_M -ideal if $\text{Ann}_M(S) \subseteq \text{Ann}_M(y)$ for some finite subset S of I and $y \in R$ implies that $y \in I$.

Proposition 2.5. *Let M be an R -module and N be a submodule of M . Then:*

- (1) *If N is a d -submodule (resp., an fd -submodule) of M , then $(N :_R K)$ is a d_M -ideal (resp., an fd_M -ideal) for every nonempty subset K of M .*
- (2) *Assume that M is a cyclic R -module. Then N is a d -submodule (resp., an fd -submodule) if and only if $(N :_R M)$ is a d_M -ideal (resp., an fd_M -ideal).*
- (3) *Suppose that M has an element with zero annihilator. If N is a 0 -submodule of M , then $(N :_R M)$ is a 0 -ideal of R .*

Proof. (1) Assume that N is a d -submodule and let K be a nonempty subset of M . Let $x \in (N :_R K)$ and $y \in R$ such that $\text{Ann}_M(x) \subseteq \text{Ann}_M(y)$. Then for each $m \in K$, $\text{Ann}_R(xm) \subseteq \text{Ann}_R(yx)$, and so $y \in (N :_R K)$ since N is a d -submodule. Similarly, we can prove that $(N :_R K)$ is an fd_M -ideal whenever N is an fd -submodule.

(2) Assume that M is generated by m and let N be a submodule of M such that $(N :_R M)$ is a d -ideal of R . Let $xm \in N$ and $ym \in M$ such that $\text{Ann}_R(xm) \subseteq \text{Ann}_R(ym)$. Then $\text{Ann}_M(x) \subseteq \text{Ann}_M(y)$. It follows that $y \in (N :_R M)$ and so $ym \in N$.

(3) This follows from [13, Lemma 2.5]. □

Remark 2.6. (1) The assertion in Proposition 2.5 (2) fails if one deletes the hypothesis that M is cyclic. For example, consider the \mathbb{Z} -module $M = \mathbb{Z} \times \mathbb{Z}$. Let $N = R(4, 0)$. Then it is clear that $(N :_R M) = 0$ and it is an fd_M -ideal of \mathbb{Z} since M is faithful. However, N is not a d -submodule (and so it is not an fd -submodule).

(2) It is interesting that the final assertion in the statement of Proposition 2.5 would fail if $T_R(M) = M$. Indeed, let $R = \mathbb{Z}$, $M = \mathbb{Z}/4\mathbb{Z}$ and $N = \langle \bar{0} \rangle$. Clearly N is a 0-submodule, but $(N :_R M) = 4\mathbb{Z}$ is not a 0-ideal.

(3) Let N be a d-submodule (resp., an fd-submodule) of an R -module M . In general, $(N :_R M)$ need not be a d-ideal (resp., an fd-ideal). For example, let (R, \mathfrak{m}) be a quasi-local domain, but not a field, and let M be a nonzero R -module such that $\mathfrak{m}M = 0$. Then the zero submodule of M is an fd-submodule but $(0 :_R M) = \mathfrak{m}$ is not a d-ideal, because R is a domain.

Proposition 2.7. *Let M be a multiplication R -module. Then the following statements hold:*

- (1) *Let N be a submodule and P be a prime submodule of M such that $N \cap P$ is a d-submodule (resp., an fd-submodule). Then either N or P is a d-submodule (resp., an fd-submodule).*
- (2) *Let P and Q be prime submodules of M which do not belong to a chain. Then P and Q are both d-submodules (resp., fd-submodules) if and only if $P \cap Q$ is a d-submodule (resp., an fd-submodule).*

Proof. (1) If $N \subseteq P$, then $N = N \cap P$ is a d-submodule. Now assume that $N \not\subseteq P$ and let $m \in P$ and $m' \in M$ such that $\text{Ann}_R(m) \subseteq \text{Ann}_R(m')$. As M is a multiplication module, we have $N = IM$ for some ideal I of R . This implies that there exists $x \in I \setminus (P :_R M)$. Therefore $\text{Ann}_R(xm) \subseteq \text{Ann}_R(xm')$ and $xm \in N \cap P$. By hypothesis, we get $xm' \in P$ and hence $m' \in P$. Similarly, we can show that N or P is an fd-submodule whenever $N \cap P$ is an fd-submodule.

(2) We need only prove the converse. Assume that $P \not\subseteq Q$. Then $(P :_R M) \not\subseteq (Q :_R M)$ since M is a multiplication module. Let $m \in Q$ and $m' \in M$ such that $\text{Ann}_R(m) \subseteq \text{Ann}_R(m')$. It follows that $\text{Ann}_R(xm) \subseteq \text{Ann}_R(xm')$ where $x \in (P :_R M) \setminus (Q :_R M)$. Since $P \cap Q$ is a d-submodule and $xm \in P \cap Q$, we have $xm' \in P \cap Q \subseteq Q$ and hence either $x \in (Q :_R M)$ or $m' \in Q$. But $x \notin (Q :_R M)$, and thus $m' \in Q$. Consequently, Q is a d-submodule and so is P via a similar argument. Similarly, we can prove that P and Q are fd-submodules whenever $P \cap Q$ is an fd-submodule. \square

Lemma 2.8. *Let M be an R -module. For every submodule N of M , define*

$$N_{fd} := \{m \in M \mid \text{Ann}_R(F) \subseteq \text{Ann}_R(m) \text{ for some finite subset } F \text{ of } N\}.$$

Then either $N_{fd} = M$ or N_{fd} is the smallest fd-submodule containing N .

Proof. Let $m, m' \in N_{fd}$. Then there exist $F_1, F_2 \subseteq N$ such that $\text{Ann}_R(F_1) \subseteq \text{Ann}_R(m)$ and $\text{Ann}_R(F_2) \subseteq \text{Ann}_R(m')$. It is obvious that $\text{Ann}_R(F_1 \cup F_2) \subseteq \text{Ann}_R(m + m')$, and so $m + m' \in N_{fd}$. Clearly $rm \in N_{fd}$ for every $r \in R$ and $m \in N_{fd}$. Hence, N_{fd} is a submodule of M . Now, suppose that $F = \{m_1, \dots, m_k\}$ is a finite subset of N_{fd} and $m \in M$ such that $\text{Ann}_R(F) \subseteq \text{Ann}_R(m)$. By definition, for each $i = 1, \dots, k$, there exists a finite subset F_i of N such that $\text{Ann}_R(F_i) \subseteq \text{Ann}_R(m_i)$. Thus $\text{Ann}_R(\bigcup_{i=1}^k F_i) \subseteq \text{Ann}_R(F) \subseteq \text{Ann}_R(m)$, and so $m \in N_{fd}$. Therefore N_{fd} is an fd-submodule of M . Finally it is clear that if N_{fd} is a proper submodule, then it is the smallest fd-submodule containing N . \square

The following proposition provides a necessary and sufficient condition for an R -module which has an element with zero annihilator to satisfy the *Property T*.

Proposition 2.9. *Let M be an R -module which has an element with zero annihilator. Then M satisfies Property T if and only if N is contained in a proper fd-submodule for every submodule $N \subseteq T_R(M)$.*

Proof. Assume that M satisfies *Property T*. Let N be a submodule of M such that $N \subseteq T_R(M)$. Then $\text{Ann}_R(S) \neq 0$ for every finite subset S of N , which implies that N_{fd} is a proper fd-submodule.

Conversely, let N be a finitely generated submodule of M such that $N \subseteq T_R(M)$ and let K be a proper fd-submodule containing N . Using the fact that $\text{Ann}_M(\text{Ann}_R(N)) \subseteq K$, we conclude that $\text{Ann}_R(N) \neq 0$. \square

Corollary 2.10. *Let M be an R -module which has an element with zero annihilator. Then M satisfies strong Property T if and only if $T_R(M)$ is an fd-submodule.*

Remark 2.11. It is well known from [1, Theorem 3.1] that an R -module M satisfies *strong Property T* if and only if M satisfies *Property T* and $T_R(M)$ is a submodule. Then Corollary 2.10 allows us to construct new original examples of d-submodules that are not fd-submodules. In fact, let M be an R -module that does not satisfy *Property T* with $T_R(M)$ is a proper submodule of M . Then $T_R(M)$ is a d-submodule that is not an fd-submodule.

We next give an example of a torsion module that does not satisfy *strong Property T*, which implies the condition “ M has an element with zero annihilator” in Corollary 2.10 is necessary.

Example 2.12. [1, Example 3.2]. Let $R := \mathbb{Z}_2[X, Y]/(X, Y)^2 = \mathbb{Z}_2[x, y]$ and $M := F/K$, where F is the free R -module on $\{e_1, e_2\}$ and $K := \langle xe_1, ye_2, ye_1 + xe_2 \rangle$.

So $M = F/K = R\bar{e}_1 + R\bar{e}_2 = \{0, \bar{e}_1, (1+y)\bar{e}_1, \bar{e}_2, (1+x)\bar{e}_2, y\bar{e}_1 = x\bar{e}_2, \bar{e}_1 + \bar{e}_2, (1+x)(\bar{e}_1 + \bar{e}_2)\}$ is a torsion R -module. Then $T_R(M) = M$ is an fd-submodule but M does not satisfy strong Property T.

We say that an R -module M satisfies the *condition* $(*)$ if $M \setminus T_R(M) = \{m \in M \mid Rm = M\}$. Note that if M satisfies the condition $(*)$, then every proper submodule of M is contained in $T_R(M)$.

Theorem 2.13. *Let M be a multiplication R -module which has an element with zero annihilator. Then M satisfies Property T and the condition $(*)$ if and only if every maximal submodule of M is an fd-submodule.*

Proof. Suppose first that M satisfies *Property T* and the condition $(*)$ and let P be a maximal submodule of M . The fact that P is a proper submodule implies that $P \subseteq T_R(M)$. Thus Proposition 2.9 ensures that P_{fd} is a proper fd-submodule which contains P and hence $P = P_{fd}$.

Conversely, assume that every maximal submodule of M is an fd-submodule. Note from [7, Theorem 2.5] that every proper submodule is contained in a maximal submodule, and so every submodule in $T_R(M)$ is contained in a proper fd-submodule. Then M satisfies *Property T* by Proposition 2.9. Next, we will prove that M satisfies the condition $(*)$. If $Rm = M$ for some $m \in M$, then $\text{Ann}_R(m) = 0$ since M is faithful. Now, let $m \in M$ such that $\text{Ann}_R(m) = 0$. If Rm is a proper submodule of M , then there exists a maximal submodule P of M containing Rm . By hypothesis, P is an fd-submodule and so $\text{Ann}_M(\text{Ann}_R(m)) \subseteq P$, whence $P = M$, a contradiction. \square

Let R be a ring and M be an R -module. We denote by $Z_R(M) = \{x \in R \mid xm = 0 \text{ for some nonzero element } m \in M\}$, the set of zero divisors of R on M and by $T_R(M) = \{m \in M \mid xm = 0 \text{ for some nonzero } x \in R\}$, the set of torsion elements of M with respect to R . Also $Q_M(R) := S_M^{-1}R$ denotes the total quotient ring of R with respect to M , where $S_M := R \setminus Z_R(M)$ and $Q_R(M) := S_M^{-1}M$ denotes the total quotient module of M .

Recall from [17] that an R -module M is reduced if and only if for any $x \in R$ and $m \in M$, $x^2m = 0$ implies that $xm = 0$. The following proposition gives a new characterization of reduced modules in terms of d-submodules.

Proposition 2.14. *Let M be an R -module. Then the following statements are equivalent.*

- (1) M is a reduced module.

- (2) $Q_R(M)$ is a reduced $Q_M(R)$ -module.
- (3) $\langle E_M(N) \rangle = N$ for every d -submodule N of M .
- (4) $\langle E_M(N) \rangle = N$ for every fd -submodule N of M .

Proof. (1) \Leftrightarrow (2) This is obvious.

(1) \Rightarrow (3) Assume that M is a reduced module and let N be a d -submodule of M . It is obvious that $N \subseteq \langle E_M(N) \rangle$. For the reverse inclusion let $m \in \langle E_M(N) \rangle$. Then $m = x_1 m_1 + \cdots + x_k m_k$ for some positive integer k and elements $x_i \in R$, $m_i \in M$ with $x_i^n m_i \in N$ for some positive integer n . As M is reduced, we have $\text{Ann}_R(x_i^n m_i) = \text{Ann}_R(x_i m_i)$ for each $i \in \{1, \dots, k\}$. This implies that $x_i m_i \in N$, and so $m \in N$.

(3) \Rightarrow (4) This is trivial.

(4) \Rightarrow (1) Suppose that $\langle E_M(N) \rangle = N$ for every fd -submodule N of M . To show that M is a reduced module, we must prove that if $x \in R$ and $m \in M$ such that $x^2 m = 0$, then $xm = 0$. But this follows from the fact that the zero submodule of M is an fd -submodule, and so $\langle E_M(0) \rangle = 0$ by the hypothesis. \square

As an immediate consequence of Proposition 2.14, we give the following corollary.

Corollary 2.15. *Let M be an R -module which satisfies the radical formula. Then the following statements are equivalent.*

- (1) M is a reduced module.
- (2) Every d -submodule of M is radical.
- (3) Every fd -submodule of M is radical.

The following theorem gives a class of modules in which the sum of two d -submodules is a d -submodule.

Theorem 2.16. *Let M be a faithful reduced $p.p.$ R -module. Then the sum of two d -submodules is a d -submodule.*

We need the following lemmas in order to prove Theorem 2.16.

Lemma 2.17. *Let M be a reduced R -module. Then,*

$$\text{Ann}_R(\text{Ann}_M(xy)) = \text{Ann}_R(\text{Ann}_M(x)) \cap \text{Ann}_R(\text{Ann}_M(y))$$

for each $x, y \in R$.

Proof. It can be easily shown the inclusion " \subseteq ". For the reverse, let

$$r \in \text{Ann}_R(\text{Ann}_M(x)) \cap \text{Ann}_R(\text{Ann}_M(y)) \text{ and } m \in \text{Ann}_M(xy).$$

Then $ym \in \text{Ann}_M(x)$ and so $rym = 0$. Similarly, we get $r^2m = 0$. As M is a reduced module, we then have $m \in \text{Ann}_R(\text{Ann}_M(xy))$. \square

Recall that an element $e \in R$ is *weak idempotent* of an R -module M if $e^2 - e \in \text{Ann}_R(M)$.

Lemma 2.18. *Let M be an R -module and let $e \in R$ be a weak idempotent of M . Then:*

- (1) $\text{Ann}_M(Re) = (1 - e)M$.
- (2) *If M is faithful, then $\text{Ann}_R(eM) = R(1 - e)$.*

Proof. Straightforward. \square

Proof of Theorem 2.16 Let N and N' be two d-submodules of M . Let $m \in N + N'$, and pick $n \in N$ and $n' \in N'$ such that $m = n + n'$. Then $\text{Ann}_R(n) \cap \text{Ann}_R(n') \subseteq \text{Ann}_R(m)$. Let e, f be idempotents of R such that $\text{Ann}_R(n) = Re$ and $\text{Ann}_R(n') = Rf$. Thus, Lemma 2.18 gives that $\text{Ann}_R(n) = \text{Ann}_R(\text{Ann}_M(e))$ and $\text{Ann}_R(n') = \text{Ann}_R(\text{Ann}_M(f))$. Consequently, by Lemma 2.17,

$$\text{Ann}_R(\text{Ann}_M(e)) \cap \text{Ann}_R(\text{Ann}_M(f)) \subseteq \text{Ann}_R(m),$$

whence $\text{Ann}_M(\text{Ann}_R(m)) \subseteq \text{Ann}_M(e) = (1 - e)M$. On the other hand, since N is a d-submodule, it follows that $\text{Ann}_R(n) = Re$ implies $(1 - e)M \subseteq \text{Ann}_M(\text{Ann}_R(n))$, and so $(1 - e)M \subseteq N$. Similarly, we have $(1 - f)M \subseteq N'$ since N' is a d-submodule. Therefore $(1 - e)m = (1 - e)fm + (1 - f)m \in N + N'$ for each $m \in M$, and so $\text{Ann}_M(\text{Ann}_R(m)) \subseteq N + N'$. Finally $N + N'$ is a d-submodule. \square

We now characterize finitely generated (resp., cyclic) modules.

Proposition 2.19. *Let M be an R -module. Then M is a finitely generated (resp., a cyclic) module if and only if $\text{Ann}_R(N) = \text{Ann}_R(M)$ for some finitely generated fd-submodule (resp., cyclic d-submodule) N of M .*

Proof. The necessity is obvious. Conversely, let N be a finitely generated fd-submodule (resp., a cyclic d-submodule) of M such that $\text{Ann}_R(N) = \text{Ann}_R(M)$. This implies that $\text{Ann}_R(N) \subseteq \text{Ann}_R(m)$ for each $m \in M$. Since N is an fd-submodule (resp., a d-submodule), we have $m \in N$. Consequently, M is a finitely generated (resp., a cyclic) module. \square

An R -module M is called *strongly duo* provided that $\text{Tr}(N, M) = N$ for each submodule N of M (see [16]). We also recall that M is called *principally quasi-injective* (pq-injective for short) if each R -morphism from a principal submodule of M to M can be extended to an endomorphism of M (see [21]).

Proposition 2.20. *Let M be an R -module. Then the following statements are equivalent.*

- (1) *Every submodule of M is a d -submodule.*
- (2) *Every cyclic submodule of M is a d -submodule.*
- (3) *M satisfies the condition $(*)$ and every submodule contained in $T_R(M)$ is a d -submodule.*
- (4) *M is a strongly duo module.*
- (5) *M is a pq-injective module.*

Proof. (1) \Rightarrow (2) This is clear.

(2) \Rightarrow (1) Let N be a submodule of M . By hypothesis, every cyclic submodule contained in N is a d -submodule. In other words, if $\text{Ann}_R(m) \subseteq \text{Ann}_R(m')$ for some $m \in N$ and $m' \in M$, then $m' \in Rm \subseteq N$.

(2) \Leftrightarrow (3) This is obvious.

(2) \Leftrightarrow (4) This follows from [16, Theorem 2.1].

(4) \Leftrightarrow (5) See [16, Theorem 3.5]. □

Recall that an R -module M is said to be *prime* if the zero submodule of M is a prime submodule of M .

Corollary 2.21. *Let M be an R -module. Then M is a prime strongly duo module if and only if M is a simple module.*

Proof. The sufficiency is trivial. Note that an R -module M is prime if and only if $\text{Ann}_R(N) = \text{Ann}_R(M)$ for every nonzero submodule N of M . If M is a prime strongly duo R -module, then by Propositions 2.19 and 2.20, $M = Rm$ for each nonzero $m \in M$. Consequently, M is a simple module, as desired. □

Let M be an R -module. We say that M is *perfect* if M satisfies the descending chain condition (*DCC*) on cyclic submodules (see [5]). Now we consider the set $\mathcal{A}_M := \{\text{Ann}_R(m) \mid m \in M\}$.

Proposition 2.22. *Let M be a strongly duo module satisfying ACC on \mathcal{A}_M . Then M is a perfect module.*

Proof. Let $Rm_1 \supseteq Rm_2 \supseteq \cdots$ be a descending chain of cyclic submodules of M . By hypothesis, $\text{Ann}_R(m_k) = \text{Ann}_R(m_{k+1})$ for some positive integer k . As M is a strongly duo module, we have $Rm_k = Rm_{k+1}$. This implies that M is a perfect module, as desired. \square

Corollary 2.23. *Let R be a Noetherian ring. Then every finitely generated strongly duo module over R is an Artinian module.*

Proof. If M is a finitely generated strongly duo module over a Noetherian ring R , then M is a Noetherian perfect module and hence M is Artinian by [6, Proposition 4.12]. \square

The following example shows that the converse of the previous corollary is not true in general.

Example 2.24. Consider the \mathbb{Z} -module $M = \mathbb{Z}_{p^\infty}$, where p is a prime number. Then, M is an Artinian module which is not Noetherian.

We now characterize finitely generated von Neumann regular modules.

Theorem 2.25. *Let M be a finitely generated R -module. Consider the following conditions:*

- (1) *M is a reduced multiplication module and every submodule is a 0-submodule.*
- (2) *M is a reduced multiplication module and every submodule is an fd-submodule.*
- (3) *M is a reduced multiplication module and satisfies any one of the conditions of Proposition 2.20.*
- (4) *M is a von Neumann regular module.*

Then:

- (i) $(1) \Rightarrow (2) \Leftrightarrow (3) \Leftrightarrow (4)$.
- (ii) *If M is a cyclic module, then the above conditions are equivalent.*

Proof. (i) $(1) \Rightarrow (2)$ and $(2) \Rightarrow (3)$ are clear.

$(3) \Rightarrow (4)$ This follows from [14, Theorem 3.10].

$(4) \Rightarrow (2)$ Assume that M is a von Neumann regular module. Let N be a submodule of M and let F be a finite subset of N . By hypothesis, there exists a weak idempotent $e \in R$ of M such that $\langle F \rangle = eM$. Then $R(1 - e) \subseteq \text{Ann}_R(F)$ by Lemma 2.18. Thus $\text{Ann}_M(\text{Ann}_R(F)) \subseteq \text{Ann}_M(1 - e) = eM$ which implies that $\text{Ann}_M(\text{Ann}_R(F)) \subseteq N$. So N is an fd-submodule.

(ii) $(4) \Rightarrow (1)$ Suppose that M is a cyclic von Neumann regular module. Let N be a submodule of M . It will first be shown that $S := \{r \in R \mid \text{Ann}_R(m) \subseteq$

$\text{Ann}_R(\text{Ann}_M(r))$ for some $m \in N$ is a multiplicative subset of R . If $r, s \in S$, then there exist $m, m' \in N$ such that $\text{Ann}_R(m) \subseteq \text{Ann}_R(\text{Ann}_M(r))$ and $\text{Ann}_R(m') \subseteq \text{Ann}_R(\text{Ann}_M(s))$. It follows by Lemma 2.17 that

$$\text{Ann}_R(m) \cap \text{Ann}_R(m') \subseteq \text{Ann}_R(\text{Ann}_M(rs)).$$

By hypothesis, there exists a weak idempotent $e \in R$ such that $Rm + Rm' = eM$ and so $Rm + Rm'$ is a cyclic submodule of M . Thus $\text{Ann}_R(m'') \subseteq \text{Ann}_R(\text{Ann}_M(rs))$ for some $m'' \in N$. We complete the proof by showing that $N = 0_S(M)$. Since N is a d -submodule, we have $0_S(M) \subseteq N$. In fact, if $m \in 0_S(M)$, then $sm = 0$ for some $s \in S$. Consequently, $\text{Ann}_R(n) \subseteq \text{Ann}_R(m)$ for some $n \in N$, and so $m \in N$. Now, let $m \in N$. By assumption, we get $Rm = eM$ for some weak idempotent $e \in R$. Then $(1 - e) \in \text{Ann}_R(m)$. Also, it is clear to see that $(1 - e) \in S$, and so $m \in 0_S(M)$. \square

We remark that finitely generated von Neumann regular modules are characterized in [14, Theorem 3.10]. In particular, it was shown that for a finitely generated module M , M is von Neumann regular if and only if M is a reduced multiplication module in which every submodule is a d -submodule.

Corollary 2.26. *Let M be a finitely generated von Neumann regular R -module. Then $\text{End}_R(M)$ is a von Neumann regular ring.*

Proof. By Theorem 2.25 and [16, Theorem 5.5]. \square

Corollary 2.27. *If R is a PID which is not a field, then finitely generated von Neumann regular R -modules are precisely non-faithful cyclic reduced R -modules.*

Proof. Combining Theorem 2.25 with [16, Corollary 3.8]. \square

Example 2.28. Finitely generated von Neumann regular \mathbb{Z} -modules are precisely $\mathbb{Z}/n\mathbb{Z}$, where $n > 1$ is square-free.

Let M_i be an R_i -module for each $i = 1, 2$. Set $M := M_1 \times M_2$ and $R := R_1 \times R_2$. Then M is clearly an R -module with componentwise addition and scalar multiplication. Also every submodule N of M is of the form $N = N_1 \times N_2$, where N_i is a submodule of M_i .

Proposition 2.29. *Let M_i be an R_i -module for each $i = 1, 2$. Set $M := M_1 \times M_2$, $R := R_1 \times R_2$, and $N := N_1 \times N_2$ be a submodule of M . Then:*

- (1) N is a d -submodule (resp., an fd -submodule) of M if and only if N_i is a d -submodule (resp., an fd -submodule) for each i .

- (2) N is a 0-submodule of M if and only if N_i is a 0-submodule of M_i for each i .

Proof. (1) It suffices to see that $\text{Ann}_R(m_1, m_2) = \text{Ann}_{R_1}(m_1) \times \text{Ann}_{R_2}(m_2)$.

(2) Assume that N is a 0-submodule of M . Then $N = 0_S(M)$ for some multiplicative subset S of R . Put $S_1 := \{s_1 \in R_1 \mid (s_1, s_2) \in S \text{ for some } s_2 \in R_2\}$ and $S_2 := \{s_2 \in R_2 \mid (s_1, s_2) \in S \text{ for some } s_1 \in R_1\}$. It is clear that S_1 and S_2 are multiplicative subsets of R_1 and R_2 , respectively. Also one can easily check that $0_S(M) = 0_{S_1}(M_1) \times 0_{S_2}(M_2)$. This implies that N is a 0-submodule of M if and only if N_1 and N_2 are 0-submodules, as desired. \square

Proposition 2.30. *Let X be an indeterminate over R and let M be an R -module. Then N is an fd-submodule if and only if $N[X]$ is an fd-submodule of $M[X]$ as an $R[X]$ -module.*

Proof. Let N be an fd-submodule of M and let $F = \{f_1, \dots, f_n\} \subseteq N[X]$ and $f \in M[X]$ such that $\text{Ann}_{R[X]}(F) \subseteq \text{Ann}_{R[X]}(f)$. Now put $f := \sum_{j=0}^p m'_j X^j$. Then it follows easily that $\text{Ann}_R(C(F)) \subseteq \text{Ann}_R(m'_1, \dots, m'_p)$ where $C(F)$ is the set of all coefficients of elements of F . Hence $\text{Ann}_R(C(F)) \subseteq \text{Ann}_R(m'_j)$ for each $j = 0, \dots, p$, and therefore $C(F) \subseteq N$ implies that $m'_j \in N$ for each $j = 0, \dots, p$. This shows that $f \in N[X]$.

Conversely, suppose that $N[X]$ is an fd-submodule. Let $\{m_1, \dots, m_n\} \subseteq N$ and $m' \in M$ such that $\bigcap_{i=1}^n \text{Ann}_R(m_i) \subseteq \text{Ann}_R(m')$. Then

$$\bigcap_{i=1}^n \text{Ann}_{R[X]}(m_i X) \subseteq \text{Ann}_{R[X]}(m' X)$$

in $M[X]$ and $\{m_1 X, \dots, m_n X\} \subseteq N[X]$ implies that $m' X \in N[X]$, and so $m' \in N$. \square

Let $f : M \rightarrow M'$ be an R -module homomorphism. By the extension of a submodule N of M , we mean $N^e = f(N)$ and by the contraction of a submodule K of M' , we mean $K^c = f^{-1}(K)$.

Proposition 2.31. *Let $f : M \rightarrow M'$ be an R -morphism. Then every d -submodule (resp., fd-submodule) of M' contracts to a d -submodule (resp., an fd-submodule) of M if and only if $\ker(f)$ is a d -submodule (resp., an fd-submodule) of M .*

Proof. The necessity is obvious. Conversely, suppose that $\ker(f)$ is a d -submodule and let K be a d -submodule of M' . If $\text{Ann}_R(m) \subseteq \text{Ann}_R(m')$ for some $m \in K^c$ and $m' \in M$, then $\text{Ann}_R(rm) \subseteq \text{Ann}_R(rm')$ for each $r \in \text{Ann}_R(f(m))$. By hypothesis, we then have $rm' \in \ker(f)$ for each $r \in \text{Ann}_R(f(m))$ and so $\text{Ann}_R(f(m)) \subseteq$

$\text{Ann}_R(f(m'))$. As K is a d -submodule, it follows that $f(m') \in K$. Similarly, we can prove that every fd -submodule of M' contracts to an fd -submodule of M whenever $\ker(f)$ is an fd -submodule of M . \square

Corollary 2.32. *Let R be a commutative ring, M be an R -module, and N be a d -submodule (resp., an fd -submodule) of M . Then every d -submodule (resp., fd -submodule) of the R -module M/N contracts to a d -submodule (resp., an fd -submodule) of M .*

Corollary 2.33. *Let R be a commutative ring, S be a multiplicative subset of R , M be an R -module, and $f : M \rightarrow S^{-1}M$ be the natural morphism of R -modules. Then every d -submodule (resp., fd -submodule) of $S^{-1}M$ contracts to a d -submodule (resp., an fd -submodule) of M .*

Proposition 2.34. *Let R be a commutative ring, S be a multiplicative subset of R , M be an R -module, and $f : M \rightarrow S^{-1}M$ be the natural morphism of R -modules. If N is a d -submodule (resp., an fd -submodule) of M with $S \cap (N :_R M) = \emptyset$, then N^{ec} is also a d -submodule (resp., an fd -submodule) containing N .*

Proof. We just observe that $N^{ec} = N_M(S)$, see Example 2.2. \square

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