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IRREDUCIBILITY OF BINOMIALS

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ABSTRACT. In this paper, we prove that the family of binomials $x_1^{a_1} \cdots x_m^{a_m} - y_1^{b_1} \cdots y_n^{b_n}$ with $\gcd(a_1, \ldots, a_m, b_1, \ldots, b_n) = 1$ is irreducible by identifying the connection between the irreducibility of a binomial in $\mathbb{C}[x_1, \ldots, x_m, y_1, \ldots, y_n]$ and $\mathbb{C}(x_2, \ldots, x_m, y_1, \ldots, y_n)[x_1]$. Then we show that the necessary and sufficient conditions for the irreducibility of this family of binomials is equivalent to the existence of a unimodular matrix U_i with integer entries such that $(a_1, \ldots, a_m, b_1, \ldots, b_n)^T = U_i \mathbf{e}_i$ for $i \in \{1, \ldots, m+n\}$, where $\mathbf{e}_i$ is the standard basis vector.

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1. Introduction

A polynomial with coefficients in a given field $\mathbb{F}$ is said to be irreducible if it cannot be factored into non-unit polynomials over $\mathbb{F}$. Polynomial factorization expresses a polynomial with coefficients in $\mathbb{F}$ as the product of irreducible factors with coefficients in $\mathbb{F}$. Polynomial rings $\mathbb{F}[x_1, \ldots, x_n]$ over a field $\mathbb{F}$ are unique factorization domains, that is, every element in $\mathbb{F}[x_1, \ldots, x_n]$ can be factored as a product of irreducible polynomials. Moreover, this decomposition is unique up to multiplication of the factors by invertible constants over the specific given field $\mathbb{F}$.

Polynomial factorization plays a significant role in many problems, in particular, it is a critical step for simplification and solving polynomial equations. There are two types of factorization of polynomials – the conventional polynomial factorization which utilizes symbolic methods to get exact factors of a polynomial, and the approximate polynomial factorization that applies numerical methods to get approximate factors of a polynomial.

Polynomial factorization has been studied for a long time. Many works have been devoted to the study of the approximate polynomial factorization. In the past few decades, dramatic progress has been made on approximate polynomial factorization, and many high efficient algorithms have been proposed [1,3,4,5,6,10, 11,13,14,15,16]. The numerical computation is famous for its high efficiency, but it only gives approximate results. Sometimes, an irreducible polynomial may be identified as a reducible polynomial. This unpleasant phenomenon occurs because when the scalars in the inputs to a symbolic computation are given as flotation point numbers, thus numerical errors and noises are introduced to the computation, and the desired properties of the problem formulations are lost. To address this issue, for multivariate polynomials with complex coefficients that contain numerical noise, new algorithms are proposed to improve the design, implementation and experimental evaluation [12,17]. With these new developments, it is still noted by researchers [17] that "Polynomial factorization in conventional sense is an ill-posed problem due to its discontinuity with respect to coefficient perturbations, making it a challenge for numerical computation using empirical data".

It is important to correctly identify irreducible polynomials. This is particular important in both algebra and geometry. Algebraically, an irreducible polynomial is identified as a prime ideal with good properties. Geometrically, an irreducible multivariate polynomial corresponds to an irreducible hypersurface, while an reducible multivariate polynomial corresponds to a union of irreducible hypersurface in the multi-dimensional space. For example, one common theme in [2] and [8] is to determine the implicit equations for surfaces given as a family of monomial parametrization in projective 3-space. In both papers, the authors devote a significant amount of work to establish the irreducibility of the polynomial obtained by eliminating the parameters. Therefore, to study the algebraic and geometric properties of a polynomial, it is important to know the irreducibility of this polynomial. However, sometimes, it is a daunting task to determine the irreducibility of an arbitrary polynomial.

The goal of this paper is two fold:

1. to study the irreducibility of the family of binomials $x_1^{a_1}\cdots x_m^{a_m} - y_1^{b_1}\cdots y_n^{b_n}$ with $\gcd(a_1,\ldots,a_m,b_1,\ldots,b_n)=1$;
2. to provide an equivalence condition of irreducibility in terms of the multiplication of a unimodular matrix and a standard basis vector.

The irreducibility of this family of binomials is confirmed in [8, Corollary 3.4], where the authors prove this by studying the relationship between the prime ideals in polynomial ring and the prime ideals in the Laurent polynomial ring through the corresponding sublattices. In this paper, we focus on an elementary purely algebraic method to prove this irreducibility. The approach in this paper is to utilize the fundamental result that: *if A is a unique factorization domain with* *quotient field K and f a primitive polynomial of positive degree in $A[X]$, then f is irreducible in $A[X]$ if and only if f is irreducible in $K[X]$.*

This approach is investigated in [2] for the case of the polynomial ring in four variables. To shed a light on this method, we begin in Section 2 by summarizing the strategies used in [2] and outline the key components of their proof. Then, we generalize their result to binomials in the polynomial ring with arbitrary variables (Theorem 2.3). Finally, we formulate our original results in Theorem 2.4 and Corollary 2.5, where we show that the irreducibility condition is equivalent to the existence of a unimodular matrix U_i with integer entries such that $(a_1, \ldots, a_m, b_1, \ldots, b_n)^T = U_i \mathbf{e}_i$ for $i \in \{1, \ldots, m+n\}$ where $\mathbf{e}_i$ is the standard basis vector. We conclude the paper by an illustrative example to flush out the new results.

2. Irreducibility

In this section, we consider the polynomial ring $\mathbb{C}[\mathbf{x}, \mathbf{y}] = \mathbb{C}[x_1, \ldots, x_m, y_1, \ldots, y_n]$ with arbitrary variables. First, we prove the irreducibility of the family of binomials $\mathbf{x}^{\mathbf{a}} - \mathbf{y}^{\mathbf{b}} = x_1^{a_1} \cdots x_m^{a_m} - y_1^{b_1} \cdots y_n^{b_n}$ with $\gcd(\mathbf{a}, \mathbf{b}) = \gcd(a_1, \ldots, a_m, b_1, \ldots, b_n) = 1$. Then we provide an equivalent condition in terms of the multiplication of a unimodular matrix and a standard basis vector.

The approach used to investigate the irreducibility of a binomial over the ring $\mathbb{C}[\mathbf{x}, \mathbf{y}]$ is to analyze its irreducibility over the quotient field using the following well-known result:

Theorem 2.1. ([9, Lemma 6.13, Page 163]) *Let A be a unique factorization domain with quotient field K and f a primitive polynomial of positive degree in $A[X]$. Then f is irreducible in $A[X]$ if and only if f is irreducible in $K[X]$.*

Theorem 2.1 is widely used to determine if a polynomial is irreducible. In [2], the authors apply Theorem 2.1 to prove a family of binomials in four variable is irreducible. We state their result and outline their proof below, for details of the proof please see [2].

Theorem 2.2. ([2, Theorem 4.1]) *If $f = x^a y^b - z^c w^d$ is a polynomial with positive integer exponents a, b, c, d, and $\gcd(a, b, c, d) = 1$, then f is irreducible in $\mathbb{C}[x, y, z, w]$.*

Proof. The key ingredient of the proof by [2] is to show that f is irreducible over $\mathbb{C}(y, z, w)[x]$. First, the authors show the following two lemmas:

Lemma A: For any divisor k of a with $k > 1$, $\sqrt[k]{\dfrac{z^c w^d}{y^b}} \notin \mathbb{C}(y, z, w)$.

Lemma B: Let K be a field that contains a primitive n-th root of unity (so in particular $charK \nmid n$). Let $\gamma \in K$ be an element such that $\sqrt[d]{\gamma} \notin K$ for any divisor d of n with $d > 1$. Then $X^n - \gamma$ is irreducible in $K[X]$.

Setting $\gamma = \dfrac{z^c w^d}{y^b}$, as a consequence of Lemma A, Lemma B, and Theorem 2.1, the authors conclude that $x^a y^b - z^c w^d$ is irreducible over the polynomial ring $\mathbb{C}[x, y, z, w]$. □

Taking advantage of Theorem 2.1 and Lemma B established in [2], we generalize Theorem 2.2 to binomials in the polynomial ring with arbitrary variables as below.

Theorem 2.3. *In the polynomial ring* $\mathbb{C}[\mathbf{x}, \mathbf{y}] = \mathbb{C}[x_1, \ldots, x_m, y_1, \ldots, y_n]$, *if*

$$a_1, \ldots, a_m, b_1, \ldots, b_n \in \mathbb{Z}_{>0}, \quad \textit{and} \quad \gcd(\mathbf{a}, \mathbf{b}) = \gcd(a_1, \ldots, a_m, b_1, \ldots, b_n) = 1,$$

then the family of binomials $\mathbf{x}^{\mathbf{a}} - \mathbf{y}^{\mathbf{b}} = x_1^{a_1} \cdots x_m^{a_m} - y_1^{b_1} \cdots y_n^{b_n}$ *is irreducible.*

Proof. The strategy used in [2] can be applied to prove this generalized result. Replacing Lemma A in the proof of Theorem 2.2 by the following:

Claim A*: For any divisor k of a_1 with $k > 1$,

$$\sqrt[k]{\frac{\mathbf{y}^{\mathbf{b}}}{\hat{\mathbf{x}}^{\hat{\mathbf{a}}}}} = \sqrt[k]{\frac{y_1^{b_1} \cdots y_n^{b_n}}{x_2^{a_2} \cdots x_m^{a_m}}} \notin \mathbb{C}(x_2, \ldots, x_m, y_1, \ldots, y_n) = \mathbb{C}(\hat{\mathbf{x}}, \mathbf{y}).$$

Then, setting $\gamma = \dfrac{\mathbf{y}^{\mathbf{b}}}{\hat{\mathbf{x}}^{\hat{\mathbf{a}}}} \in \mathbb{C}(\hat{\mathbf{x}}, \mathbf{y})$, as a consequence of Claim A*, Lemma B, and Theorem 2.1, we conclude that the family of binomials $\mathbf{x}^{\mathbf{a}} - \mathbf{y}^{\mathbf{b}}$ with $\gcd(\mathbf{a}, \mathbf{b}) = 1$ is irreducible.

The proof of Claim A* can be directly modified from the proof of Lemma A in [2]. For the convenience of the reader, we include the detailed proof for Claim A* below. The proof of Lemma B remains the same since it is independent of the number of variables, and we refer the reader to [2] for a detailed proof.

Proof of Claim A*: Suppose $\sqrt[k]{\dfrac{\mathbf{y}^{\mathbf{b}}}{\hat{\mathbf{x}}^{\hat{\mathbf{a}}}}} = \dfrac{P}{Q} \in \mathbb{C}(\hat{\mathbf{x}}, \mathbf{y})$ where $P, Q \in \mathbb{C}[\hat{\mathbf{x}}, \mathbf{y}]$, and $\gcd(P, Q) = 1$, then

$$\hat{\mathbf{x}}^{\hat{\mathbf{a}}} P^k = \mathbf{y}^{\mathbf{b}} Q^k. \tag{1}$$

Since $\gcd(\mathbf{a}, \mathbf{b}) = 1$, we have that k cannot divide all of $a_2, \ldots, a_m, b_1, \ldots, b_m$.

Suppose $k \nmid a_i$ for some $i \in \{2, \ldots, m\}$, then Equation (1) shows that $x_i | Q$. Let $Q = x_i^e \bar{Q}$ with x_i prime to $\bar{Q}$, then Equation (1) becomes

$$\hat{\mathbf{x}}^{\hat{\mathbf{a}}} P^k = \mathbf{y}^{\mathbf{b}} x_i^{ek} \bar{Q}^k. \tag{2}$$

Equation (2) implies that $ek \geq a_i$, but $ek = a_i$ is impossible since $k \nmid a_i$, hence $ek > a_i$. Therefore we obtain the following equation by cancelling $x_i^{a_i}$

$$x_2^{a_2} \cdots a_{i-1}^{a_{i-1}} x_{i+1}^{a_{i+1}} \cdots x_m^{a_m} P^k = \mathbf{y}^{\mathbf{b}} x_i^{a_i'} \bar{Q}^k, \quad a_i' \geq 1. \tag{3}$$

Equation (3) yields that $x_i | P$, which is a contradiction, since $\gcd(P, Q) = 1$. Therefore $k | a_i$ for all $i \in \{2, \ldots, m\}$.

Suppose $k \nmid b_j$ for some $j \in \{1, \ldots, n\}$, then Equation (1) implies that $y_j | P$; that is, $P = y_j^{e'} \bar{P}$ with y_j prime to $\bar{P}$. Therefore, Equation (1) becomes

$$\hat{\mathbf{x}}^{\hat{\mathbf{a}}} y_j^{e'k} \bar{P}^k = \mathbf{y}^{\mathbf{b}} Q^d. \tag{4}$$

Equation (4) yields that $e'k \geq c$. Since $k \nmid b_j$, we must have $e'k > b_j$. Cancelling $y_j^{b_j}$, we have

$$\hat{\mathbf{x}}^{\hat{\mathbf{a}}} y_j^{b_j'} \bar{P}^k = y_1^{b_1} \cdots y_{j-1}^{b_{j-1}} y_{j+1}^{b_{j+1}} \cdots y_n^{b_n} Q^k, \quad b_j' \geq 1. \tag{5}$$

Equation (5) shows that $y_j | Q$, which again is a contradiction, since $\gcd(P, Q) = 1$. Therefore $k | b_j$ for all $j \in \{1, \ldots, n\}$.

This says that if $\sqrt[k]{\dfrac{\mathbf{y}^{\mathbf{b}}}{\hat{\mathbf{x}}^{\hat{\mathbf{a}}}}} = \dfrac{P}{Q} \in \mathbb{C}(\hat{\mathbf{x}}, \mathbf{y})$, then $\gcd(\mathbf{a}, \mathbf{b}) \neq 1$, contradicting our condition. Therefore, our assumption is false and we complete the proof of our claim. □

Now, we shall provide an equivalent condition of irreducibility in terms of the multiplication of a unimodular matrix and a standard basis vector. We include an illustrative example to flush out the new results.

Theorem 2.4. *Let* $\mathbf{v} = (\mathbf{a}, \mathbf{b})^T = (a_1, \ldots, a_m, b_1, \ldots, b_n)^T$*, and* $\mathbf{w} = (1, 0, \ldots, 0)^T$*. Then the family of binomials* $\mathbf{x}^{\mathbf{a}} - \mathbf{y}^{\mathbf{b}} = x_1^{a_1} \cdots x_m^{a_m} - y_1^{b_1} \cdots y_n^{b_n} \in \mathbb{C}[\mathbf{x}, \mathbf{y}]$ *is irreducible if and only if there exists a unimodular matrix* U *(i.e., a square matrix with integer entries and determinant* ± 1*) with integer entries such that* $\mathbf{w} = U\mathbf{v}$ *(or* $\mathbf{v} = U^{-1}\mathbf{w}$*).*

Proof. ($\Rightarrow$) Suppose $\mathbf{x}^{\mathbf{a}} - \mathbf{y}^{\mathbf{b}} \in \mathbb{C}[\mathbf{x}, \mathbf{y}]$ is irreducible, we will construct a unimodular matrix U such that $\mathbf{w} = U\mathbf{v}$.

First, by Theorem 2.3, $\gcd(\mathbf{v}) = \gcd(\mathbf{a}, \mathbf{b}) = \gcd(a_1, \ldots, a_m, b_1, \ldots, b_n) = 1$, hence there exists a vector $\mathbf{u}_1 = (\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_n) \in \mathbb{Z}^{m+n}$ such that $\mathbf{u}_1 \cdot \mathbf{v} = \sum_{i=1}^m \alpha_i a_i + \sum_{i=1}^n \beta_i b_i = 1$.

Second, by Hilbert Syzygy Theorem [7], there exist $m+n-1$ linearly independent vectors $\mathbf{u}_2, \ldots, \mathbf{u}_{m+n}$ in $\mathbb{Z}^{m+n}$ such that $\mathbf{u}_i \cdot \mathbf{v} = 0$, and $\mathbf{v}$ is the outer product of the vectors $\mathbf{u}_2, \ldots, \mathbf{u}_{m+n}$, that is, the j-th coordinate of the vector $\mathbf{v}$ is the determinant of the $(m+n-1)\times(m+n-1)$ submatrix of $\begin{bmatrix}\mathbf{u}_2\\ \vdots\\ \mathbf{u}_{m+n}\end{bmatrix}$ by deleting j-th column with alternating signs. Note that these $m+n-1$ linearly independent vectors can be selected from the Koszul syzygies of $\mathbf{v}$, and the outer product of the vectors $\mathbf{u}_2,\ldots,\mathbf{u}_{m+n}$ is denoted by $\{\mathbf{u}_2,\ldots,\mathbf{u}_{m+n}\}_o$.

Finally, let $U=\begin{bmatrix}\mathbf{u}_1\\ \mathbf{u}_2\\ \vdots\\ \mathbf{u}_{m+n}\end{bmatrix}$, where the rows of U are the row vectors $\mathbf{u}_1,\ldots,\mathbf{u}_{m+n}$. It is easy to observe that U is an $(m+n)\times(m+n)$ matrix with integer entries, and $\det(U)=1$ since by expanding the first row

$$\det(U)=\det\begin{bmatrix}\mathbf{u}_1\\ \mathbf{u}_2\\ \vdots\\ \mathbf{u}_{m+n}\end{bmatrix}=\mathbf{u}_1\cdot\{\mathbf{u}_2,\ldots,\mathbf{u}_{m+n}\}_o=\mathbf{u}_1\cdot\mathbf{v}=1.$$

Therefore, by definition, U is a unimodular matrix, and $\mathbf{w}=U\mathbf{v}$.

($\Leftarrow$) Suppose there exists a unimodular matrix U with integer entries such that $\mathbf{w}=U\mathbf{v}$, we will prove $\mathbf{x}^{\mathbf{a}}-\mathbf{y}^{\mathbf{b}}\in\mathbb{C}[\mathbf{x},\mathbf{y}]$ is irreducible.

To see this, note that since U is a unimodular matrix, U^{-1} exists and has only integer entries. Hence, $\mathbf{v}=U^{-1}\mathbf{w}$, that is, $\mathbf{v}$ the first column of the matrix U^{-1}. Since $\det(U)=\pm1$, $\gcd(\mathbf{v})=\gcd(\mathbf{a},\mathbf{b})=\gcd(a_1,\ldots,a_m,b_1,\ldots,b_n)=1$. Therefore, by Theorem 2.3, $\mathbf{x}^{\mathbf{a}}-\mathbf{y}^{\mathbf{b}}\in\mathbb{C}[\mathbf{x},\mathbf{y}]$ is irreducible. ☐

The following result is a direct consequence of Theorem 2.4.

Corollary 2.5. $\mathbf{x}^{\mathbf{a}}-\mathbf{y}^{\mathbf{b}}\in\mathbb{C}[\mathbf{x},\mathbf{y}]$ *is irreducible if and only if there exists a unimodular matrix* U_j *with integer entries such that* $\mathbf{e}_j=U_j\mathbf{v}$ *(or* $\mathbf{v}=U_j^{-1}\mathbf{e}_j$*) for* $j\in\{1,\ldots,m+n\}$, *where* $\mathbf{e}_j$ *is the standard basis vector.*

Proof. We observe that the case $j=1$ is exactly Theorem 2.4, and the unimodular matrix U_1 is the matrix U constructed in the proof of Theorem 2.4. For $j\in\{2,\ldots,m+n\}$, the unimodular matrix U_j can be obtained by exchanging the first row and the j-th row of the matrix U_1, and $\det(U_j)=-1$. ☐

We will use the following example to illustrate Theorem 2.3 and Corollary 2.5.

Example 2.6. Consider binomial $x_1^{12}x_2^5 - y_1^2y_2^{15}$. This binomial is irreducible in $\mathbb{C}[x_1, x_2, y_1, y_2]$ by Theorem 2.3, since $\gcd(\mathbf{v}) = \gcd(12, 5, 2, 15) = 1$. Let

$$\mathbf{u}_1 = (-2, 2, 0, 1), \text{ where } \mathbf{u}_1 \cdot \mathbf{v} = (-2, 2, 0, 1) \cdot (12, 5, 2, 15) = -24 + 10 + 15 = 1.$$

Select three linearly independent Koszul syzygies $\mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4$ of $\mathbf{v}$ as below:

$$\mathbf{u}_2 = (1, 0, -6, 0),\ \mathbf{u}_3 = (0, 3, 0, -1),\ \mathbf{u}_4 = (0, 2, -5, 0).$$

Note that $\mathbf{v} = \{\mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}_o$, since

$$\begin{aligned}
\{\mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}_o &= \begin{Bmatrix} 1 & 0 & -6 & 0 \\ 0 & 3 & 0 & -1 \\ 0 & 2 & -5 & 0 \end{Bmatrix}_o \\
= &\left(\det \begin{bmatrix} 0 & -6 & 0 \\ 3 & 0 & -1 \\ 2 & -5 & 0 \end{bmatrix}, -\det \begin{bmatrix} 1 & -6 & 0 \\ 0 & 0 & -1 \\ 0 & -5 & 0 \end{bmatrix}, \det \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & 2 & 0 \end{bmatrix}, -\det \begin{bmatrix} 1 & 0 & -6 \\ 0 & 3 & 0 \\ 0 & 2 & -5 \end{bmatrix} \right) \\
= &\begin{bmatrix} 12 & 5 & 2 & 15 \end{bmatrix}^T = \mathbf{v}.
\end{aligned}$$

Set $U_1 = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 & \mathbf{u}_4 \end{bmatrix}^T$, then $\det(U_1) = 1$, and

$$\mathbf{e}_1 = \begin{bmatrix} -2 & 2 & 0 & 1 \\ 1 & 0 & -6 & 0 \\ 0 & 3 & 0 & -1 \\ 0 & 2 & -5 & 0 \end{bmatrix} \begin{bmatrix} 12 \\ 5 \\ 2 \\ 15 \end{bmatrix} = U_1\mathbf{v};\ \mathbf{v} = U_1^{-1}\mathbf{e}_1 = \begin{bmatrix} 12 & 25 & 12 & -30 \\ 5 & 10 & 5 & -12 \\ 2 & 4 & 2 & -5 \\ 15 & 30 & 14 & -36 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 12 \\ 5 \\ 2 \\ 15 \end{bmatrix}.$$

Similarly,

$$\mathbf{v} = U_i\mathbf{e}_i,\ i = 2, 3, 4, \text{ where } U_2 = \begin{bmatrix} \mathbf{u}_2 \\ \mathbf{u}_1 \\ \mathbf{u}_3 \\ \mathbf{u}_4 \end{bmatrix}, U_3 = \begin{bmatrix} \mathbf{u}_3 \\ \mathbf{u}_2 \\ \mathbf{u}_1 \\ \mathbf{u}_4 \end{bmatrix}, U_4 = \begin{bmatrix} \mathbf{u}_4 \\ \mathbf{u}_2 \\ \mathbf{u}_3 \\ \mathbf{u}_1 \end{bmatrix}.$$

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