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TWO GENERALIZED DERIVATIONS ON LIE IDEALS IN PRIME RINGS

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Abstract. Let R be a prime ring of characteristic not equal to 2, U be the Utumi quotient ring of R and C be the extended centroid of R. Let G and F be two generalized derivations on R and L be a non-central Lie ideal of R. If $F\Big(G(u)\Big)u = G(u^2)$ for all $u \in L$, then one of the following holds:

(1) $G = 0$.
(2) There exist $p, q \in U$ such that $G(x) = px$, $F(x) = qx$ for all $x \in R$ with $qp = p$.
(3) R satisfies s_4.

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1. Introduction

R always stands for the prime ring with center $Z(R)$ throughout this article. The Utumi quotient ring of R is denoted by U. The center of U is called the extended centroid of R and it is denoted by C. The definition and construction of U can be found in [3]. An additive mapping $d : R \to R$ is said to be a derivation if $d(xy) = d(x)y + xd(y)$ for all $x, y \in R$. For a fixed $p \in R$, the mapping $\delta_p : R \to R$, defined by $\delta_p(x) = [p, x]$ for all $x \in R$ is a derivation, known as inner derivation induced by an element p. A derivation that is not inner is called outer derivation. An additive mapping $F : R \to R$ is said to be a generalized derivation if there exists a derivation d on R such that $F(xy) = F(x)y + xd(y)$ for all $x, y \in R$. For fixed $a, b \in R$, the mapping $F_{(a,b)} : R \to R$ defined by $F_{(a,b)}(x) = ax + xb$ is a generalized derivation on R. The mapping $F_{(a,b)}$ is usually called generalized inner derivation on R. An additive subgroup L of R is said to be a Lie ideal of R if $[L, R] \subseteq L$. The standard polynomial identity s_4 in four variables is defined as

$$s_4(x_1, x_2, x_3, x_4) = \sum_{\sigma \in S_4} (-1)^\sigma x_{\sigma(1)}x_{\sigma(2)}x_{\sigma(3)}x_{\sigma(4)}$$

where $(-1)^\sigma$ is $+1$ or -1 according as σ being even or odd permutation in symmetric group S_4.

In [19], Posner demonstrated that if d is a derivation of a prime ring R such that $[d(x), x] \in Z(R)$ for all $x \in R$, then either $d = 0$ or R is a commutative ring. Many mathematicians have extended Posner's result in various ways. In [14], Lanski generalized Posner's result by proving it on a Lie ideal L of R. In [4], Bresar proved that if f_1 and f_2 are two derivations such that $f_1(x)x - xf_2(x) = 0$ for all $x \in L$ then either $f_1 = f_2 = 0$ or R is commutative. More recently in [21], Tiwari has given the entire structure of F, G and H if they satisfy the identity $F(G(u)u) = H(u^2)$ for all $u \in S$, where F, G and H are generalized derivations and S is a suitable subset of R. Generalized derivations on Lie ideals and left ideals have been studied in [1,3,6,7], where further references can be found. Motivated by the above cited results, it is a very natural question what would be the structure of F and G if they satisfy the identity $F(G(u))u = G(u^2)$. Our main result is the following:

Theorem 1.1. *Let R be a prime ring of characteristic not equal to 2, U be the Utumi quotient ring of R and C be the extended centroid of R. Let G and F be two generalized derivations on R and L be a non-central Lie ideal of R. If $F(G(u))u = G(u^2)$ for all $u \in L$, then one of the following holds:*

(1) $G = 0$.
(2) *There exist $p, q \in U$ such that $G(x) = px$, $F(x) = qx$ for all $x \in R$ with $qp = p$.*
(3) *R satisfies s_4.*

We use the following remarks in the sequel to prove our result.

Remark 1. [6] Let K be an infinite field and $m \geq 2$ be an integer. If $P_1, \ldots, P_k$ are non-scalar matrices in $M_m(K)$ then there exists an invertible matrix $P \in M_m(K)$ such that each matrix $PP_1P^{-1}, \ldots, PP_kP^{-1}$ has all non-zero entries.

Remark 2. Let K be any field and $R = M_m(K)$ be the algebra of all $m \times m$ matrices over K with $m \geq 2$. Then the matrix unit e_{ij} is an element of $[R, R]$ for all $1 \leq i \neq j \leq m$.

Remark 3. [2] Every generalized derivation F of R can be uniquely extended to a generalized derivation of U and it assumes the form $F(x) = ax + d(x)$, for some $a \in U$ and a derivation d on U.

Remark 4. [16] If I is a two-sided ideal of R, then R, I and U satisfy the same differential identities.

Remark 5. [2] If I is a two-sided ideal of R, then R, I and U satisfy the same generalized polynomial identities with coefficients in U.

Remark 6. [13, Kharchenko Theorem] Let R be a prime ring, d be a non-zero derivation on R and I be a non-zero ideal of R. If I satisfies the differential identity

$$f(r_1, \ldots, r_n, d(r_1), \ldots, d(r_n)) = 0$$

for all $r_1, \ldots, r_n \in I$, then either

(i) I satisfies the generalized polynomial identity $f(r_1, \ldots, r_n, x_1, \ldots, x_n) = 0$

or

(ii) d is U-inner i.e., for some $q \in U, d(x) = [q, x]$ and I satisfies the generalized polynomial identity $f(r_1, \ldots, r_n, [q, r_1], \ldots, [q, r_n]) = 0$.

Remark 7. [3, Theorem 4.2.1, (Jacobson density theorem)] Let R be a primitive ring with V_R a faithful irreducible R-module and $D = End(V_R)$, then for any positive integer n if $v_1, v_2, \ldots, v_n$ are D-independent in V and $w_1, w_2, \ldots, w_n$ are arbitrary in V then there exists $r \in R$ such that $v_i r = w_i$ for $i = 1, 2, \ldots, n$.

Remark 8. [5] Let $X = \{x_1, x_2, \ldots\}$ be a countable set consisting of non-commuting indeterminates $x_1, x_2, \ldots$. Let $C\{X\}$ be the free algebra over C on the set X. We denote $T = U *_C C\{X\}$, the free product of the C-algebras U and C{X}. The elements of T are called the generalized polynomials with coefficients in U. Let B be a set of C-independent vectors of U. Then any element $f \in T$ can be represented in the form $f = \sum_i a_i n_i$, where $a_i \in C$ and n_i are B-monomials of the form $p_0 u_1 p_1 u_2 p_2 \cdots u_n p_n$, with $p_0, p_1, \ldots, p_n \in B$ and $u_1, u_2, \ldots, u_n \in X$. Any generalized polynomial $f = \sum_i a_i n_i$ is trivial i.e., zero element in T if and only if $a_i = 0$ for each i.

2. F and G are generalized inner derivations

In this section we study the case when F and G are generalized inner derivations. Suppose $F(x) = ax+xb$ and $G(x) = cx+xd$ for all $x \in R$ and for some $a, b, c, d \in U$. To prove our main result, we prove the following proposition.

Proposition 2.1. *Let R be a prime ring of characteristic not equal to 2, U be the Utumi quotient ring of R and C be the extended centroid of R. Let G and F be two generalized inner derivations on R and $L = [R, R]$ be a non-central Lie ideal of R. If $F\Big(G(u)\Big)u = G(u^2)$ for all $u \in L$, then one of the following holds:*

(1) $G = 0$.

(2) *There exist $p, q \in U$ such that $G(x) = px$, $F(x) = qx$ for all $x \in R$ with $qp = p$.*

(3) *R satisfies s_4.*

We need the following results to prove Proposition 2.1.

Lemma 2.2. *Let $R = M_m(K)$ be the ring of all $m \times m$ matrices over an infinite field K with characteristic not equal to 2 and $m \geq 3$ and $L = [R, R]$. Let $a_1, a_2, a_3, a_4, a_5, a_6 \in R$ such that*

$$a_1[x_1, x_2]^2 + a_2[x_1, x_2]a_3[x_1, x_2] + a_4[x_1, x_2]a_5[x_1, x_2] + [x_1, x_2]a_6[x_1, x_2] \quad (1)$$

$$= a_4[x_1, x_2]^2 + [x_1, x_2]^2 a_3$$

for all $x_1, x_2 \in R$. Then one of the following holds:

(1) $a_3, a_5, a_6 \in K.I_m$.

(2) $a_3, a_4, a_4a_5 + a_6 \in K.I_m$.

Proof. By the hypothesis R satisfies

$$a_1u^2 + a_2ua_3u + a_4ua_5u + ua_6u - a_4u^2 - u^2a_3 = 0 \quad (2)$$

for all $u \in [R, R]$.

First, we assume that a_3 is not central. Since equation (2) is invariant under the action of all automorphisms of R, a_3 may be assumed to have all non-zero entries by Remark 1. For three different indices i, j, h, let $u = [e_{ij}, e_{ji}] = e_{ii} - e_{jj}$, then from equation (2), we have

$$a_1(e_{ii} + e_{jj}) + a_2(e_{ii} - e_{jj})a_3(e_{ii} - e_{jj}) + a_4(e_{ii} - e_{jj})a_5(e_{ii} - e_{jj}) \quad (3)$$

$$+(e_{ii} - e_{jj})a_6(e_{ii} - e_{jj}) - a_4(e_{ii} + e_{jj}) - (e_{ii} + e_{jj})a_3 = 0.$$

Left multiplying by e_{ii} and right multiplying by e_{hh} in equation (3), we get the following

$$e_{ii}a_3e_{hh} = (a_3)_{ih}e_{ih} = 0$$

i.e. $(a_3)_{ih} = 0$, which is a contradiction. Thus $a_3 \in K.I_m$.

Now, if a_4 and a_5 are non-central elements, then by the previous arguments, we may assume that all the entries of a_4 and a_5 are non-zero. Since $a_3 \in K.I_m$ therefore equation (2) reduces to:

$$(a_1 + a_2a_3 - a_4 - a_3)u^2 + a_4ua_5u + ua_6u = 0. \tag{4}$$

Substituting $u = [e_{ii}, e_{ij}] = e_{ij}$ in equation (4), we get

$$a_4e_{ij}a_5e_{ij} + e_{ij}a_6e_{ij} = 0. \tag{5}$$

Left multiplying by e_{ij} in equation (5), we have

$$e_{ij}a_4e_{ij}a_5e_{ij} = (a_4)_{ji}(a_5)_{ji}e_{ij} = 0$$

which implies either $(a_4)_{ji} = 0$ or $(a_5)_{ji} = 0$. In each case we get a contradiction thus either a_4 is central or a_5 is central.
Case 1: If $a_4 \in K.I_m$ then equation (4) reduces to

$$(a_1 + a_2a_3 - a_4 - a_3)u^2 + u(a_4a_5 + a_6)u = 0. \tag{6}$$

Again choosing $u = e_{ij}$ in equation (6), we get that

$$(a_4a_5 + a_6)_{ji}e_{ij} = 0$$

i.e. $(a_4a_5 + a_6)_{ji} = 0$ and thus again by Remark 1, $a_4a_5 + a_6 \in K.I_m$. Hence in this case we have $a_3, a_4, a_4a_5 + a_6 \in C$, which is our conclusion (2).
Case 2: If $a_5 \in K.I_m$ then from equation (4), we get

$$(a_1 + a_2a_3 - a_4 - a_3 + a_4a_5)u^2 + ua_6u = 0 \tag{7}$$

for all $u \in L$. Thus by similar arguments as above, we get $a_6 \in K.I_m$. Hence in this case we get our conclusion (1). □

Lemma 2.3. *Let $R = M_m(K)$ be the ring of all $m \times m$ matrices over a field K with characteristic not equal to 2 and $m \geq 3$ and $L = [R, R]$. Let $a_1, a_2, a_3, a_4, a_5, a_6 \in R$. If*

$$a_1[x_1, x_2]^2 + a_2[x_1, x_2]a_3[x_1, x_2] + a_4[x_1, x_2]a_5[x_1, x_2] + [x_1, x_2]a_6[x_1, x_2] \tag{8}$$

$$= a_4[x_1, x_2]^2 + [x_1, x_2]^2a_3$$

for all $x_1, x_2 \in R$. Then one of the following holds:

(1) $a_3, a_5, a_6 \in K.I_m$.
(2) $a_3, a_4, a_4a_5 + a_6 \in K.I_m$.

Proof. If K is an infinite field, then the conclusion follows from Lemma 2.2. Otherwise, let F be an infinite field which is an extension of K and let $\bar{R} = M_t(F) \cong R \otimes_K F$. It is worth noting that a multilinear polynomial is an identity for R if and only if it is an identity for $\bar{R}$. So $\bar{R}$ does not satisfy s_4 and we may assume that $t \geq 3$. Consider the following generalized polynomial identity

$$f(Y_1, Y_2) = a_1[Y_1, Y_2]^2 + a_2[Y_1, Y_2]a_3[Y_1, Y_2] + a_4[Y_1, Y_2]a_5[Y_1, Y_2] + [Y_1, Y_2]a_6[Y_1, Y_2]$$
$$-a_4[Y_1, Y_2]^2 - Y_1, Y_2]^2 a_3 \tag{9}$$

which is a generalized polynomial identity for R with multi-homogeneous of multi degree $(2, 2)$ in indeterminates Y_1 and Y_2. Thus the complete linearization of $f(Y_1, Y_2)$ gives a multilinear generalized polynomial $H(Y_1, Y_2, X_1, X_2)$ in 4 indeterminates. Moreover,

$$H(Y_1, Y_2, Y_1, Y_2) = 4f(Y_1, Y_2).$$

Clearly the multilinear polynomial $H(Y_1, Y_2, X_1, X_2)$ is a generalized polynomial identity for R and $\bar{R}$ too. Since $char(K) \neq 2$, we obtain $f(Y_1, Y_2) = 0$, for all $Y_1, Y_2 \in \bar{R}$, and the conclusion follows from Lemma 2.2. □

Lemma 2.4. *Let R be a prime ring of characteristic not equal to 2 and a_1, a_2, a_3, a_4, a_5, $a_6 \in R$ such that*

$$a_1[x_1, x_2]^2 + a_2[x_1, x_2]a_3[x_1, x_2] + a_4[x_1, x_2]a_5[x_1, x_2] + [x_1, x_2]a_6[x_1, x_2] \tag{10}$$
$$-a_4[x_1, x_2]^2 - [x_1, x_2]^2 a_3 = 0$$

for all $x_1, x_2 \in R$. If R does not satisfy any non-trivial generalized polynomial identity, then one of the following holds:

(1) $a_3, a_5, a_6 \in C$.

(2) $a_3, a_4, a_4a_5 + a_6 \in C$.

Proof. Suppose that a_3, a_4, a_5 are not central elements. From hypothesis R satisfies the following generalized polynomial identity

$$h(x_1, x_2) = a_1[x_1, x_2]^2 + a_2[x_1, x_2]a_3[x_1, x_2] + a_4[x_1, x_2]a_5[x_1, x_2] \tag{11}$$
$$+[x_1, x_2]a_6[x_1, x_2] - a_4[x_1, x_2]^2 - [x_1, x_2]^2 a_3$$

for all $x_1, x_2 \in R$. Since R and U satisfy same generalized polynomial identity (GPI), U satisfies $h(x_1, x_2) = 0_T$. Suppose that $h(x_1, x_2)$ is a trivial GPI for U. Let $T = U *_C C\{x_1, x_2\}$, the free product of U and $C\{x_1, x_2\}$, the free C-algebra in non-commuting indeterminates x_1, x_2. Then, $h(x_1, x_2)$ is zero element in $T = U *_C C\{x_1, x_2\}$. Since $\{1, a_3\}$ is linearly C-independent therefore by Remark

8, we get $[x_1, x_2]^2 a_3 = 0 \in T$, which is a contradiction. Thus a_3 must be in C. Then U satisfies

$$h(x_1, x_2) = p[x_1, x_2]^2 + a_4[x_1, x_2]a_5[x_1, x_2] + [x_1, x_2]a_6[x_1, x_2] \tag{12}$$

where $p = a_1 + a_2a_3 - a_4 - a_3 \in U$. It implies that $\{1, a_5, a_6\}$ is linearly C-dependent otherwise $a_4[x_1, x_2]a_5[x_1, x_2]$ will appear as a non-trivial polynomial identity. Then there exist $\alpha_1, \alpha_2, \alpha_3 \in C$ such that $\alpha_1 + \alpha_2 a_5 + \alpha_3 a_6 = 0$. If $\alpha_3 = 0$ then $a_5 = -\alpha_2^{-1}\alpha_1$, a contradiction. Therefore $\alpha_3 \neq 0$. Then $a_6 = \beta_1 + \beta_2 a_5$, where $\beta_1 = -\alpha_1\alpha_3^{-1}$ and $\beta_2 = -\alpha_2\alpha_3^{-1}$. Thus from equation (12), we get

$$(p + \beta_1)[x_1, x_2]^2 + a_4[x_1, x_2]a_5[x_1, x_2] + \beta_2[x_1, x_2]a_5[x_1, x_2] = 0 \tag{13}$$

for all $x_1, x_2 \in R$. Since $\{1, a_5\}$ is linearly C-independent, by using Remark 8 in equation (13), we get

$$a_4[x_1, x_2]a_5[x_1, x_2] + \beta_2[x_1, x_2]a_5[x_1, x_2] = 0.$$

Again since $\{1, a_4\}$ is linearly C-independent, by previous arguments we get

$$a_4[x_1, x_2]a_5[x_1, x_2] = 0,$$

a contradiction. Hence either $a_4 \in C$ or $a_5 \in C$.

Case 1: If $a_5 \in C$ then from $a_6 = \beta_1 + \beta_2 a_5$, we get $a_6 \in C$. Thus in this case we get our conclusion (1).

Case 2: If $a_4 \in C$ then equation (12) reduces to

$$p[x_1, x_2]^2 + [x_1, x_2](a_4a_5 + a_6)[x_1, x_2] = 0 \tag{14}$$

for all $x_1, x_2 \in R$. Now if $a_4a_5 + a_6 \notin C$ then from Remark 8 in equation (14), U satisfies the non-trivial identity $[x_1, x_2](a_4a_5 + a_6)[x_1, x_2] = 0$, a contradiction. Thus $a_4a_5 + a_6 \in C$. Hence in this case we get our conclusion (2). □

Lemma 2.5. *Let R be a prime ring of characteristic not equal to 2 and a_1, a_2, a_3, a_4, a_5, $a_6 \in R$ such that*

$$a_1[x_1, x_2]^2 + a_2[x_1, x_2]a_3[x_1, x_2] + a_4[x_1, x_2]a_5[x_1, x_2] + [x_1, x_2]a_6[x_1, x_2] \tag{15}$$

$$-a_4[x_1, x_2]^2 - [x_1, x_2]^2 a_3 = 0$$

for all $x_1, x_2 \in R$. Then one of the following holds:

(1) $a_3, a_5, a_6 \in C$.
(2) $a_3, a_4, a_4a_5 + a_6 \in C$.

Proof. We may assume that R does not satisfy standard identity s_4. By Lemma 2.4, R satisfies the non-trivial generalized polynomial identity

$$h(x_1,x_2) = a_1[x_1,x_2]^2 + a_2[x_1,x_2]a_3[x_1,x_2] + a_4[x_1,x_2]a_5[x_1,x_2] \quad (16)$$
$$+[x_1,x_2]a_6[x_1,x_2] - a_4[x_1,x_2]^2 - [x_1,x_2]^2a_3.$$

Since R and U satisfy the same polynomial identity (see Remark 4) therefore equation (16) is also satisfied by U. In case C is infinite, we have $h(x_1,x_2) = 0$ for all $x_1, x_2 \in U \otimes_C \bar{C}$, where $\bar{C}$ is the algebraic closure of C. Since both U and $U \otimes_C \bar{C}$ are prime and centrally closed [9], we may replace R by U or $U \otimes_C \bar{C}$ according to C is finite or infinite. Then R is centrally closed over C and $h(x_1,x_2) = 0$ for all $x_1, x_2 \in R$. By Martindale's theorem, [18], R is then a primitive ring with non-zero socle $soc(R)$ and with C as its associated division ring. Then, by Jacobson's theorem [12, p.75], R is isomorphic to a dense ring of linear transformations of a vector space V over C.

Assume first that V is finite-dimensional over C. Then the density of R on V implies $R \cong M_k(C)$, the ring of all $k \times k$ matrices over C. Since R does not satisfy s_4, we have $k \geq 3$. Thus $M_k(C)$ satisfies the following polynomial identity:

$$a_1[x_1,x_2]^2 + a_2[x_1,x_2]a_3[x_1,x_2] + a_4[x_1,x_2]a_5[x_1,x_2] + [x_1,x_2]a_6[x_1,x_2] \quad (17)$$
$$-a_4[x_1,x_2]^2 - [x_1,x_2]^2a_3 = 0.$$

Now suppose that $\dim_C V$ is infinite. Suppose that $a_3, a_4, a_5, a_6, a_4a_5 + a_6 \notin C$. By Martindale's theorem [18], there exists a non-zero idempotent $e^2 = e \in R$ such that $eRe \cong M_n(C)$ with $n = \dim_C Ve$. Since a_3, a_4, $a_5, a_6, a_4a_5 + a_6 \notin C$, there exist $h_1, h_3, h_2, h_4, h_5 \in soc(R)$ such that $[a_3,h_1] \neq 0$, $[a_4,h_3] \neq 0$, $[a_5,h_2] \neq 0$, $[a_6,h_4] \neq 0$ and $[a_4a_5 + a_6, h_5] \neq 0$. By Litoff's theorem [10], there exists a non-trivial idempotent $e \in soc(R)$ such that $a_3h_1, h_1a_3, a_2h_3, h_3a_2, a_5h_2, h_2a_5 \in eRe$. Since we have

$$e\big(a_1[ex_1e,ex_2e]^2 + a_2[ex_1e,ex_2e]a_3[ex_1e,ex_2e] + a_4[ex_1e,ex_2e]a_5[ex_1e,ex_2e] \quad (18)$$
$$+[ex_1e,ex_2e]a_6[ex_1e,ex_2e] - a_4[ex_1e,ex_2e]^2 - [ex_1e,ex_2e]^2a_3\big)e = 0.$$

Thus the subring eRe satisfies the following equation

$$ea_1e[x_1,x_2]^2 + ea_2e[x_1,x_2]ea_3e[x_1,x_2] + ea_4e[x_1,x_2]ea_5e[x_1,x_2] \quad (19)$$
$$+[x_1,x_2]ea_6e[x_1,x_2] - ea_4e[x_1,x_2]^2 - [x_1,x_2]^2ea_3e = 0.$$

Then by Lemma 2.2, we have either ea_3e, ea_5e, $ea_6e \in C$ or ea_3e,ea_4e, $e(a_4a_5 + a_6)e \in C$. If $ea_3e \in C$ then $a_3h_1 = (ea_3e)h_1 = h_1(ea_3e) = h_1a_3$, which contradict the fact that $a_3 \in C$. Therefore we must have $a_3 \in C$. Thus in the case when ea_3e,

ea_5e, $ea_6e \in C$ we get $a_3, a_5, a_6 \in C$ and when ea_3e, ea_4e, $e(a_4a_5 + a_6)e \in C$ then we get $a_3, a_4, a_4a_5 + a_6 \in C$. □

Proof of Proposition 2.1: We assume that R does not satisfy s_4. Since F, G are generalized inner derivations, there exist $a, b, c, d \in U$ such that $F(x) = ax + xb$ and $G(x) = cx + xd$ for all $x \in R$. From the hypothesis U satisfies the following generalized identity

$$ac[x_1, x_2]^2 + a[x_1, x_2]d[x_1, x_2] + c[x_1, x_2]b[x_1, x_2] + [x_1, x_2]db[x_1, x_2] \qquad (20)$$

$$-c[x_1, x_2]^2 - [x_1, x_2]^2 d = 0$$

for all $x_1, x_2 \in R$. From Lemma 2.5, either $d, b, db \in C$ or $d, c, cb + db \in C$.

Case 1: If $d, db, b \in C$ then equation (20) reduces to following

$$\big((a + b)(c + d) - d - c\big)[x_1, x_2]^2 = 0 \qquad (21)$$

which implies $(a + b)(c + d) = (c + d)$. Thus in this case we get $F(x) = (a + b)x$, $G(x) = (c + d)x$ for all $x \in R$ with $(a + b)(c + d) = (c + d)$, which is our conclusion (2).

Case 2: Again if $c, d, cb + db \in C$ then equation (20) reduces to following

$$\big((a + b)(c + d) - d - c\big)[x_1, x_2]^2 = 0 \qquad (22)$$

which implies $(a + b)(c + d) = (c + d)$. If $c + d = 0$ then $G(x) = (c + d)x = 0$ for all $x \in R$, which is our conclusion (1). If $(c + d) \neq 0$ then we get $F(x) = (a + b)x$, $G(x) = (c + d)x$ for all $x \in R$ with $(a + b)(c + d) = (c + d)$, which is our conclusion (2). □

3. Proof of Theorem 1.2

We may assume that R does not satisfy s_4. If $G = 0$, then we are done. Suppose that $G \neq 0$. In view of [17], we may assume that, for some $a, b \in U$, there exist derivations d and g on U such that $G(x) = ax + d(x)$ and $F(x) = bx + g(x)$, for all $x \in R$. Now since L is not central and $char(R) \neq 2$, there exists a non-zero ideal I of R such that $0 \neq [I, R] \subseteq L$ ([11, p.45], [8, Lemma 2 and Proposition 1], [15, Theorem 4]). Therefore we have $F(G(u))u - G(u^2) = 0$, for all $u \in [I, I]$. Since R and I satisfy the same generalized differential identities, we also have $F(G(u))u - G(u^2) = 0$ for all $u \in [R, R]$. Then by the hypothesis, we have

$$\Big(b\big(a[x_1, x_2] + d([x_1, x_2])\big) + g\big(a[x_1, x_2] + d([x_1, x_2])\big)\Big)[x_1, x_2] \qquad (23)$$

$$= \big(a[x_1, x_2]^2 + d([x_1, x_2]^2)\big).$$

If d and g both are inner derivations then the result follows from Proposition 2.1. So assume that both d and g are not inner derivations. Now we have the following cases.

CASE 1: Let d be an inner derivation and g be an outer derivation. Then for some $q \in U$, $d(x) = [q, x]$ for all $x \in R$, then from equation (23), we have

$$b\Big(a[x_1, x_2] + \big[q, [x_1, x_2]\big]\Big)[x_1, x_2] + g\Big(a[x_1, x_2] + \big[q, [x_1, x_2]\big]\Big)[x_1, x_2] \\ = \Big(a[x_1, x_2]^2 + \big[q, [x_1, x_2]^2\big]\Big). \tag{24}$$

That is

$$b\Big(a[x_1, x_2] + \big[q, [x_1, x_2]\big]\Big)[x_1, x_2] + \Big(g(a+q)[x_1, x_2] \\ +(a+q)\big([g(x_1), x_2] + [x_1, g(x_2)][x_1, x_2]\big)\Big)[x_1, x_2] \\ -\big(([g(x_1), x_2] + [x_1, g(x_2)])q + [x_1, x_2]g(q)\big)[x_1, x_2] \\ = \Big(a[x_1, x_2]^2 + \big[q, [x_1, x_2]^2\big]\Big). \tag{25}$$

Since g is an outer derivation on R, by Kharchenko's theorem (see Remark 6) in equation (25), we obtain

$$b\Big(a[x_1, x_2] + \big[q, [x_1, x_2]\big]\Big)[x_1, x_2] + \Big(g(a+q)[x_1, x_2] \\ +(a+q)\big([y_1, x_2] + [x_1, y_2]\big)\Big)[x_1, x_2] \\ -\Big(([y_1, x_2] + [x_1, y_2])q + [x_1, x_2]g(q)\Big)[x_1, x_2] \\ = \Big(a[x_1, x_2]^2 + \big[q, [x_1, x_2]^2\big]\Big) \tag{26}$$

for all $x_1, x_2, y_1, y_2 \in R$. In particular R satisfies

$$(a+q)\big([y_1, x_2] + [x_1, y_2]\big)[x_1, x_2] - \Big(([y_1, x_2] + [x_1, y_2])q\Big)[x_1, x_2]. \tag{27}$$

It follows from Posner's theorem [20] that there exist a suitable field K and a positive integer t such that R and $M_t(K)$ satisfy the same polynomial identities i.e. $M_t(K)$ also satisfies equation (27). Since R does not satisfies s_4 therefore we must have $t \geq 3$. Choosing $x_1 = y_1 = e_{ij}, x_2 = e_{ji}$ and $y_2 = 0$ in equation (27), we get

$$e_{ij}qe_{ij} = q_{ij}e_{ij} = 0$$

i.e. $q_{ij} = 0$. Thus q is a diagonal matrix. By standard argument we can show that $q \in C$. Hence equation (27) reduces to

$$a\big([y_1, x_2] + [x_1, y_2]\big)[x_1, x_2] = 0.$$

In particular, we have $2a[x_1, x_2]^2 = 0$, which implies $a = 0$. Thus we have $G(x) = 0$, which is a contradiction.

CASE 2: Let g be an inner derivation and d be an outer derivation. Then for some $p \in U$, $g(x) = [p, x]$, for all $x \in R$. Then from equation (23), we have

$$b\Big(a[x_1, x_2] + d\big([x_1, x_2]\big)\Big)[x_1, x_2] + \Big[p, a[x_1, x_2] + d\big([x_1, x_2]\big)\Big][x_1, x_2] \tag{28}$$
$$= \Big(a[x_1, x_2]^2 + d\big([x_1, x_2]^2\big)\Big).$$

We can rewrite equation (28) as

$$b\Big(a[x_1, x_2] + [d(x_1), x_2] + [x_1, d(x_2)]\Big)[x_1, x_2] \tag{29}$$
$$+\Big[p, a[x_1, x_2] + \big([d(x_1), x_2] + [x_1, d(x_2)]\big)\Big][x_1, x_2]$$
$$= \Big(a[x_1, x_2]^2 + \big([d(x_1), x_2] + [x_1, d(x_2)]\big)[x_1, x_2] + [x_1, x_2]\big([d(x_1), x_2] + [x_1, d(x_2)]\big)\Big).$$

Since d is an outer derivation on R, by Kharchenko's theorem (Remark 6) in equation (29), we get

$$b\Big(a[x_1, x_2] + [z_1, x_2] + [x_1, z_2]\Big)[x_1, x_2] \tag{30}$$
$$+\Big[p, a[x_1, x_2] + \big([z_1, x_2] + [x_1, z_2]\big)\Big][x_1, x_2]$$
$$= \Big(a[x_1, x_2]^2 + \big([z_1, x_2] + [x_1, z_2]\big)[x_1, x_2] + [x_1, x_2]\big([z_1, x_2] + [x_1, z_2]\big)\Big)$$

for all $x_1, x_2, z_1, z_2 \in R$. In particular, if we choose $z_1 = z_2 = 0$ then R satisfies the following blended component

$$b\Big([z_1, x_2] + [x_1, z_2]\Big)[x_1, x_2] + \Big[p, \big([z_1, x_2] + [x_1, z_2]\big)\Big][x_1, x_2] \tag{31}$$
$$= \Big(\big([z_1, x_2] + [x_1, z_2]\big)[x_1, x_2] + [x_1, x_2]\big([z_1, x_2] + [x_1, z_2]\big)\Big).$$

Then by Posner's theorem there exist a suitable field K and positive integer t such that $M_t(K)$ and R satisfy equation (31). Since R does not satisfy s_4, we may assume that $t \geq 3$. Now if we choose $z_1 = x_1 = e_{ij}$, $x_2 = e_{jj}$ and $z_2 = 0$ in equation (31) then we get

$$-e_{ij}pe_{ij} = -p_{ij}e_{ij} = 0$$

i.e. $p_{ij} = 0$ which implies p is a diagonal matrix. By standard arguments one can show that p is a central element. Thus equation (31) reduces to

$$b\Big([z_1, x_2] + [x_1, z_2]\Big)[x_1, x_2] \tag{32}$$
$$= \Big(\big([z_1, x_2] + [x_1, z_2]\big)[x_1, x_2] + [x_1, x_2]\big([z_1, x_2] + [x_1, z_2]\big)\Big).$$

Now for $i \neq j$, choosing $x_1 = e_{ij}$, $x_2 = e_{ji}$, $z_1 = 0$ and $z_2 = e_{jh}$, we reach to the following contradiction

$$0 = e_{ih}.$$

Case 3: Now suppose that none of d and g is an inner derivation, then following two subcases arises.

Subcase 1: Assume both d and g are C-independent modulo inner derivations of R. Then from Kharchenko's theorem on d in equation (23), R satisfies the following identity

$$\Big(b\big(a[x_1,x_2]+[z_1,x_2]+[x_1,z_2]\big)+g\big(a[x_1,x_2]+[z_1,x_2]+[x_1,z_2]\big)\Big)[x_1,x_2] \quad (33)$$
$$=\big(a[x_1,x_2]^2+([z_1,x_2]+[x_1,z_2])[x_1,x_2]+[x_1,x_2]([z_1,x_2]+[x_1,z_2])\big).$$

In particular for $z_1=z_2=0$, we obtain

$$\Big(b\big([z_1,x_2]+[x_1,z_2]\big)+g\big([z_1,x_2]+[x_1,z_2]\big)\Big)[x_1,x_2] \quad (34)$$
$$=\big(([z_1,x_2]+[x_1,z_2])[x_1,x_2]+[x_1,x_2]([z_1,x_2]+[x_1,z_2])\big).$$

That is

$$\Big(b\big([z_1,x_2]+[x_1,z_2]\big)+\big([g(z_1),x_2]+[z_1,g(x_2)]+[g(x_1),z_2]+[x_1,g(z_2)]\big)\Big)[x_1,x_2] \quad (35)$$
$$=\big(([z_1,x_2]+[x_1,z_2])[x_1,x_2]+[x_1,x_2]([z_1,x_2]+[x_1,z_2])\big)$$

Again from Kharchenko's theorem on g in equation (35), R satisfies

$$\Big(b\big([z_1,x_2]+[x_1,z_2]\big)+\big([y_1,x_2]+[z_1,w_2]+[w_1,z_2]+[x_1,y_2]\big)\Big)[x_1,x_2] \quad (36)$$
$$=\big(([z_1,x_2]+[x_1,z_2])[x_1,x_2]+[x_1,x_2]([z_1,x_2]+[x_1,z_2])\big).$$

In particular, for $y_1=y_2=0$, R satisfies the blended component $([y_1,x_2]+[x_1,y_2])[x_1,x_2]=0$. Then by Posner's theorem there exist a positive integer $t\geq 3$ and a suitable field K such that $M_t(K)$ satisfies $([y_1,x_2]+[x_1,y_2])[x_1,x_2]=0$. In particular, for $i\neq j$ if we choose $y_1=x_1=e_{ij}$, $x_2=e_{ji}$ and $y_2=0$, we get

$$([e_{ij},e_{ji}]+[e_{ij},0])[e_{ij},e_{ji}]=0$$

i.e. $e_{ii}+e_{jj}=0$, a contradiction.

Subcase 2: Suppose that both d and g are linearly C-dependent modulo inner derivations. Then there exist γ, $\delta\in C$ such that $\gamma d+\delta g=[p_1,x]$ for some $p_1\in U$. Now if $\gamma=0$ then g will be an inner derivation, which is a contradiction. Similarly, if $\delta=0$ then d will be an inner derivation, which is again a contradiction. Thus both δ, γ are non-zero which gives $g=\alpha d(x)+[q_1,x]$, where $0\neq\alpha_1=-\delta^{-1}\gamma$ and $q_1=\delta^{-1}p_1$. Then from equation (23), R satisfies

$$b\big(a[x_1,x_2]+[d(x_1),x_2]+[x_1,d(x_2)]\big)[x_1,x_2]+\alpha_1\big(d(a)[x_1,x_2] \quad (37)$$
$$+a[d(x_1),x_2]+a[x_1,d(x_2)]+[d^2(x_1),x_2]+2[d(x_1),d(x_2)]$$
$$+[x_1,d^2(x_2)]\big)[x_1,x_2]+\big[q_1,a[x_1,x_2]+[d(x_1),x_2]+[x_1,d(x_2)]\big][x_1,x_2]$$

$$= \Big(a[x_1, x_2]^2 + \big([d(x_1), x_2] + [x_1, d(x_2)]\big)[x_1, x_2] + [x_1, x_2]\big([d(x_1), x_2] + [x_1, d(x_2)]\big)\Big).$$

From Kharchenko's theorem in equation (37), R satisfies the following identity

$$\begin{aligned} b\big(a[x_1, x_2] + [z_1, x_2] + [x_1, z_2]\big)[x_1, x_2] + \alpha_1\big(d(a)[x_1, x_2] \\ +a[z_1, x_2] + a[x_1, z_2] + [w_1, x_2] + 2[z_1, z_2] \\ +[x_1, w_2])\big)[x_1, x_2] + \big[q_1, a[x_1, x_2] + [z_1, x_2] + [x_1, z_2]\big][x_1, x_2] \\ = \Big(a[x_1, x_2]^2 + \big([z_1, x_2] + [x_1, z_2]\big)[x_1, x_2] + [x_1, x_2]\big([z_1, x_2] + [x_1, z_2]\big)\Big). \end{aligned} \tag{38}$$

In particular for $w_1 = 0$, R satisfies the following blended component

$$\alpha_1[w_1, x_2][x_1, x_2] = 0.$$

Again by Posner's theorem there exist a suitable field K and a fixed integer $t \geq 3$ such that $M_t(K)$ satisfies $\alpha_1[w_1, x_2][x_1, x_2] = 0$. In particular, for $w_1 = x_1 = e_{ij}$, and $x_2 = e_{ji}$, we get

$$\alpha_1[e_{ij}, e_{ji}][e_{ij}, e_{ji}] = e_{ii} + e_{jj} = 0$$

which is a contradiction. □

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