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MINIMAL RINGS RELATED TO GENERALIZED QUATERNION RINGS

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Abstract. The family of rings of the form

$$\frac{\mathbb{Z}_4\langle x, y\rangle}{\langle x^2 - a, y^2 - b, yx - xy - 2(c + dx + ey + fxy)\rangle}$$

is investigated which contains the generalized Hamilton quaternions over $\mathbb{Z}_4$. These rings are local rings of order 256. This family has 256 rings contained in 88 distinct isomorphism classes. Of the 88 non-isomorphic rings, 10 are minimal reversible nonsymmetric rings and 21 are minimal abelian reflexive nonsemicommutative rings. Few such examples have been identified in the literature thus far. The computational methods used to identify the isomorphism classes are also highlighted. Finally, some generalized Hamilton quaternion rings over $\mathbb{Z}_{p^s}$ are characterized.

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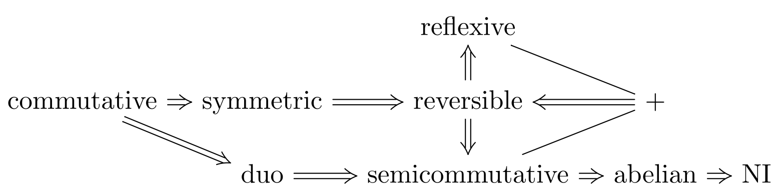
1. Introduction

In recent years there has been much attention on ring properties related to commutativity, namely, symmetric, reversible, semicommutative, reflexive and duo. Cohn introduced reversible rings in [2]. A ring R is *reversible* if for any $a, b \in R$, $ab = 0$ implies $ba = 0$. Similar to reversible rings, Lambek studied symmetric rings in [8]. A ring R is *symmetric* if for any $a, b, c \in R$, $abc = 0$ implies $bac = 0$. Marks in [9], investigated the connection between reversible and symmetric rings with and without identity. It is easy to see that a symmetric ring with identity is reversible, but, Marks gave the example $\mathbb{F}_2Q_8$ of a reversible nonsymmetric ring. Incidentally, he also showed that these two properties are independent when considering rings without identity. In [12], Szabo answered a question posed by Marks which essentially asked "What is the minimal possible order of a reversible nonsymmetric ring (with identity)?"

Reflexive rings were first studied by Mason in [11]. A ring R is *reflexive* if for any $a, b \in R$, $aRb = 0$ implies $bRa = 0$. A similar notion is that of semicommutative rings. A ring R is *semicommutative* if for any $a, b \in R$, $ab = 0$ implies $aRb = 0$. There is a nice connection between reversible, reflexive and semicommutative rings. A ring is reversible if and only if the ring is reflexive and semicommutative. In the recent paper [1], Chimal and Szabo showed that a minimal abelian reflexive nonsemicommutative ring also has order 256. A ring R is *abelian* if each idempotent of R is central. Interestingly enough, an example of such a ring is $\mathbb{F}_2D_8$. Note, Q_8 and D_8 are the only two nonabelian groups of order 8. The mentioned results show that $\mathbb{F}_2D_8$ and $\mathbb{F}_2Q_8$ are not isomorphic. Generalizations of these group rings and their ring properties were given in [3]. A ring R is *right (left) duo* if for all $a, b \in R$, $ba \in aR$ ($ba \in Rb$) (equivalently, every right (left) ideal of R is 2-sided and a ring that is both right and left duo is simply a *duo* ring).

A taxonomy relating the ring properties mentioned thus far which also included 2-primal rings was given in [10]. For finite rings, 2-primal rings and NI rings are equivalent. A ring is *NI* if its set of nilpotent elements form an ideal. The following diagram is taken from [13] showing the various finite ring class containments. These class containments are strict as has been shown in [1,7,9,10,12,13].



In both cases of the minimal rings mentioned so far, only a couple of examples have been given. In this work, using the family of rings of the form

$$\frac{\mathbb{Z}_4\langle x, y\rangle}{\langle x^2 - a, y^2 - b, yx - xy - 2(c + dx + ey + fxy)\rangle}$$

and some computational ingenuity, many examples of these two types of rings will be given. Notice that any element can be represented as $s_1 + s_2x + s_3y + s_4xy$ where $s_1, s_2, s_3, s_4 \in \mathbb{Z}_4$. This allows for the study of these rings computationally. Finding rings with certain properties is generally an arduous task. The rings in this family are 4-dimensional algebras over $\mathbb{Z}_4$, so they have a simple matrix representation as sub-rings of $M_4(\mathbb{Z}_4)$, the main fact that makes them computationally easily treatable.

For a ring R and $a, b \in R$, the *generalized Hamilton quaternion ring of* (a, b) *over* R is defined as

$$\left(\frac{a,b}{R}\right) = \frac{R\langle x,y\rangle}{\langle x^2-a, y^2-b, yx+xy\rangle}.$$

For more on these rings see [4,5,6,14]. The class of generalized Hamilton quaternion rings over $\mathbb{Z}_4$ is contained within the family of rings we are considering. Foreshadowing, we point out that $\left(\frac{3,3}{\mathbb{Z}_4}\right)$ and $\left(\frac{1,1}{\mathbb{Z}_4}\right)$ are minimal reversible nonsymmetric and abelian reflexive nonsemisommutative rings respectively. It turns out that generalized Hamilton quaternion rings of (p, p) over $\mathbb{Z}_{p^s}$ for an odd prime p are also local 4-dimensional algebras over $\mathbb{Z}_{p^s}$ and possess other similar characteristics to the specific case when $s = 2$ and $p = 2$. As with many other results related to primes, there are two main cases, when $p \equiv 1 \pmod 4$ and $p \equiv 3 \pmod 4$. Having a square root of -1 as usual, is the reason for this distinction playing a crucial role in the underlying results.

Throughout, rings are assumed to be associative with identity $1 \neq 0$. Given a ring R, $J(R)$ is the Jacobson radical of R, $U(R)$ is the set of invertible elements of R and $N(R)$ is the set of nilpotent elements of R. In Section 2, the main family of rings will be considered. Each ring in the family will be shown whether or not they posses certain ring properties. Specifically, the properties considered are abelian, commutative, symmetric, reversible, semicommutative and reflexive. In Section 3 many examples of minimal rings from the family are given. Finally, in Section 4 all generalized quaternion algebras over $\mathbb{Z}_4$ are characterized. Furthermore, the generalized Hamilton quaternion rings of (p, p) over $\mathbb{Z}_{p^s}$ for an odd prime p are also characterized.

2. The ring $\frac{\mathbb{Z}_4\langle x,y\rangle}{\langle x^2-a,y^2-b,yx-xy-2(c+dx+ey+fxy)\rangle}$

Let $a_1, a_0, b_1, b_0, c, d, e, f \in \{0, 1\} \subset \mathbb{Z}_4$, $a = 2a_1 + a_0$, $b = 2b_1 + b_0$ and

$$S = \frac{\mathbb{Z}_4\langle x,y\rangle}{\langle x^2-a, y^2-b, yx-xy-2(c+dx+ey+fxy)\rangle}.$$

Then with the substitutions $x = u + a_0$ and $y = v + b_0$ and then collecting terms, we see that the previous ring is isomorphic to

$$R = \frac{\mathbb{Z}_4\langle u,v\rangle}{\langle u^2+2(a_1+a_0u), v^2+2(b_1+b_0v), vu-uv-2(\alpha_1+\beta_1u+\gamma_1v+\delta_1uv)\rangle}$$

where $\alpha_1 = c + fa_0b_0 + da_0 + eb_0 \pmod 2$, $\beta_1 = d + fb_0 \pmod 2$, $\gamma_1 = e + fa_0 \pmod 2$, and $\delta_1 = f$. In this section, we present two propositions which collectively characterize each of the rings in the form of R. First, some common calculations are given to simplify the proofs to come.

It can be shown that R is a $\mathbb{Z}_4$-algebra with basis $\{1, u, v, uv\}$. Also,

$$
\begin{aligned}
u^2 &= -2(a_0u + a_1) = 2(a_0u + a_1), \\
v^2 &= -2(b_0v + b_1) = 2(b_0v + b_1), \\
vu &= uv + 2(\alpha_1 + \beta_1u + \gamma_1v + \delta_1uv), \\
(uv)u = u(vu) &= u(uv + 2(\alpha_1 + \beta_1u + \gamma_1v + \delta_1uv)) \\
&= u^2v + 2(\alpha_1u + \gamma_1uv) \\
&= 2a_0uv + 2a_1v + 2\alpha_1u + 2\gamma_1uv \qquad (2.1) \\
&= 2\alpha_1u + 2a_1v + 2(a_0 + \gamma_1)uv, \\
u(uv) = u^2v &= 2a_1v + 2a_0uv, \\
v(uv) = (vu)v &= 2b_1u + 2\alpha_1v + 2(b_0 + \beta_1)uv, \\
(uv)v = uv^2 &= 2b_1u + 2b_0uv, \text{ and} \\
(uv)^2 &= 2\alpha_1uv.
\end{aligned}
$$

Note, u, v and uv are nilpotent. Therefore, R is local.

In the proofs we will also need the following tedious calculations multiple times. Let $x = x_1 + x_2u + x_3v + x_4uv, y = y_1 + y_2u + y_3v + y_4uv \in R$ where $x_i, y_i \in \mathbb{Z}_4$. Then

$$
\begin{aligned}
xy = {} & x_1y_1 + 2(a_1x_2y_2 + \alpha_1x_3y_2 + b_1x_3y_3) + \\
& (x_2y_1 + x_1y_2 + 2(\alpha_1x_4y_2 + \beta_1x_3y_2 + a_0x_2y_2 + b_1(x_4y_3 + x_3y_4)))u + \\
& (x_3y_1 + x_1y_3 + 2(\alpha_1x_3y_4 + \gamma_1x_3y_2 + b_0x_3y_3 + a_1(x_4y_2 + x_2y_4)))v + \\
& (x_4y_1 + x_1y_4 + x_3y_2 + x_2y_3 + \\
& 2(\delta_1x_3y_2 + (a_0 + \gamma_1)x_4y_2 + b_0x_4y_3 + a_0x_2y_4 + (b_0 + \beta_1)x_3y_4 + \alpha_1x_4y_4))uv.
\end{aligned} \qquad (2.2)
$$

Swapping the roles of x and y we can calculate yx and see that

$$
\begin{aligned}
yx - xy = {} & 2\alpha_1(x_2y_3 - x_3y_2) + \\
& (2\beta_1(x_2y_3 - x_3y_2) + 2\alpha_1(x_2y_4 - x_4y_2))u + \\
& (2\gamma_1(x_2y_3 - x_3y_2) + 2\alpha_1(x_4y_3 - x_3y_4))v + \\
& (2\delta_1(x_2y_3 - x_3y_2) + 2\gamma_1(x_2y_4 - x_4y_2) + 2\beta_1(x_4y_3 - x_3y_4))uv.
\end{aligned} \qquad (2.3)
$$

Before proceeding, a simple result on reflexive rings is given, which simplifies the proof of reflexivity in the coming proposition.

Lemma 2.1. *A ring T is reflexive if and only if for any $a, b \in T$, $aTb = 0$ implies $ba = 0$.*

Proof. Let T be a ring with the property that for any $a, b \in T$, $aTb = 0$ implies $ba = 0$. Let $a, b, r \in T$ such that $aTb = 0$. Then $raTb = 0$. By the assumed property of R, $bra = 0$ showing $bTa = 0$. Hence, T is reflexive. The reverse implication is obvious. □

Proposition 2.2. *Let $a_0, a_1, b_0, b_1, \alpha_1, \beta_1, \gamma_1, \delta_1 \in \{0, 1\} \subset \mathbb{Z}_4$ and*

$$R = \frac{\mathbb{Z}_4 \langle u, v\rangle}{\langle u^2 + 2(a_1 + a_0 u), v^2 + 2(b_1 + b_0 v), vu - uv - 2(1 + \beta_1 u + \gamma_1 v + \delta_1 uv)\rangle}.$$

Then R is local reflexive nonsymmetric of order 256. Furthermore, R is semicommutative (hence reversible) if and only if $a_1 = 1 = b_1$.

Proof. The order and localness were shown in the discussion preceding the propositions. Notice in this proposition, the cases where $\alpha_1 = 1$ are considered. By Equations 2.1,

$$\begin{aligned} u^2 &= 2(a_0 u + a_1),\\ v^2 &= 2(b_0 v + b_1),\\ vu &= uv + 2(1 + \beta_1 u + \gamma_1 v + \delta_1 uv),\\ uvu &= 2u + 2a_1 v + 2(a_0 + \gamma_1)uv,\\ u^2 v &= 2a_1 v + 2a_0 uv,\\ vuv &= 2b_1 u + 2v + 2(b_0 + \beta_1)uv,\\ uv^2 &= 2b_1 u + 2b_0 uv \text{ and}\\ (uv)^2 &= 2uv. \end{aligned} \tag{2.4}$$

Note, u, v and uv are nilpotent and $2vu = 2uv$.

Since $u(uv)v = 0$ but $(uv)uv = 2uv \neq 0$, R is nonsymmetric. If $a_1 = 0$ then $u(u + 2a_0) = 0$ and $uv(u + 2a_0) = uvu + 2a_0 uv = 2u + 2\gamma_1 uv \neq 0$. Hence, R is nonsemicommutative in this case. Similarly, if $b_1 = 0$ then $(v + 2b_0)v = 0$ and $(v + 2b_0)uv = 2v + 2\beta_1 uv \neq 0$ showing R is nonsemicommutative.

Next, we show R is reflexive. Let $x = x_1 + x_2 u + x_3 v + x_4 uv, y = y_1 + y_2 u + y_3 v + y_4 uv \in R$ where $x_i, y_i \in \mathbb{Z}_4$ and assume $xRy = 0$. If $x \in U(R)$ then $y = 0$ and $yx = 0$. Similarly, if $y \in U(R)$, $yx = 0$. So, assume x and y are not units. Then $x_1, y_1 \in \langle 2\rangle$. So, from Equation 2.2, $0 = x2y = 2xy = 2(x_3 y_2 + x_2 y_3)uv$ showing $x_3 y_2 + x_2 y_3 \in \langle 2\rangle$.

By Equation 2.2 with appropriate substitutions,

$$\begin{aligned} xu =& 2(a_1x_2 + x_3) + (x_1 + 2(x_2a_0 + \beta_1x_3 + x_4))u + \\ & 2(\gamma_1x_3 + a_1x_4)v + (x_3 + 2(\delta_1x_3 + a_0x_4 + \gamma_1x_4))uv, \\ xv =& 2x_3b_1 + 2b_1x_4u + (x_1 + 2b_0x_3)v + (x_2 + 2b_0x_4)uv, \\ xuv =& 2b_1x_3u + 2(x_3 + a_1x_2)v + (x_1 + 2(x_2a_0 + b_0x_3 + \beta_1x_3 + x_4))uv. \end{aligned}$$

Now again using Equation 2.2 with appropriate substitutions (remember $x_3y_2 + x_2y_3 \in \langle 2\rangle$),

$$\begin{aligned} 0 = xuy =& (2(a_1x_2 + x_3)y_2 + 2(x_3y_2 + b_1x_3y_3))u + (2(a_1x_2 + x_3)y_3 + 2a_1x_3y_2)v + \\ & (x_3y_1 + 2(a_1x_2 + x_3)y_4 + 2(\gamma_1x_3 + a_1x_4)y_2 + \\ & (x_1 + 2(x_2a_0 + \beta_1x_3 + x_4))y_3 + 2((a_0 + \gamma_1)x_3y_2 + b_0x_3y_3 + x_3y_4))uv \\ =& (2a_1x_2y_2 + 2b_1x_3y_3)u + 2x_3y_3v + \\ & (x_3y_1 + x_1y_3 + 2a_1(x_4y_2 + x_2y_4) + 2(\beta_1 + b_0)x_3y_3 + 2x_4y_3)uv \end{aligned}$$

and similarly

$$\begin{aligned} 0 = xvy =& 2x_2y_2u + (2b_1x_3y_3 + 2a_1x_2y_2)v + \\ & (x_2y_1 + x_1y_2 + 2b_1(x_4y_3 + x_3y_4) + 2(a_0 + \gamma_1)x_2y_2 + 2x_2y_4)uv. \end{aligned}$$

From the previous 2 equations, $x_2y_2, x_3y_3 \in \langle 2\rangle$, so,

$$\begin{aligned} 0 = xuy =& (x_3y_1 + x_1y_3 + 2a_1(x_4y_2 + x_2y_4) + 2x_4y_3)uv \\ 0 = xvy =& (x_2y_1 + x_1y_2 + 2b_1(x_4y_3 + x_3y_4) + 2x_2y_4)uv. \end{aligned}$$

This shows

$$\begin{aligned} x_3y_1 + x_1y_3 + 2a_1(x_4y_2 + x_2y_4) =& 2x_4y_3 \\ x_2y_1 + x_1y_2 + 2b_1(x_4y_3 + x_3y_4) =& 2x_2y_4 \end{aligned} \tag{2.5}$$

Since $0 = xy$, we have that $x_1y_1 + 2(a_1x_2y_2 + x_3y_2 + b_1x_3y_3) = 0$ and since $x_1, y_1, x_2y_2, x_3y_3, x_3y_2 + x_2y_3 \in \langle 2\rangle$, $x_3y_2, x_2y_3 \in \langle 2\rangle$. Then using Equations 2.5,

$$\begin{aligned} 0 = xy = & 2(x_4y_2 + x_2y_4)u + \\ & 2(x_4y_3 + x_3y_4)v + \\ & (x_4y_1 + x_1y_4 + x_3y_2 + x_2y_3 + \\ & 2((a_0 + \gamma_1)x_4y_2 + b_0x_4y_3 + a_0x_2y_4 + (b_0 + \beta_1)x_3y_4 + x_4y_4))uv. \end{aligned} \tag{2.6}$$

This shows

$$\begin{aligned} 2(x_4y_2 + x_2y_4) =& 0 \\ 2(x_4y_3 + x_3y_4) =& 0 \qquad (2.7) \\ (x_4y_1 + x_1y_4 + x_3y_2 + x_2y_3 + 2(\gamma_1 x_4y_2 + \beta_1 x_3y_4 + x_4y_4) =& 0. \end{aligned}$$

Finally, switching the roles of x and y in Equation 2.2 and applying all of the developed relations, $yx = 0$. Hence, by Lemma 2.1, R is reflexive.

Assume $a_1 = b_1 = 1$. We now show R is reversible in this case. Let $x = x_1 + x_2u + x_3v + x_4uv, y = y_1 + y_2u + y_3v + y_4uv \in R$ where $x_i, y_i \in \mathbb{Z}_4$ and assume $xy = 0$. If $x \in U(R)$ the $y = 0$ and if $y \in UR)$ then $x = 0$. In either case, $yx = 0$. So, assume neither is a unit. Then $x_1 \in \langle 2\rangle$ and $y_1 \in \langle 2\rangle$. Then

$$\begin{aligned} 0 = xy \quad = \quad & x_1y_1 + 2(x_2y_2 + x_3y_2 + x_3y_3) + \\ & (x_2y_1 + x_1y_2 + 2(\beta_1 x_3y_2 + a_0x_2y_2 + x_4y_2 + x_4y_3 + x_3y_4))u + \\ & (x_3y_1 + x_1y_3 + 2(\gamma_1 x_3y_2 + b_0x_3y_3 + x_3y_4 + x_4y_2 + x_2y_4))v + \\ & (x_4y_1 + x_1y_4 + x_3y_2 + x_2y_3 + 2(a_0(x_4y_2 + x_2y_4) + b_0(x_4y_3 + x_3y_4) + \\ & \beta_1 x_3y_4 + \gamma_1 x_4y_2 + \delta_1 x_3y_2 + x_4y_4))uv. \end{aligned}$$

First, $2x_2y_2 + 2x_3y_2 + 2x_3y_3 = 0$ showing at least one of x_2, x_3, y_2, or y_3 is in $\langle 2\rangle$. Since $x_1, y_1 \in \langle 2\rangle$, $0 = 2xy = 2(x_3y_2 + x_2y_3)uv$ showing $(x_3y_2 + x_2y_3) \in \langle 2\rangle$. Thus at least two of x_2, x_3, y_2, or y_3 are in $\langle 2\rangle$. If $x_2, y_2 \in (2)$ then

$$2x_2y_2 + 2x_3y_2 + 2x_3y_3 = 2x_3y_3 = 0$$

so, $x_3 \in \langle 2\rangle$ or $y_3 \in \langle 2\rangle$. Similarly, if $x_3, y_3 \in (2)$ then $x_2 \in \langle 2\rangle$ or $y_2 \in \langle 2\rangle$. Assume $x_2, x_3 \in \langle 2\rangle$ but $y_2 \notin \langle 2\rangle$ and $y_3 \notin \langle 2\rangle$. Then

$$\begin{aligned} 0 = xy \quad = \quad & (x_1y_2)u + \\ & (x_1y_3 + 2x_4y_2)v + \\ & (x_4y_1 + x_3y_2 + x_2y_3 + x_1y_4 + \\ & 2(a_0x_4y_2 + \gamma_1 x_4y_2 + b_0x_4y_3 + x_4y_4))uv. \end{aligned}$$

Since $y_2 \notin \langle 2\rangle$ and $y_3 \notin \langle 2\rangle$, $0 = x_1y_2 + x_1y_3 + 2x_4y_2 = x_1(y_2 + y_3) + 2x_4y_2 = 2x_4y_2$ showing $x_4 \in \langle 2\rangle$. This shows in this case $x \in \langle 2\rangle$. Similarly, if $y_2, y_3 \in \langle 2\rangle$ but $x_2 \notin \langle 2\rangle$ and $x_3 \notin \langle 2\rangle$ then $y \in \langle 2\rangle$.

Now, if $x_2, y_2, x_3 \in \langle 2\rangle$ then

$$\begin{aligned} 0 \quad = \quad & x_2y_1 + x_1y_2 + 2(\beta_1 x_3y_2 + a_0x_2y_2 + x_4y_2 + x_4y_3 + x_3y_4) \\ = \quad & 2x_4y_3 \end{aligned}$$

showing $x_2, x_3, y_2, y_3 \in \langle 2 \rangle$ or $x \in \langle 2 \rangle$. Similarly, it can be shown that if three of the four members of $\{x_2, x_3, y_2, y_3\}$ are in $\langle 2 \rangle$ then $x_2, x_3, y_2, y_3 \in \langle 2 \rangle$, $x \in \langle 2 \rangle$ or $y \in \langle 2 \rangle$ and so this is true in any case. Since $x_2, x_3, y_2, y_3 \in \langle 2 \rangle$, $x \in \langle 2 \rangle$ or $y \in \langle 2 \rangle$, by Equation 2.3, we have that $yx = yx - xy = 0$. Hence, R is reversible. □

Proposition 2.3. *Let $a_1, a_0, b_1, b_0, \beta_1, \gamma_1, \delta_1 \in \mathbb{Z}_2$ and*

$$R = \frac{\mathbb{Z}_4 \langle u, v \rangle}{\langle u^2 + 2(a_1 + a_0 u), v^2 + 2(b_1 + b_0 v), vu - uv - 2(\beta_1 u + \gamma_1 v + \delta_1 uv) \rangle}.$$

Then R is local of order 256.

(1) *If $\beta_1 = 0$ and $\gamma_1 = 0$ then R is symmetric. In addition,*
   (a) *if $\delta_1 = 0$ then R is commutative and*
   (b) *if $\delta_1 = 1$ then R is noncommutative.*

(2) *If $\beta_1 \neq 0$ or $\gamma_1 \neq 0$ and $\beta_1 = b_1$ and $\gamma_1 = a_1$ then R is symmetric noncommutative.*

(3) *If $\beta_1 \neq 0$ or $\gamma_1 \neq 0$ and $\beta_1 \neq b_1$ or $\gamma_1 \neq a_1$ then R is nonreflexive. In addition,*
   (a) *if $a_1 = b_1 = 1$ and $a_0 = b_0$ then R is semicommutative and*
   (b) *if $a_1 \neq 1$, $b_1 \neq 1$ or $a_0 \neq b_0$ then R is nonsemicommutative.*

Proof. The order and localness were shown in the discussion preceding the propositions. Notice in this proposition, the cases where $\alpha_1 = 0$ are considered. By Equations 2.1,

$$\begin{aligned}
u^2 &= 2(a_0 u + a_1),\\
v^2 &= 2(b_0 v + b_1),\\
vu &= uv + 2(\beta_1 u + \gamma_1 v + \delta_1 uv),\\
uvu &= 2a_1 v + 2(a_0 + \gamma_1)uv,\\
u^2 v &= 2a_1 v + 2a_0 uv,\\
vuv &= 2b_1 u + 2(b_0 + \beta_1)uv,\\
uv^2 &= 2b_1 u + 2b_0 uv \text{ and}\\
(uv)^2 &= 0.
\end{aligned} \tag{2.8}$$

Note, u, v and uv are nilpotent and $2vu = 2uv$.

For use throughout the proof, let $x = x_1 + x_2 u + x_3 v + x_4 uv, y = y_1 + y_2 u + y_3 v + y_4 uv, z = z_1 + z_2 u + z_3 v + z_4 uv \in R \setminus 0$ where $x_i, y_i, z_i \in \mathbb{Z}_4$.

case $\beta_1 = \gamma_1 = 0$: Assume $\beta_1 = \gamma_1 = 0$. As in the proof of Proposition 2.2, if $xy = 0$ then $x_3 y_2 + x_2 y_3 \in \langle 2 \rangle$ and then Equation 2.3 shows that $yx = yx - xy =$

$2\delta_1(x_2y_3 - x_3y_2)uv = 0$. So, R is reversible. Assume $xyz = 0$. If $z \in U(R)$ then $xy = 0$ and since R is reversible, $yx = 0$ so, $yxz = 0$. So, in addition, assume $z \notin U(R)$. Then $z_1 \in \langle 2\rangle$ and so, $yxz = (yx - xy)z = 2\delta_1(x_2y_3 - x_3y_2)uvz = 0$. Hence, R is symmetric. Clearly here R is commutative if and only if $\delta_1 = 0$.

case $\beta_1 = b_1 = 0$ **and** $\gamma_1 = a_1 = 1$: Assume $\beta_1 = b_1 = 0$ and $\gamma_1 = a_1 = 1$. Assume $xy = 0$. Then from Equation 2.2, considering the constant term, $x_2 \in \langle 2\rangle$ or $y_2 \in \langle 2\rangle$. Considering this in the u-term, both x_2 and y_2 must be divisible by 2. Then using Equation 2.3, $yx = yx - xy = 0$ showing R is reversible. Now, assume instead $xyz = 0$. If $z \in U(R)$ then $xy = 0$ and since R is reversible, $yx = 0$ so, $yxz = 0$. So, in addition, assume $z \notin U(R)$. Then $z_1 \in \langle 2\rangle$ which implies $0 = 2xyz = 2(x_2y_3 + x_3y_2)z = 2(x_2y_3 - x_3y_2)z$ and $2uvz = 0$. So,

$$\begin{aligned} yxz &= (yx - xy)z \\ &= (2\gamma_1(x_2y_3 - x_3y_2)v + 2(\delta_1(x_2y_3 - x_3y_2) + \gamma_1(x_2y_4 - x_4y_2))uv)z \\ &= 0. \end{aligned}$$

Hence, R is symmetric.

case $\beta_1 = b_1 = 1$ **and** $\gamma_1 = a_1 = 0$: Similar to the previous case, if $\beta_1 = b_1 = 1$ and $\gamma_1 = a_1 = 0$ then R is symmetric.

case $\beta_1 = b_1 = 1$ **and** $\gamma_1 = a_1 = 1$: Assume $\beta_1 = b_1 = 1$ and $\gamma_1 = a_1 = 1$. As in the previous cases, we first show that R is reversible. To that end, assume $xy = 0$. Again, since x and y are nonzero, $x_1, y_1 \in \langle 2\rangle$. Equation 2.2 with $xy = 0$ shows that $x_2y_2 + x_3y_3 \in \langle 2\rangle$. Since $2xy = 0$, from Equation 2.2, $x_2y_3 + x_3y_2 \in \langle 2\rangle$. If $x_2, x_3, y_2, y_3 \in U(\mathbb{Z}_4)$ (2 divides none of them) then $yx = (2(y_4 - x_4) + 2(y_4 - x_4))uv = 0$ and if $x_2, x_3, y_2, y_3 \notin U(\mathbb{Z}_4)$ (2 divides all of them) then $yx = 0$. Also, if 2 divides 3 of the four of them then it divides them all. For instance, if $2|x_2$, $2|x_3$ and $2|y_2$ then $0 = x_1y_3 + x_3y_1 + 2x_3y_3b_0 + 2x_3y_2 + 2(x_2y_4 + x_4y_2) = x_1y_3$ showing $2|y_3$. Since $x_2y_2 + x_3y_3, x_2y_3 + x_3y_2 \in \langle 2\rangle$, it cannot be the case that 2 divides exactly one of x_2, x_3, y_2, y_3 nor exactly one of x_2, x_3 and one of y_2, y_3. Without loss of generality we may assume $x_2, x_3 \in \langle 2\rangle$. Then either $y_2, y_3 \in \langle 2\rangle$ or $y_2, y_3 \notin \langle 2\rangle$. Note that in either case $y_2 + y_3 \in \langle 2\rangle$. Fianlly, since $2(x_2y_3 - x_3y_2) = 2(x_2y_3 + x_3y_2) = 0$, from Equation 2.3,

$$\begin{aligned} yx = yx - xy &= (2(x_2y_4 - x_4y_2) + 2(x_3y_4 - x_4y_3))uv \\ &= 2x_4(y_2 + y_3)uv = 0 \end{aligned}$$

showing R is reversible.

Now assume $xyz = 0$. By reversibility, we may assume, x, y and z are not units so, $x_1, y_1, z_1 \in \langle 2 \rangle$. So, $2uvz = 0$ and by Equation 2.3,

$$\begin{aligned} yxz = (yx - xy)z &= 2(x_2y_3 - x_3y_2)(u+v)z \\ &= (x_2y_3 - x_3y_2)2(u+v)(z_2u + z_3v) \\ &= (x_2y_3 - x_3y_2)2(z_2 + z_3)uv. \end{aligned}$$

If $z_2, z_3 \in \langle 2 \rangle$ or $z_2, z_3 \notin \langle 2 \rangle$, $yxz = 0$. Assume $2|z_2$ but $2 \nmid z_3$. Then

$$\begin{aligned} 0 = xyz &= 2(x_2y_2 + x_3y_3)(z_3v + z_4uv) + \\ &\quad (x_1y_2 + x_2y_1 + 2x_2y_2a_0 + 2x_3y_2 + 2x_3y_4 + 2x_4y_3)z_3uv + \\ &\quad (x_3y_2 + x_2y_3)z_1uv + (x_3y_2 + x_2y_3)z_3(2b_0uv + 2u). \end{aligned}$$

So, $(x_2y_3 + x_3y_2)z_3$ is the coefficient of u above showing $(x_2y_3 + x_3y_2)z_3 = 0$. So, in this case $yxz = 0$. Similarly, if $2 \nmid z_2$ but $2|z_3$ then $yxz = 0$. Hence, in any case, $yxz = 0$ showing R is symmetric. Hence, R is symmetric.

case $\beta_1 \neq 0$ **or** $\gamma_1 \neq 0$ **and** $\beta_1 \neq b_1$ **or** $\gamma_1 \neq a_1$: Now, assume $\beta_1 \neq 0$ or $\gamma_1 \neq 0$ and that $\beta_1 \neq b_1$ or $\gamma_1 \neq a_1$.

Assume $a_1 = 0$ and $\gamma_1 = 1$. If $a_0 = 0$ then $(uv)u = 2uv \neq 0$ and $uR(uv) = 0$ and if $a_0 = 1$ then $u(uv) = 2uv \neq 0$ and $(uv)Ru = 0$. Also, $u(u + 2a_0) = 0$ but $uv(u + 2a_0) = 2uv$. Then R is nonreflexive nonsemicommutative in this case. Similarly, $b_1 = 0$ and $\beta_1 = 1$, R is nonreflexive nonsemicommutative.

Now, if $a_1 = 0 = \gamma_1$, since $\beta_1 \neq 0$ or $\gamma_1 \neq 0$ and $\beta_1 \neq b_1$ or $\gamma_1 \neq a_1$, $b_1 = 0$ and $\beta_1 = 1$. Similarly, if $b_1 = 0 = \beta_1$ then $a_1 = 0$ and $\gamma_1 = 1$. So, if $a_1 = 0$ or $b_1 = 0$ then R is nonreflexive nonsemicommutative.

Assume $a_1 = 1 = b_1$. Without loss of generality, we may assume $\beta_1 = 1$ and $\gamma_1 = 0$. Then

$$(u^2 + uv + 2b_0u)R(u+v) = 0$$

but

$$(u+v)(u^2 + uv + 2b_0u) = 2uv \neq 0$$

so, R is nonreflexive in this case. If $a_0 = 0$ and $b_0 = 1$ then

$$\begin{aligned} (2 + 2\delta_1u + (u+v))(u+v) &= 2u + 2v + 2\delta_1uv + (u+v)^2 \\ &= 2u + 2v + 2\delta_1uv + 2 + uv + 2u + uv + 2\delta_1uv + 2v + 2 \\ &= 0 \end{aligned}$$

but $(2 + 2\delta_1u + (u+v))u(u+v) = 2uv \neq 0$, showing R is nonsemicommutative in this case. Similarly, if $a_0 = 1$ and $b_0 = 0$ then R is nonsemicommutative. Finally, we show if $a_0 = b_0$ then R is semicommutative. So, to finish, assume $a_0 = b_0$ (and remember $a_1 = b_1 = \beta_1 = 1$ and $\gamma_1 = 0$).

Assume $xy = 0$. Then

$$\begin{aligned} 0 = xy &= x_1y_1 + 2(x_2y_2 + x_3y_3) + \\ &\quad (x_2y_1 + x_1y_2 + 2(x_3y_2 + a_0x_2y_2 + x_4y_3 + x_3y_4))u + \\ &\quad (x_3y_1 + x_1y_3 + 2(a_0x_3y_3 + x_4y_2 + x_2y_4))v + \\ &\quad (x_4y_1 + x_1y_4 + x_3y_2 + x_2y_3 + \\ &\quad 2(x_3y_2\delta_1 + a_0(x_4y_2 + x_2y_4 + x_4y_3 + x_3y_4) + x_3y_4)uv. \end{aligned}$$

Similar to previous cases, we can deduce that $x_1, y_1, x_2y_3 + x_3y_2, x_2y_2 + x_3y_3 \in \langle 2 \rangle$. If $x_2, x_3, y_2, y_3 \in U(\mathbb{Z}_4)$ (2 divides none of them) then $y_1 + x_1 + 2(1 + a_0 + x_4 + y_4) = 0$ and $y_1 + x_1 + 2(a_0 + x_4 + y_4) = 0$. Adding these equations gives $2 = 0$. So, 2 divides at least one of them. Since $x_2y_3 + x_3y_2, x_2y_2 + x_3y_3 \in \langle 2 \rangle$, it cannot be the case that 2 divides exactly one of x_2, x_3, y_2, y_3 nor exactly one of x_2, x_3 and one of y_2, y_3. If 2 divides 3 of the four of them then it divides them all. For instance, if $2|x_2$, $2|x_3$ and $2|y_2$ then $0 = x_3y_1 + x_1y_3 + 2(a_0x_3y_3 + x_4y_2 + x_2y_4) = x_1y_3$ showing $y_3 \in \langle 2 \rangle$. Without loss of generality we may assume $x_2, x_3 \in \langle 2 \rangle$. Then we can also deduce that $x_1y_2 = 2x_4y_3$ and $x_1y_3 = 2x_4y_2$. Let $r = r_1 + r_2u + r_3v + r_4uv$ where $r_1, r_2, r_3, r_4 \in \mathbb{Z}_4$. Then $xr_1y = r_1xy = 0$, $xr_4uvy = r_4x_1uvy = 0$,

$$\begin{aligned} xr_2uy &= r_2(x_1u + x_3vu + x_4uvu)y \\ &= r_2(x_1u + x_3uv + x_4(2v + 2a_0uv))y \\ &= r_2(x_1u + 2x_4v + (2x_4a_0 + x_3)uv))y \\ &= r_2(2x_4y_2 + x_1y_3)uv = 0 \end{aligned}$$

and

$$\begin{aligned} xr_3vy &= r_3(x_1v + x_2uv + x_4uv^2)y \\ &= r_3(x_1v + x_2uv + x_4(2u + 2a_0uv))y \\ &= r_3(x_1v + 2x_4u + (2x_4a_0 + x_2)uv))y \\ &= r_3(2x_4y_3 + x_1y_2)uv = 0. \end{aligned}$$

So, $xry = 0$. Hence, if $xy = 0$, $xRy = 0$ showing R is semicommutative. □

3. Isomorphism classes

Let

$$R_n = \frac{\mathbb{Z}_4\langle u,v\rangle}{\langle u^2+2(a_1+a_0u), v^2+2(b_1+b_0v), vu-uv-2(\alpha_1+\beta_1u+\gamma_1v+\delta_1uv)\rangle}$$

where

$$n = a_1*2^7 + a_0*2^6 + b_1*2^5 + b_0*2^4 + \alpha_1*2^3 + \beta_1*2^2 + \gamma_1*2 + \delta_1 \in \mathbb{N}$$

and $\mathcal{R} = \{R_n | 0 \le n \le 255\}$. Using Mathematica we have identified 21 non-isomorphic reflexive nonsemicommutative rings in $\mathcal{R}$ and 10 non-isomorphic reversible nonsymmetric rings in $\mathcal{R}$. By Proposition 2.2 it can be seen that for such rings in $\mathcal{R}$, $\alpha_1 = 1$. By Theorem 2 in [1] and Theorem 3.1 in [12] these are minimal rings of their respective types.

Proposition 3.1. *The following rings are a subset of $\mathcal{R}$ which are non-isomorphic representatives of each of the isomorphic classes of reflexive nonsemicommutative rings in $\mathcal{R}$.*

(1) $\frac{\mathbb{Z}_4\langle u,v\rangle}{\langle u^2,v^2,vu-uv-2(1+v)\rangle}$

(2) $\frac{\mathbb{Z}_4\langle u,v\rangle}{\langle u^2,v^2,vu-3uv-2(1+v)\rangle}$

(3) $\frac{\mathbb{Z}_4\langle u,v\rangle}{\langle u^2,v^2+2v,vu-uv-2\rangle}$

(4) $\frac{\mathbb{Z}_4\langle u,v\rangle}{\langle u^2,v^2+2v,vu-3uv-2\rangle}$

(5) $\frac{\mathbb{Z}_4\langle u,v\rangle}{\langle u^2,v^2+2,vu-uv-2(1+v)\rangle}$

(6) $\frac{\mathbb{Z}_4\langle u,v\rangle}{\langle u^2,v^2+2,vu-3uv-2(1+v)\rangle}$

(7) $\frac{\mathbb{Z}_4\langle u,v\rangle}{\langle u^2,v^2+2,vu-uv-2(1+u)\rangle}$

(8) $\frac{\mathbb{Z}_4\langle u,v\rangle}{\langle u^2,v^2+2,vu-3uv-2(1+u)\rangle}$

(9) $\frac{\mathbb{Z}_4\langle u,v\rangle}{\langle u^2,v^2+2(1+v),vu-uv-2\rangle}$

(10) $\frac{\mathbb{Z}_4\langle u,v\rangle}{\langle u^2,v^2+2(1+v),vu-3uv-2\rangle}$

(11) $\frac{\mathbb{Z}_4\langle u,v\rangle}{\langle u^2+2u,v^2+2,vu-uv-2\rangle}$

(12) $\frac{\mathbb{Z}_4\langle u,v\rangle}{\langle u^2+2u,v^2+2,vu-3uv-2\rangle}$

(13) $\frac{\mathbb{Z}_4\langle u,v\rangle}{\langle u^2,v^2,vu-uv-2\rangle}$

(14) $\frac{\mathbb{Z}_4\langle u,v\rangle}{\langle u^2,v^2,vu-3uv-2\rangle} \cong \frac{\mathbb{Z}_4\langle u,v\rangle}{\langle u^2,v^2+2,vu-3uv-2\rangle}$

(15) $\frac{\mathbb{Z}_4\langle u,v\rangle}{\langle u^2,v^2+2,vu-uv-2\rangle}$

(16) $\frac{\mathbb{Z}_4\langle u,v\rangle}{\langle u^2+2u,v^2+2v,vu-uv-2\rangle}$

(17) $\frac{\mathbb{Z}_4\langle u,v\rangle}{\langle u^2+2,v^2,vu-3uv-2(1+u+v)\rangle}$

(18) $\frac{\mathbb{Z}_4\langle u,v\rangle}{\langle u^2,v^2,vu-uv-2(1+u+v)\rangle}$

(19) $\frac{\mathbb{Z}_4\langle u,v\rangle}{\langle u^2,v^2,vu-3uv-2(1+u+v)\rangle}$

(20) $\frac{\mathbb{Z}_4\langle u,v\rangle}{\langle u^2,v^2+2v,vu-uv-2(1+v)\rangle}$

(21) $\frac{\mathbb{Z}_4\langle u,v\rangle}{\langle u^2+2u,v^2+2v,vu-3uv-2\rangle}$

Proposition 3.2. *The following rings are a subset of $\mathcal{R}$ which are non-isomorphic representatives of each of the isomorphic classes of reversible nonsymmetric rings in $\mathcal{R}$.*

(1) $\frac{\mathbb{Z}_4\langle u,v\rangle}{\langle u^2+2,v^2+2,vu-uv-2\rangle}$

(2) $\frac{\mathbb{Z}_4\langle u,v\rangle}{\langle u^2+2,v^2+2,vu-3uv-2\rangle}$

(3) $\frac{\mathbb{Z}_4\langle u,v\rangle}{\langle u^2+2,v^2+2,vu-uv-2(1+v)\rangle}$

(4) $\frac{\mathbb{Z}_4\langle u,v\rangle}{\langle u^2+2,v^2+2,vu-3uv-2(1+v)\rangle}$

(5) $\frac{\mathbb{Z}_4\langle u,v\rangle}{\langle u^2+2,v^2+2(1+v),vu-uv-2\rangle}$

(6) $\frac{\mathbb{Z}_4\langle u,v\rangle}{\langle u^2+2,v^2+2(1+v),vu-3uv-2\rangle}$

(7) $\frac{\mathbb{Z}_4\langle u,v\rangle}{\langle u^2+2(1+u),v^2+2(1+v),vu-3uv-2\rangle}$

(8) $\frac{\mathbb{Z}_4\langle u,v\rangle}{\langle u^2+2(1+u),v^2+2(1+v),vu-uv-2\rangle}$

(9) $\frac{\mathbb{Z}_4\langle u,v\rangle}{\langle u^2+2,v^2+2,vu-3uv-2(1+u+v\rangle}$

(10) $\frac{\mathbb{Z}_4\langle u,v\rangle}{\langle u^2+2,v^2+2,vu-uv-2(1+u+v\rangle}$

Here we list the precise isomorphic classes of $\mathcal{R}$ (along with the representative from the list above). The subscripts are listed as a 2-digit hexadecimal so as to make it easy to read what the precise ring is in the class. This then also shows how closely related the classes are.

- reflexive nonsemicommutative
  (1) $\{R_{0A_h}, R_{0C_h}, R_{1E_h}, R_{4E_h}\}, \frac{\mathbb{Z}_4\langle u,v\rangle}{\langle u^2,v^2,vu-uv-2(1+v)\rangle}$
  (2) $\{R_{0B_h}, R_{0D_h}, R_{1F_h}, R_{4F_h}\}, \frac{\mathbb{Z}_4\langle u,v\rangle}{\langle u^2,v^2,vu-3uv-2(1+v)\rangle}$
  (3) $\{R_{18_h}, R_{48_h}, R_{5A_h}, R_{5C_h}\}, \frac{\mathbb{Z}_4\langle u,v\rangle}{\langle u^2,v^2+2v,vu-uv-2\rangle}$
  (4) $\{R_{19_h}, R_{49_h}, R_{5D_h}, R_{5B_h}\}, \frac{\mathbb{Z}_4\langle u,v\rangle}{\langle u^2,v^2+2v,vu-3uv-2\rangle}$
  (5) $\{R_{2A_h}, R_{8C_h}, R_{3E_h}, R_{CE_h}\}, \frac{\mathbb{Z}_4\langle u,v\rangle}{\langle u^2,v^2+2,vu-uv-2(1+v)\rangle}$
  (6) $\{R_{2B_h}, R_{8D_h}, R_{3F_h}, R_{CF_h}\}, \frac{\mathbb{Z}_4\langle u,v\rangle}{\langle u^2,v^2+2,vu-3uv-2(1+v)\rangle}$
  (7) $\{R_{2C_h}, R_{6E_h}, R_{8A_h}, R_{9E_h}\}, \frac{\mathbb{Z}_4\langle u,v\rangle}{\langle u^2,v^2+2,vu-uv-2(1+u)\rangle}$
  (8) $\{R_{2D_h}, R_{6F_h}, R_{8B_h}, R_{9F_h}\}, \frac{\mathbb{Z}_4\langle u,v\rangle}{\langle u^2,v^2+2,vu-3uv-2(1+u)\rangle}$
  (9) $\{R_{38_h}, R_{7A_h}, R_{C8_h}, R_{DC_h}\}, \frac{\mathbb{Z}_4\langle u,v\rangle}{\langle u^2,v^2+2(1+v),vu-uv-2\rangle}$
  (10) $\{R_{39_h}, R_{7B_h}, R_{C9_h}, R_{DD_h}\}, \frac{\mathbb{Z}_4\langle u,v\rangle}{\langle u^2,v^2+2(1+v),vu-3uv-2\rangle}$
  (11) $\{R_{68_h}, R_{7C_h}, R_{98_h}, R_{DA_h}\}, \frac{\mathbb{Z}_4\langle u,v\rangle}{\langle u^2+2u,v^2+2,vu-uv-2\rangle}$
  (12) $\{R_{69_h}, R_{7D_h}, R_{99_h}, R_{DB_h}\}, \frac{\mathbb{Z}_4\langle u,v\rangle}{\langle u^2+2u,v^2+2,vu-3uv-2\rangle}$
  (13) $\{R_{08_h}, R_{1C_h}, R_{4A_h}, R_{5E_h}\}, \frac{\mathbb{Z}_4\langle u,v\rangle}{\langle u^2,v^2,vu-uv-2\rangle}$
  (14) $\{R_{09_h}, R_{1D_h}, R_{4B_h}, R_{5F_h},$
  $R_{29_h}, R_{3D_h}, R_{6B_h}, R_{7F_h}, R_{89_h}, R_{9D_h}, R_{CB_h}, R_{DF_h}\}, \frac{\mathbb{Z}_4\langle u,v\rangle}{\langle u^2,v^2,vu-3uv-2\rangle} \cong \frac{\mathbb{Z}_4\langle u,v\rangle}{\langle u^2,v^2+2,vu-3uv-2\rangle}$
  (15) $\{R_{28_h}, R_{3C_h}, R_{6A_h}, R_{7E_h}, R_{88_h}, R_{9C_h}, R_{CA_h}, R_{DE_h}\}, \frac{\mathbb{Z}_4\langle u,v\rangle}{\langle u^2,v^2+2,vu-uv-2\rangle}$
  (16) $\{R_{58_h}, R_{6C_h}, R_{9A_h}\}, \frac{\mathbb{Z}_4\langle u,v\rangle}{\langle u^2+2u,v^2+2v,vu-uv-2\rangle}$
  (17) $\{R_{1B_h}, R_{6D_h}, R_{9B_h},$
  $R_{4D_h}, R_{2F_h}, R_{8F_h}\}, \frac{\mathbb{Z}_4\langle u,v\rangle}{\langle u^2+2,v^2,vu-3uv-2(1+u+v)\rangle}$
  (18) $\{R_{0E_h}, R_{2E_h}, R_{8E_h}\}, \frac{\mathbb{Z}_4\langle u,v\rangle}{\langle u^2,v^2,vu-uv-2(1+u+v)\rangle}$
  (19) $\{R_{0F_h}, R_{3B_h}, R_{CD_h}\}, \frac{\mathbb{Z}_4\langle u,v\rangle}{\langle u^2,v^2,vu-3uv-2(1+u+v)\rangle}$
  (20) $\{R_{1A_h}, R_{3A_h}, R_{CC_h},$
  $R_{4C_h}, R_{78_h}, R_{D8_h}\}, \frac{\mathbb{Z}_4\langle u,v\rangle}{\langle u^2,v^2+2v,vu-uv-2(1+v)\rangle}$
  (21) $\{R_{59_h}, R_{79_h}, R_{D9_h}\}, \frac{\mathbb{Z}_4\langle u,v\rangle}{\langle u^2+2u,v^2+2v,vu-3uv-2\rangle}$

- reversible nonsymmetric
  (1) $\{R_{A8_h}, R_{BC_h}, R_{EA_h}, R_{FE_h}\}, \frac{\mathbb{Z}_4\langle u,v\rangle}{\langle u^2+2,v^2+2,vu-uv-2\rangle}$
  (2) $\{R_{A9_h}, R_{BD_h}, R_{EB_h}, R_{FF_h}\}, \frac{\mathbb{Z}_4\langle u,v\rangle}{\langle u^2+2,v^2+2,vu-3uv-2\rangle}$
  (3) $\{R_{AA_h}, R_{AC_h}, R_{BE_h}, R_{EE_h}\}, \frac{\mathbb{Z}_4\langle u,v\rangle}{\langle u^2+2,v^2+2,vu-uv-2(1+v)\rangle}$
  (4) $\{R_{AB_h}, R_{AD_h}, R_{BF_h}, R_{EF_h}\}, \frac{\mathbb{Z}_4\langle u,v\rangle}{\langle u^2+2,v^2+2,vu-3uv-2(1+v)\rangle}$
  (5) $\{R_{B8_h}, R_{E8_h}, R_{FA_h}, R_{FC_h}\}, \frac{\mathbb{Z}_4\langle u,v\rangle}{\langle u^2+2,v^2+2(1+v),vu-uv-2\rangle}$
  (6) $\{R_{B9_h}, R_{E9_h}, R_{FB_h}, R_{FD_h}\}, \frac{\mathbb{Z}_4\langle u,v\rangle}{\langle u^2+2,v^2+2(1+v),vu-3uv-2\rangle}$
  (7) $\{R_{F9_h}\}, \frac{\mathbb{Z}_4\langle u,v\rangle}{\langle u^2+2(1+u),v^2+2(1+v),vu-3uv-2\rangle}$
  (8) $\{R_{F8_h}, R_{BA_h}, R_{EC_h}\}, \frac{\mathbb{Z}_4\langle u,v\rangle}{\langle u^2+2(1+u),v^2+2(1+v),vu-uv-2\rangle}$
  (9) $\{R_{AF_h}, R_{BB_h}, R_{ED_h}\}, \frac{\mathbb{Z}_4\langle u,v\rangle}{\langle u^2+2,v^2+2,vu-3uv-2(1+u+v)\rangle}$
  (10) $\{R_{AE_h}\}, \frac{\mathbb{Z}_4\langle u,v\rangle}{\langle u^2+2,v^2+2,vu-uv-2(1+u+v)\rangle}$

4. Generalized Hamilton quaternions over a ring $\mathbb{Z}_{p^s}$

In this section we will consider generalized Hamilton quaternion rings which we define next.

Definition 4.1. For a ring R and $a, b \in R$, the generalized Hamilton quaternion ring of (a, b) over R is defined as

$$\left(\frac{a,b}{R}\right) = \frac{R\langle x,y\rangle}{\langle x^2-a, y^2-b, yx+xy\rangle}.$$

We first consider the set of generalized Hamilton quaternion rings over $\mathbb{Z}_4$. This is a subset of the family of rings considered in Section 2. Taking $a, b \in \mathbb{Z}_4$ there are 16 rings. Using Mathematica, the isomorphism classes have been determined while using Propositions 2.2 and 2.3 the ring types can be deduced. These results are collected here.

Proposition 4.2. *A generalized Hamilton quaternion ring over $\mathbb{Z}_4$ is local and there are 16 such rings contained in 7 distinct isomorphism classes. The classes along with their ring type are*

(1) $\left\{\left(\frac{0,0}{\mathbb{Z}_4}\right)\right\}$, *symmetric*
(2) $\left\{\left(\frac{0,3}{\mathbb{Z}_4}\right), \left(\frac{3,0}{\mathbb{Z}_4}\right)\right\}$, *symmetric*
(3) $\left\{\left(\frac{2,2}{\mathbb{Z}_4}\right), \left(\frac{0,2}{\mathbb{Z}_4}\right), \left(\frac{2,0}{\mathbb{Z}_4}\right)\right\}$, *symmetric*
(4) $\left\{\left(\frac{3,3}{\mathbb{Z}_4}\right)\right\}$, *reversible nonsymmetric*
(5) $\left\{\left(\frac{1,1}{\mathbb{Z}_4}\right), \left(\frac{1,3}{\mathbb{Z}_4}\right), \left(\frac{3,1}{\mathbb{Z}_4}\right)\right\}$, *reflexive nonsemicommutative*
(6) $\left\{\left(\frac{0,1}{\mathbb{Z}_4}\right), \left(\frac{1,0}{\mathbb{Z}_4}\right)\right\}$, *nonreflexive nonsemicommutative*
(7) $\left\{\left(\frac{1,2}{\mathbb{Z}_4}\right), \left(\frac{2,1}{\mathbb{Z}_4}\right), \left(\frac{2,3}{\mathbb{Z}_4}\right), \left(\frac{3,2}{\mathbb{Z}_4}\right)\right\}$, *nonreflexive nonsemicommutative.*

Proof. With $a = 2a_1 + a_0$ and $b = 2b_1 + b_0$ where $a_0, a_1, b_0, b_1 \in \{0, 1\} \subset \mathbb{Z}_4$, from Section 2, we can deduce that

$$\left(\frac{a,b}{R}\right) = \frac{R\langle x,y\rangle}{\langle x^2-a, y^2-b, yx+xy\rangle}$$
$$= \frac{R\langle u,v\rangle}{\langle u^2+2(a_1+a_0u), v^2+2(b_1+b_0v), vu-uv-2(a_0b_0+b_0u+a_0v+uv)\rangle}. \tag{4.1}$$

Then the ring types can be determined by Propositions 2.2 and 2.3. □

Next, we determine the ring type of the generalized Hamilton quaternion ring of (p,p) over $\mathbb{Z}_{p^s}$ i.e. $\left(\frac{p,p}{\mathbb{Z}_{p^s}}\right)$ for any $s \in \mathbb{N}$ and odd prime p.

Definition 4.3. For $w \in \left(\frac{p,p}{\mathbb{Z}}\right)$ s.t. $w = a_0 + a_1x + a_2y + a_3xy$ where $a_i \in \mathbb{Z}$ define

$$\|w\| = a_0^2 - pa_1^2 - pa_2^2 + p^2a_3^2.$$

The next two results are well-known.

Lemma 4.4. *Let $w, z \in \left(\frac{a,b}{\mathbb{Z}}\right)$. Then $\|wz\| = \|w\| \cdot \|z\|$.*

Lemma 4.5. *Let p be an odd prime. Then the only solutions of the equation $x^2 + y^2 \equiv 0 \pmod p$ are $x, y \equiv 0 \pmod p$ if and only if $p \equiv 3 \pmod 4$.*

The next five lemmas are the keys to the main results which immediately follow them.

Lemma 4.6. *Let p be an odd prime and $w = a_0 + a_1x + a_2y + a_3xy \in \left(\frac{p,p}{\mathbb{Z}}\right)$. If $\|w\| \equiv 0 \pmod{p^2}$ then $p|a_0$ and $p|a_1^2 + a_2^2$.*

Proof. Assume $\|w\| \equiv 0 \pmod{p^2}$. Then $a_0^2 - p(a_1^2 + a_2^2) \equiv \|w\| \equiv 0 \pmod{p^2}$ which implies $p|a_0$ and $p|a_1^2 + a_2^2$. □

Lemma 4.7. *Let p be an odd prime s.t. $p \equiv 3 \pmod 4$ and $w = a_0 + a_1x + a_2y + a_3xy \in \left(\frac{p,p}{\mathbb{Z}}\right)$.*

(1) *If $\|w\| \equiv 0 \pmod{p^2}$ then $a_0 \equiv a_1 \equiv a_2 \equiv 0 \pmod p$.*
(2) *If $\|w\| \equiv 0 \pmod{p^3}$ then $p|w$.*

Proof. First, assume $\|w\| \equiv 0 \pmod{p^2}$. By Lemma 4.6, $p|a_0$ and $p|a_1^2 + a_2^2$. Since $p \equiv 3 \pmod 4$, by Lemma 4.5, $p|a_1$ and $p|a_2$.

Next, assume $\|w\| \equiv 0 \pmod{p^3}$. This implies $\|w\| \equiv 0 \pmod{p^2}$, so from above, $p|a_0$, $p|a_1$ and $p|a_2$. Remember, $\|w\| = a_0^2 - p(a_1^2 + a_2^2) + p^2a_3$. Since $p|a_0$, there exists $a_0' \in \mathbb{Z}$ s.t. $a_0 = pa_0'$. Then $p|a_0'^2 + a_3^2$. Since $p \equiv 3 \pmod 4$, by Lemma 4.5, $p|a_3$. □

Lemma 4.8. *Let p be an odd prime and $w, z \in \left(\frac{0,0}{\mathbb{Z}}\right)$. Then $wz \equiv 0 \pmod{p}$ if and only if $zw \equiv 0 \pmod{p}$.*

Proof. Can be verified by direct computation. □

Lemma 4.9. *Let p be an odd prime s.t. $p \equiv 3 \pmod{4}$, $s \geq 2$ and $w, z \in \left(\frac{p,p}{\mathbb{Z}}\right)$. Then $wz \equiv 0 \pmod{p^s}$ if and only if $zw \equiv 0 \pmod{p^s}$.*

Proof. First, $w = \alpha_1 + \beta_1 x + \gamma_1 y + \delta_1 xy$ and $z = \alpha_2 + \beta_2 x + \gamma_2 y + \delta_2 xy$ for some $\alpha_1, \beta_1, \gamma_1, \delta_1, \alpha_2, \beta_2, \gamma_2, \delta_2 \in \mathbb{Z}$. Assume $wz \equiv 0 \pmod{p^s}$. It should be clear that $p|\alpha_1$ and $p|\alpha_2$. Note, for $a \in \left(\frac{p,p}{\mathbb{Z}}\right)$, if $a \equiv 0 \pmod{p^s}$ then $\|a\| \equiv 0 \pmod{p^{2s}}$. By assumption $wz \equiv 0 \pmod{p^s}$, so by Lemma 4.4, $\|w\| \equiv 0 \pmod{p^3}$ or $\|z\| \equiv 0 \pmod{p^3}$ or $s = 2$, $\|w\| \equiv 0 \pmod{p^2}$ and $\|z\| \equiv 0 \pmod{p^2}$.

Then by Lemma 4.7, $p|w$, $p|z$ or we have that $s = 2$ and $p|\beta_1$, $p|\beta_2$, $p|\gamma_1$ and $p|\gamma_2$. We proceed by induction. Assume $s = 2$. Assume $p|w$ so, $w = pw'$ for some w'. Then $w'z \equiv 0 \pmod{p}$. By Lemma 4.8, $zw' \equiv 0 \pmod{p}$ showing $zw \equiv 0 \pmod{p^2}$. Similarly, if $p|z$, the result holds. Now assume $p|\beta_1$, $p|\beta_2$, $p|\gamma_1$ and $p|\gamma_2$. Then

$$
\begin{aligned}
0 \equiv & wz \\
\equiv & (\alpha_1 + \beta_1 x + \gamma_1 y + \delta_1 xy)(\alpha_2 x + \beta_2 x + \gamma_2 xy + \delta_2 xy) \\
\equiv & (\alpha_1 \delta_2 + \delta_1 \alpha_2) xy \pmod{p^2},
\end{aligned}
$$

showing $\alpha_1 \delta_2 + \delta_1 \alpha_2 \equiv 0 \pmod{p^2}$ and so,

$$
\begin{aligned}
zw \equiv & (\alpha_2 + \beta_2 x + \gamma_2 y + \delta_2 xy)(\alpha_1 x + \beta_1 x + \gamma_1 xy + \delta_1 xy) \\
\equiv & (\alpha_1 \delta_2 + \delta_1 \alpha_2) xy \\
\equiv & 0 \pmod{p^2}.
\end{aligned}
$$

Now, assume $s > 2$ and for $2 \geq k < s$, $ab \equiv 0 \pmod{p^k}$ if and only if $ba \equiv 0 \pmod{p^k}$ for $a, b \in \left(\frac{p,p}{\mathbb{Z}}\right)$. Assume $w = pw'$ for some $w' \in \mathbb{Z}$ from which we have $w'z \equiv 0 \pmod{p^{s-1}}$. By the assumption then $zw' \equiv 0 \pmod{p^{s-1}}$. Finally, $zw = z(pw') = (pw')z = wz \equiv 0 \pmod{p^s}$. Similarly, if $p|z$ the result holds. □

Lemma 4.10. *Let p be an odd prime, $s \geq 2$ and $w, z \in \left(\frac{p,p}{\mathbb{Z}}\right)$. If for any $r \in \left(\frac{p,p}{\mathbb{Z}}\right)$ $wrz \equiv 0 \pmod{p^s}$ then $zw \equiv 0 \pmod{p^s}$.*

Proof. First, $w = \alpha_1 + \beta_1 x + \gamma_1 y + \delta_1 xy$ and $z = \alpha_2 + \beta_2 x + \gamma_2 y + \delta_2 xy$ for some $\alpha_1, \beta_1, \gamma_1, \delta_1, \alpha_2, \beta_2, \gamma_2, \delta_2 \in \mathbb{Z}$. Assume for any $r \in \left(\frac{p,p}{\mathbb{Z}}\right)$ $wrz \equiv 0 \pmod{p^s}$. If $w \equiv 0 \pmod{p^s}$ or $z \equiv 0 \pmod{p^s}$ then the result holds so we assume this is not the case. Since $s > 1$, it should be clear that $p|\alpha_1$ and $p|\alpha_2$. Now,

$$\begin{aligned}
0 \equiv & wz \\
\equiv & (\alpha_1 + \beta_1 x + \gamma_1 y + \delta_1 xy)(\alpha_2 + \beta_2 x + \gamma_2 y + \delta_2 xy) \\
\equiv & (\alpha_1\alpha_2 + \beta_1\beta_2 p + \gamma_1\gamma_2 p - \delta_1\delta_2 p^2) + \\
& ((\alpha_1\beta_2 + \beta_1\alpha_2) + (-\gamma_1\delta_2 + \delta_1\gamma_2)p)x + \\
& ((\alpha_1\gamma_2 + \gamma_1\alpha_2) + (-\delta_1\beta_2 + \beta_1\delta_2)p)y + \\
& ((\alpha_1\delta_2 + \delta_1\alpha_2) + (-\gamma_1\beta_2 + \beta_1\gamma_2))xy \pmod{p^s}
\end{aligned} \tag{4.2}$$

$$\begin{aligned}
0 \equiv & wxz \\
\equiv & (\alpha_1 + \beta_1 x + \gamma_1 y + \delta_1 xy)(\alpha_2 x + \beta_2 p + \gamma_2 xy + \delta_2 py) \\
\equiv & (\alpha_1\alpha_2 + \beta_1\beta_2 p - \gamma_1\gamma_2 p + \delta_1\delta_2 p^2)x + \\
& ((\alpha_1\beta_2 + \beta_1\alpha_2)p + (\gamma_1\delta_2 - \delta_1\gamma_2)p^2) + \\
& ((\alpha_1\gamma_2 - \gamma_1\alpha_2) + (\delta_1\beta_2 + \beta_1\delta_2)p)xy + \\
& ((\alpha_1\delta_2 - \delta_1\alpha_2)p + (\gamma_1\beta_2 + \beta_1\gamma_2)p)y \pmod{p^s}
\end{aligned} \tag{4.3}$$

$$\begin{aligned}
0 \equiv & wyz \\
\equiv & (\alpha_1 y + \beta_1 xy + \gamma_1 p + \delta_1 px)(\alpha_2 + \beta_2 x + \gamma_2 y + \delta_2 xy) \\
\equiv & (\alpha_1\alpha_2 - \beta_1\beta_2 p + \gamma_1\gamma_2 p + \delta_1\delta_2 p^2)y + \\
& ((-\alpha_1\beta_2 + \beta_1\alpha_2) + (\gamma_1\delta_2 + \delta_1\gamma_2)p)xy + \\
& ((\alpha_1\gamma_2 + \gamma_1\alpha_2)p + (\delta_1\beta_2 - \beta_1\delta_2)p^2) + \\
& ((-\alpha_1\delta_2 + \delta_1\alpha_2)p + (\gamma_1\beta_2 + \beta_1\gamma_2)p)x \pmod{p^s}
\end{aligned} \tag{4.4}$$

$$\begin{aligned}
0 \equiv & wxyz \\
\equiv & (\alpha_1 xy + \beta_1 py - \gamma_1 px + \delta_1 p^2)(\alpha_2 + \beta_2 x + \gamma_2 y + \delta_2 xy) \\
\equiv & (\alpha_1\alpha_2 - \beta_1\beta_2 p - \gamma_1\gamma_2 p + \delta_1\delta_2 p^2)xy + \\
& ((-\alpha_1\beta_2 + \beta_1\alpha_2)p + (-\gamma_1\delta_2 + \delta_1\gamma_2)p^2)y + \\
& ((\alpha_1\gamma_2 - \gamma_1\alpha_2)p + (\delta_1\beta_2 - \beta_1\delta_2)p^2)x + \\
& ((-\alpha_1\delta_2 + \delta_1\alpha_2)p^2 + (-\gamma_1\beta_2 + \beta_1\gamma_2)p^2) \pmod{p^s}
\end{aligned} \tag{4.5}$$

Using sums and differences of 4.2, 4.3, 4.4 and 4.5 we can establish the following:

$$\begin{aligned} \beta_1\beta_2 p &\equiv 0 \pmod{p^s} \\ \gamma_1\gamma_2 p &\equiv 0 \pmod{p^s} \\ \delta_1\delta_2 p^2 &\equiv 0 \pmod{p^s} \\ \beta_1\alpha_2 + \delta_1\gamma_2 p &\equiv 0 \pmod{p^s} \\ \alpha_1\beta_2 + \gamma_1\delta_2 p &\equiv 0 \pmod{p^s} \\ \alpha_1\gamma_2 + \beta_1\delta_2 p &\equiv 0 \pmod{p^s} \\ \gamma_1\alpha_2 + \delta_1\beta_2 p &\equiv 0 \pmod{p^s} \end{aligned} \tag{4.6}$$

Note, for $a \in \left(\frac{p,p}{\mathbb{Z}}\right)$, if $a \equiv 0 \pmod{p^s}$ then $\|a\| \equiv 0 \pmod{p^{2s}}$. By assumption $wz \equiv 0 \pmod{p^s}$, so by Lemma 4.4, $\|w\| \equiv 0 \pmod{p^2}$ or $\|z\| \equiv 0 \pmod{p^2}$. By (4.6), $p|\beta_1$ or $p|\beta_2$ and $p|\gamma_1$ or $p|\gamma_2$ and if $s > 2$, $p|\delta_1$ or $p|\delta_2$.

(1) If $p \nmid \beta_2$ and $p \nmid \gamma_2$ then by (4.6), $p|\beta_1$, $p|\gamma_1$ and $p|\delta_1$. Hence, $p|w$.
(2) If $p \nmid \beta_1$ and $p \nmid \gamma_1$ then by (4.6), $p|\beta_2$, $p|\gamma_2$ and $p|\delta_2$. Hence, $p|z$.
(3) Since $\|w\| \equiv 0 \pmod{p^2}$ or $\|z\| \equiv 0 \pmod{p^2}$,
 (a) If $p|\beta_1$ and $p|\beta_2$ then $p|\gamma_1$ or $p|\gamma_2$ by Lemma 4.6.
 (b) If $p|\beta_1$ and $p|\gamma_2$ then $p|\beta_2$ or $p|\gamma_1$ by Lemma 4.6.
 (c) if $p|\beta_2$ and $p|\gamma_1$ then $p|\gamma_2$ or $p|\beta_2$ by Lemma 4.6.
 (d) If $p|\gamma_1$ and $p|\gamma_2$ then $p|\beta_1$ or $p|\beta_2$ by Lemma 4.6.
 (e) If p divides exactly 3 elements of $\{\beta_1, \beta_2, \gamma_1, \gamma_2\}$, then by (4.6) $p|w$ or $p|z$.
 (f) If p divides all 4 elements of $\{\beta_1, \beta_2, \gamma_1, \gamma_2\}$ and $s > 2$, then by (4.6) $p|w$ or $p|z$.

In any case, $p|w$, $p|z$ or we have that $s = 2$ and $p|\beta_1$, $p|\beta_2$, $p|\gamma_1$ and $p|\gamma_2$. From here the same inductive argument can be used as in the proof of Lemma 4.9. □

Proposition 4.11. *Let p be an odd prime and*

$$R = \frac{\mathbb{Z}_p \langle x, y\rangle}{\langle x^2, y^2, yx + xy\rangle}.$$

Then R is symmetric.

Proof. Since $xyx = -x^2y = 0$ and $yxy = -y^2x = 0$, we see that $J(R)^3 = 0$. Since R is local, by Proposition 2.4 in [12] we need only show R is reversible. To that end, let $a, b \in R \setminus 0$ and assume $ab = 0$. Since R is local, a nor b is a unit. So, $a = \beta_1 x + \gamma_1 y + \delta_1 xy$ and $b = \beta_2 x + \gamma_2 y + \delta_2 xy$ for $\beta_1, \gamma_1, \delta_1, \beta_2, \gamma_2, \delta_2 \in \mathbb{Z}_p$. Then $0 = ab = (\beta_1\gamma_2 - \gamma_1\beta_2)xy$ and so $ba = (\gamma_1\beta_2 - \beta_1\gamma_2))xy = 0$. Hence, R is symmetric. □

Lemma 4.12. *Let $s > 1$, p be an odd prime and*

$$R = \frac{\mathbb{Z}_{p^s}\langle x, y\rangle}{\langle x^2 - p, y^2 - p, yx + xy\rangle}.$$

Then R is local reflexive nonsymmetric.

Proof. By Lemma 4.10, R is reflexive. Since $p^{s-2}(p+xy)y(x+y) = 0$ but $p^{s-2}(p+xy)(x+y)y = 2p^{s-1}xy \neq 0$, R is nonsymmetric. □

Proposition 4.13. *Let $s > 1$, p be an odd prime s.t. $p \equiv 1 \pmod 4$ and*

$$R = \frac{\mathbb{Z}_{p^s}\langle x, y\rangle}{\langle x^2 - p, y^2 - p, yx + xy\rangle}.$$

Then R is local reflexive nonsemicommutative.

Proof. By Lemma 4.12, R is reflexive. By Lemma 4.5, there exist $a, b \in \mathbb{Z}_{p^s}$ s.t. $p \nmid a$, $p \nmid b$ and $p | a^2 + b^2 \neq 0$. Then $p^{s-2}(ax + by)^2 = (a^2 + b^2)p^{s-1} = 0$ and $p^{s-2}(ax + by)x(ax + by) = p^{s-2}(a^2px + abpy + abpy - b^2px) = (a^2 - b^2)p^{s-1}x + 2abp^{s-1}y \neq 0$. Hence, R is nonsemicommutative. □

Proposition 4.14. *Let $s > 1$, p be an odd prime s.t. $p \equiv 3 \pmod 4$ and*

$$R = \frac{\mathbb{Z}_{p^s}\langle x, y\rangle}{\langle x^2 - p, y^2 - p, yx + xy\rangle}.$$

Then R is reversible and nonsymmetric.

Proof. By Lemma 4.9, R is reversible and by Lemma 4.12 R is nonsymmetric. □

5. Computational considerations

In this section, the computational considerations are discussed to give insight into how the isomorphism classes were found in previous sections using Mathematica. A Mathematica notebook for the code can be found at https://oeis.org/A342305. In this discussion, the family of rings considered is slightly larger than that considered in Sections 2 and 3. Let R be a commutative ring with identity and let $a, b, c, d, e, f \in R$. Throughout this section, we will denote by $R(a, b, c, d, e, f)$, the quotient ring

$$\frac{R\langle x, y\rangle}{\langle x^2 - a, y^2 - b, yx - c - dx - ey - fxy\rangle}.$$

Note that $R(a, b, c, d, e, f)$ is always generated by the set $\{1, x, y, xy\}$ and, consequently, it has at most $|R|^4$ elements. In the case that it has exactly $|R|^4$, every element can be expressed uniquely in the form $x_0 + x_1x + x_2y + x_3xy$ where $x_i \in R$, i.e., it is a free algebra over R. Hence, these rings in this case, can be embedded in a standard way into the ring $M_4(R)$ using right multiplication and the canonical basis, $\{1, x, y, xy\}$. Moreover, given a ring $R(a, b, c, d, e, f)$, we can define

$$X = \begin{pmatrix} 0 & 1 & 0 & 0 \\ a & 0 & 0 & 0 \\ c & d & e & f \\ ad & c & af & e \end{pmatrix} \text{ and } Y = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ b & 0 & 0 & 0 \\ 0 & b & 0 & 0 \end{pmatrix}.$$

Then the set of matrices $\{I_4, X, Y, XY\}$ are linearly independent and $Y^2 = bI_4$ in any case. If $X^2 = aI_4$ and $YX = cI_4 + dX + eY + fXY$, the set $\{x_0 I_4 + x_1 X + x_2 Y + x_3 XY : x_i \in R\}$ forms a ring which is then isomorphic to $R(a, b, c, d, e, f)$. As a consequence, we get the following proposition.

Proposition 5.1. *Let R be a ring and $a, b, c, d, e, f \in R$. Then the ring $R(a, b, c, d, e, f)$ has order $|R|^4$ if and only if*

$$X = \begin{pmatrix} 0 & 1 & 0 & 0 \\ a & 0 & 0 & 0 \\ c & d & e & f \\ ad & c & af & e \end{pmatrix} \text{ and } Y = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ b & 0 & 0 & 0 \\ 0 & b & 0 & 0 \end{pmatrix}$$

satisfy the equations $X^2 = aI_4$ and $YX = cI_4 + dX + eY + fXY$.

In addition to enumerating all the rings of the form $R(a, b, c, d, e, f)$, we will also seek to determine the isomorphism classes. In the cases where $|R(a, b, c, d, e, f)| = |R|^4$, we used the following result which can easily be proven using Proposition 5.1.

Corollary 5.2. *Let R be a ring, $a_1, b_1, c_1, d_1, e_1, f_1, a_2, b_2, c_2, d_2, e_2, f_2 \in R$, $R_1 = R(a_1, b_1, c_1, d_1, e_1, f_1)$, $R_2 = R(a_2, b_2, c_2, d_2, e_2, f_2)$,*

$$X = \begin{pmatrix} 0 & 1 & 0 & 0 \\ a_1 & 0 & 0 & 0 \\ c_1 & d_1 & e_1 & f_1 \\ a_1 d_1 & c_1 & a_1 f_1 & e_1 \end{pmatrix} \text{ and } Y = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ b_1 & 0 & 0 & 0 \\ 0 & b_1 & 0 & 0 \end{pmatrix}.$$

Assume $|R_1| = |R_2| = |R|^4$. Then $R_1 \cong R_2$ if and only if there exist matrices $A, B \in \{x_0 I_4 + x_1 X + x_2 Y + x_3 XY : x_i \in R\}$ satisfying the equations $A^2 = a_2 I_4$, $B^2 = b_2 I_4$ and $BA = c_2 I_4 + d_2 A + e_2 B + f_2 AB$ and the set $|\{x_0 I_4 + x_1 A + x_2 B + x_3 AB : x_i \in R\}| = |\mathrm{R}|^4$.

Now, we turn to anti-isomorphisms. Recall that two rings R_1 and R_2 are anti-isomorphic if and only if $R_1 \cong R_2^{op}$. Hence, in order to be able to use Proposition 5.1 to determine anti-isomorphism classes, we need to describe the ring $\mathrm{R}(a, b; \alpha, \beta, \gamma, \delta)^{op}$. This is done in the following result.

Proposition 5.3. *Let us assume that the rings* $\mathrm{R}(a, b; \alpha, \beta, \gamma, \delta)$ *and* $\mathrm{R}(b, a; \alpha, \gamma, \beta, \delta)$ *are of order* $|\mathrm{R}|^4$. *Then,*

$$\mathrm{R}(a, b; \alpha, \beta, \gamma, \delta)^{op} \cong \mathrm{R}(b, a; \alpha, \gamma, \beta, \delta).$$

Proof. Since both rings are of order $|\mathrm{R}|^4$, they are free R-modules. By their definition, we can put $\mathrm{R}(a, b; \alpha, \beta, \gamma, \delta) = \{x_0 + x_1 x + x_2 y + x_3 xy : x_i \in \mathrm{R}\}$, with $x^2 = a, y^2 = b, yx = \alpha + \beta x + \gamma y + \delta xy$ and $\mathrm{R}(b, a; \alpha, \gamma, \beta, \delta) = \{x_0 + x_1 \overline{x} + x_2 \overline{y} + x_3 \overline{x}\,\overline{y} : x_i \in \mathrm{R}\}$, with $\overline{x}^2 = b, \overline{y}^2 = a, \overline{y}\,\overline{x} = \alpha + \gamma\overline{x} + \beta\overline{y} + \delta\overline{x}\,\overline{y}$. Now, if we define an R-module homomorphism $\phi : \mathrm{R}(b, a; \alpha, \gamma, \beta, \delta) \longrightarrow \mathrm{R}(a, b; \alpha, \beta, \gamma, \delta)$ by $\phi(1) = 1$, $\phi(\overline{x}) = y$, $\phi(\overline{y}) = x$, $\phi(\overline{x}\,\overline{y}) = xy$, it is straightforward that it induces the claimed isomorphism and the result follows. ☐

In this situation we get the following corollary, which does not require any proof. It implies that, once when know the isomorphism classes of our family of rings, we also know the anti-isomorphism classes.

Corollary 5.4. *Let us assume that the rings* $\mathrm{R}(a, b; \alpha, \beta, \gamma, \delta)$ *and* $\mathrm{R}(\widehat{a}, \widehat{b}; \widehat{\alpha}, \widehat{\beta}, \widehat{\gamma}, \widehat{\delta})$ *are of order* $|\mathrm{R}|^4$. *Then, they are anti-isomorphic if and only if*

$$\mathrm{R}(\widehat{a}, \widehat{b}; \widehat{\alpha}, \widehat{\beta}, \widehat{\gamma}, \widehat{\delta}) \cong \mathrm{R}(b, a; \alpha, \gamma, \beta, \delta).$$

Algorithm 1 Isomorphy Test

Let $R_1 = R(a, b, \alpha, \beta, \gamma, \delta)$; R_2 a ring of order $|R|^4$
Let $\Omega(a) = \{X \in R_2 : X^2 = a\} = \{X_1, X_2, ..., X_n\}$
Let $\Omega(b) = \{X \in R_2 : X^2 = b\} = \{Y_1, Y_2, ..., Y_m\}$
for $i = 1, 2, \dots n$ **do**
  for $j = 1, 2, \dots m$ **do**
    if $Y_j X_i = \alpha + \beta X_i + \gamma Y_j + \delta X_i Y_j$ and $| < I, X_i, Y_j, X_i Y_j > | = |R|^4$ **then**
      return True
    end if
  end for
end for
return False

Sometimes it is easy and fast to check that two rings are not isomorphic, just by verifying that certain equations have a different number of solutions in each ring. However, this approach fails in many cases, for example when the rings are anti-isomorphic but not isomorphic. In such situations it is necessary to resort to the algorithm above. This is also the case if we want to prove that two rings are isomorphic. The computational complexity of this test is easily seen to be of order $O(n^8)$, and hence it can only be used for moderately low values of n. The reflexibility, reversibility, semi-commutativity and abelianity tests are also of complexity $O(n^8)$. The symmetry test is of even higher complexity, $O(n^{12})$, since the definition of symmetry involves 3 elements.

In the case $R = \mathbb{Z}_4$, Algorithm 1, allows to determine that there are 97 isomorphic classes of this family and there are 88 in the family considered in Sections 2 and 3. The extra 9 classes that were not part of our main analysis were either nonabelian or nonreflexive and nonsymmetric.

Algorithm 1 has also made it possible to find numerous interesting examples using other coefficient rings. Some results are given in the following table.

R	isomorphic classes	comm.	non-auto-anti-isom.	reversible nonsymm.	reflexive abelian nonsemicomm.
$\mathbb{F}_2$	3	1	0	0	0
$\mathbb{F}_3$	13	5	2	0	0
$\mathbb{F}_4$	4	1	0	0	0
$\mathbb{Z}_4$	97	9	4	10	21
$\mathbb{F}_5$	14	5	2	0	0
$\mathbb{F}_7$	15	5	2	0	0
$\mathbb{F}_8$	5	1	0	0	0
$\mathbb{Z}_8$	624	29	16	34	166
$\mathbb{F}_9$	16	5	2	0	0
$\mathbb{Z}_9$	67	13	28	5	9

In all cases except for $\mathbb{Z}_9$ in the table, all non-auto-anti-isomorphic rings are nonreversible. In the $\mathbb{Z}_9$ case, there are 8 reversible non-auto-anti-isomorphic rings.

6. Conclusions and challenges

In this work we have presented some novel examples of minimal reversible nonsymmetric rings and minimal abelian reflexive nonsemicommutative rings. Few such rings have been previously presented and finding them without computational aids is indeed a very hard task. Our computational approach has allowed us to find 10 new minimal reversible nonsymmetric rings and 21 new minimal abelian reflexive nonsemicommutative rings. In addition, our computations allowed us to find examples of characteristic p^2 for $p = 2, 3, 5$ that led us to conjecture, and then prove, the existence of other examples of characteristic p^s for any prime and

$s \geq 2$. To close the paper, we propose some challenges related to this work that will hopefully foster further research.

- Find new examples of minimal nonsymmetric reversible (or abelian reflexive nonsemicommutative) rings of characteristic 2.
- Find some minimal reversible nonsymmetric (or abelian reflexive nonsemicommutative) rings of characteristic 8, 16 and 32; or prove that such rings do not exist.
- In [9], an example of a 13-dimensional reversible nonsymmetric K-algebra (K a field) was given. Does there exist a family of reversible nonsymmetric K-algebras of lesser dimension? Similarly, in [1], Example 4.2 is a 12-dimensional abelian reflexive nonsemicommutative $\mathbb{F}_2$-algebra. Replacing $\mathbb{F}_2$ with K will be a K-algebra of the same type. Does there exist a family of reflexive nonsemicommutative K-algebras of lesser dimension?

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