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EXTENSIONS OF THE CATEGORY OF COMODULES OF THE TAFT ALGEBRA

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Abstract. We construct a family of non-equivalent pairwise extensions of the category of comodules of the Taft algebra, which are equivalent to representation categories of non-triangular quasi-Hopf algebras.

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1. Introduction

Given a finite group G and a fusion category $\mathcal{C}$, the G-extensions of $\mathcal{C}$ were classified in [3], however to give concrete examples of these classification in general is complicated. In the literature there are few examples of these extensions when $\mathcal{C}$ is non-semisimple. A different version, called Crossed Products was introduced in [4], where the parameters to construct those extensions are calculable when $\mathcal{C}$ is the category of comodules over a Hopf algebra H. The main difference with the work of [3] is that we need to calculate the Brauer-Picard group of the category $comod(H)$, and for the work in [4] we only need the group of biGalois objects of the Hopf algebra.

Following this idea, in [5], we construct eight tensor categories which are extensions of the category of comodules over a supergroup algebra and in [6], we analyze when these categories are braided.

In this work we construct an infinite family (non equivalent pairwise) of C_2-extensions of the category of comodules over the Taft algebra $T(q)$, where C_2 is the cyclic group of two elements. As Abelian categories, they are two copies of the category of comodules of $T(q)$ with tensor product described in Equations (6) and (7), and non-trivial associativity constrains.

Since $T(q)$ is not a co-quasitriangular Hopf algebra, $Comod(T(q))$ is not braided then [6, Theorem 2.6] any extension of $Comod(T(q))$ is not braided. Therefore the categories described here are not braided. Nevertheless, each one is equivalent to the category of representations over some non-triangular quasi-Hopf algebra, using Frobenius-Perron dimension. In particular, this is another example of how results obtained in a categorical context produce results (of existence) in a context of Hopf algebras. Several examples of this have been introduced in the literature, for example in [2], the classification of braided unipotent tensor categories gives place to the classification of coconnected coquasitriangular Hopf algebras; and in [1] a result over modular categories allows to prove the Kaplansky's conjecture for quasitriangular semisimple Hopf algebras.

2. Preliminaries

2.1. Hopf algebras and BiGalois objects. In this work we work over the complex field $\mathbb{C}$. Let H be a Hopf algebra and $g \in G(H)$ be a group-like element. We denote $\mathbb{C}_g$ the one-dimensional vector space generated by w_g with left H-comodule given by $\lambda : \mathbb{C}_g \to H \otimes \mathbb{C}_g, \lambda(w_g) = g \otimes w_g$.

Let A be an H-biGalois object with left H-comodule structure $\lambda : A \to H \otimes A$. If g is a group-like element we can define a new H-biGalois object A^g on the same underlying algebra A with unchanged right comodule structure and a new left H-comodule structure given by $\lambda^g : A^g \to H \otimes A^g, \lambda^g(a) = g^{-1}a_{(-1)}g \otimes a_{(0)}$ for all $a \in A$. Recall [7] that two H-biGalois objects A, B are equivalent, if there exists an element $g \in G(H)$ such that $A^g \simeq B$ as biGalois objects.

$BiGal(H)$ is a group with the cotensor product $\square_H$, where H is the unit, and for $L \in BiGal(H)$, $\overline{\lambda} : L \to H\square_H L$ and $\overline{\rho} : L \to L\square_H H$ are the isomorphisms induced by the left and right coactions, with inverses induced by the counit. The subgroup of BiGal(H) consisting of H-biGalois objects equivalent to H is denoted by InnbiGal(H). This group is a normal subgroup of BiGal(H). We denote

$$\mathrm{OutbiGal(H)} = \mathrm{BiGal(H)}/\mathrm{InnbiGal(H)}.$$

If $g \in G(H)$ and L is a (H, H)-biGalois object then the cotensor product $L\square_H\mathbb{C}_g$ is one-dimensional. Let $\phi(L, g) \in G(H)$ such that $L\square_H\mathbb{C}_g \simeq \mathbb{C}_{\phi(L,g)}$ as left H-comodules.

2.2. Autoequivalences on categories. Let $\mathcal{C}$ be a finite tensor category. Given an invertible object $\sigma \in \mathcal{C}$, we define the monoidal equivalence

$$\begin{aligned} Ad_\sigma : \mathcal{C} &\to \mathcal{C} \\ V &\mapsto \sigma \otimes V \otimes \sigma^{-1}, \end{aligned}$$

and $Ad_\sigma(V \otimes W) = Ad_\sigma(V) \otimes Ad_\sigma(W)$ for all $V, W \in \mathcal{C}$. The functor Ad_σ is a monoidal isomorphism with inverse $Ad_{\sigma^{-1}}$. In fact, $Ad_\sigma \circ Ad_{\sigma^{-1}} = Ad_{\sigma^{-1}} \circ Ad_\sigma =$

$\mathrm{id}_{\mathcal{C}}$, also Ad_σ is pseudonatural **isomorphic** to $\mathrm{id}_{\mathcal{C}}$, where $(\sigma, \mathrm{id}) : Ad_\sigma \to id_{\mathcal{C}}$ and $(\sigma^{-1}) : \mathrm{id}_{\mathcal{C}} \to Ad_\sigma$.

Proposition 2.1. *Let $F : \mathcal{C} \to \mathcal{C}$ be an autoequivalence.*

(1) *F is pseudonatural equivalent to $\mathrm{id}_{\mathcal{C}}$ if and only if F is monoidal equivalent to Ad_σ for some invertible object $\sigma \in \mathcal{C}$.*

(2) *Ad_σ is monoidal equivalent to Ad_τ if and only if $\sigma^{-1} \otimes \tau$ admits an structure $(\sigma^{-1} \otimes \tau, \psi)$ of object in the Drinfeld center $\mathcal{Z}(\mathcal{C})$ of $\mathcal{C}$, such that $\sigma \otimes \psi \otimes \tau^{-1} : Ad_\sigma \to Ad_\tau$ is a monoidal isomorphism.*

Proof. If $(\sigma, \sigma_{(-)}) : F \to \mathrm{id}_{\mathcal{C}}$ is an invertible pseudonatural transformation, then exist an other pseudonatural transformation $(\tau, \tau) : \mathrm{id}_{\mathcal{C}} \to F$ and invertible modifications $\alpha : \sigma \bar{\circ} \tau \to Id_G$ and $\beta : \tau \bar{\circ} \sigma \to Id_F$, then $\sigma \otimes \tau \cong \tau \otimes \sigma \cong 1$. Then is implies that σ is invertible and $\tau = \sigma^{-1}$. By the definition we have natural isomorphisms

$$\sigma_V : F(V) \otimes \sigma \to \sigma \otimes V,$$

so the functor F is natural isomorphic to Ad_σ. Let $\sigma, \tau \in \mathcal{C}$ be invertible objects and $(F, \psi) : Ad_\sigma \to Ad_\tau$ a monoidal equivalence, then the composition

$$\mathrm{id}_{\mathcal{C}} \to Ad_\sigma \to Ad_\tau \to \mathrm{id}_{\mathcal{C}},$$

defines a pseudonatural equivalence of $\mathrm{id}_{\mathcal{C}}$, i.e., an invertible object in $\mathcal{Z}(\mathcal{C})$, that satisfies the condition of the proposition. □

2.3. C_2-crossed product tensor categories. In [4], Galindo introduced a way to construct extensions of a given category. When the graded group is C_2, the cyclic group of order 2, in [5], the authors give a complete classification if the tensor category in degree zero is the category of comodules of supergroup algebras.

Theorem 2.2. [5, Section 5.1, Lemma 5.9] *There is a correspondence between C_2-crossed product tensor categories over $Comod(H)$ and collections (L, g, f, γ) where*

(1) *L is a (H, H)-biGalois object;*

(2) *$g \in G(H)$ such that $L \square_H \mathbb{C}_g \simeq \mathbb{C}_g$ as left H-comodules;*

(3) *$f : (L \square_H L)^g \to H$ is a bicomodule algebra isomorphism;*

(4) *$\gamma \in \mathbb{C}^\times, \gamma^2 = 1$.*

The tensor categories associated to two collections (L, g, f, γ) and (L', g', f', γ') are monoidally equivalent if, and only if, there exist a collection (A, h, φ, τ) where

(1) *A is a (H, H)-biGalois object,*

(2) *$h \in G(H)$ and $\varphi : (A \square_H L)^h \to L' \square_H A$ is a biGalois isomorphism,*

(3) *$\tau \in \mathbb{C}^\times$,*

and also the following equations are fulfilled

$$\phi(L,g) = h\phi(L',h)g', \tag{1}$$

$$\overline{\rho}^{-1}(id_A \otimes f) = \overline{\lambda}^{-1}(f' \otimes id_A)(id_{L'} \otimes \varphi)(\varphi \otimes id_L). \tag{2}$$

3. Taft algebra

Let $N \geq 2$ be an integer and let $q \in \mathbb{C}$ be a primitive N-th root of unity. The *Taft algebra* $T(q)$ is the $\mathbb{C}$-algebra presented by generators X and Y with relations $X^N = 1$, $Y^N = 0$ and $YX = qXY$. The algebra $T(q)$ carries a Hopf algebra structure, determined by

$$\Delta X = X \otimes X, \qquad \Delta Y = 1 \otimes Y + Y \otimes X.$$

Then $\varepsilon(X) = 1$, $\varepsilon(Y) = 0$, $\mathcal{S}(X) = X^{-1}$, and $\mathcal{S}(Y) = -q^{-1}X^{-1}Y$. It is known that

(1) $T(q)$ is a pointed non-semisimple Hopf algebra,
(2) the group of group-like elements of $T(q)$ is $G(T(q)) = \langle X \rangle \simeq \mathbb{Z}/(N)$,
(3) $T(q) \simeq T(q)^*$,
(4) $T(q) \simeq T(q')$ if and only if $q = q'$.

For all $\alpha \in \mathbb{C}^*$ and $\beta \in \mathbb{C}$, Schauenburg [7] proved that the $T(q)$-biGalois objects are the algebras

$$A_{\alpha,\beta} := k\langle x, y\rangle/(x^N = \alpha, y^N = \beta, yx = qxy),$$

with right ρ and left λ comodule structures

$$\rho(x) = x \otimes X, \rho(Y) = 1 \otimes Y + y \otimes X, \lambda(x) = X \otimes x, \lambda(y) = 1 \otimes y + Y \otimes x.$$

The biGalois objects $A_{\alpha,\beta}$ are representative sets of equivalence classes of biGalois objects [7, Theorem 2.2]. There exists a group isomorphism

$$\begin{aligned} \psi : \mathbb{C}^* \rtimes \mathbb{C} &\to \mathrm{BiGal}(T(q)) \\ (\alpha, \beta) &\mapsto A_{\alpha,\beta}, \end{aligned}$$

then $A_{\alpha,\beta} \square_{T(q)} A_{\alpha',\beta'} \simeq A_{\alpha\alpha',\beta\alpha'+\beta'}$ and there is a canonical isomorphism

$$\delta_0 : A_{\alpha\alpha',\beta\alpha'+\beta'} \to A_{\alpha,\beta} \square_H A_{\alpha',\beta'}, \quad x \mapsto x \otimes x, y \mapsto 1 \otimes y + y \otimes x. \tag{3}$$

Schauenburg also calculates the group of Hopf algebra automorphism [7, Lemma 2.1], where

$$\begin{aligned} \varphi : \mathbb{C}^* &\to Aut_{Hopf}(T(q)) \\ r &\mapsto f_r \end{aligned}$$

with $f_r(X) = X$ and $f_r(Y) = rY$ is a group isomorphism; and for $X^r \in G(T(q))$, $\varphi(Ad_{X^r}) = f_{q^{-r}}$. Also there exists a group homomorphism $Aut_{Hopf}(T(q)) \to BiGal(T(q))$ given by $f \mapsto {}^fH$ where as a vector space is H with left coaction given by $v \mapsto f(v_{(-1)}) \otimes v_{(0)}$. Regarding about bicomodule algebra isomorphisms of $T(q)$, by [7, Theorem 2.2.3], they are precisely ι_p for $p^N = 1$ where

$$\iota_p(X) = pX, \quad \iota_p(Y) = Y. \tag{4}$$

Now, it is possible to calculate the inner and outer biGalois objects.

Theorem 3.1. *$InnbiGal(T(q))$ is trivial, and $OutbiGal(T(q)) \simeq \mathbb{C}^* \rtimes \mathbb{C}$.*

Proof. If $f_r \in Aut_{Hopf}(T(q))$ then by [7, Theorem 2.5], ${}^{f_r}A_{\alpha,\beta} \cong A_{\alpha r^N,\beta}$ as $T(q)$-biGalois objects. Let $X^r \in G(T(q))$, since $Ad_{X^r} = f_{q^{-1}}$ and $q^N = 1$ we have that

$$A_{\alpha,\beta}^{X^r} \cong A_{\alpha q^{-Nr},\beta} = A_{\alpha,\beta}. \tag{5}$$

Then every inner biGalois object is trivial. $\square$

4. C_2-Crossed product tensor categories

Now, we apply Theorem 2.2 to $H = T(q)$, where the biGalois objects are parametrized by $L = A_{\alpha,\beta}$ with $\alpha \in \mathbb{C}^*$ and $\beta \in \mathbb{C}$. Fix $A_{\alpha,\beta}$, since $G(T(q)) = \{X^s | s = 0, \dots, N-1\}$, for a given $s < N$, $A_{\alpha,\beta}\square_{T(q)}\mathbb{C}_{X^s} = \mathbb{C}\{x^s \otimes 1\}$ is one-dimensional; moreover the left coaction of T over ${}^{\bullet}A_{\alpha,\beta}\square_{T(q)}\mathbb{C}_{X^s}$ is given by $x^s \otimes 1 \mapsto X^s \otimes x^s \otimes 1$, therefore as left H-comodules

$$A_{\alpha,\beta}\square_{T(q)}\mathbb{C}_{X^s} \simeq \mathbb{C}_{X^s},$$

and $\phi(A_{\alpha,\beta}, X^s) = X^s$. Now, $(A_{\alpha,\beta}\square_{T(q)}A_{\alpha,\beta})^{X^s} \simeq A^{X^s}_{\alpha^2,\beta\alpha+\beta} \simeq A_{\alpha^2,\beta\alpha+\beta}$ by Equation (5), then there exists such f if, and only if,

$$A_{\alpha^2,\beta\alpha+\beta} \simeq A_{1,0}.$$

We obtain $L \in \{T(q), A_{-1,\beta} | \beta \in \mathbb{C}\}$ and $g \in \{X^s | s < N\}$. Now, we explicitly need to determine all comodule algebra morphism f. Since $(L\square_{T(q)}L)^g \simeq T(q)$, f is parametrized by the bicomodule algebra automorphisms of $T(q)$ described in Equation (4).

Lemma 4.1. *Each collection (L, g, f, γ) where $L \in \{T(q), A_{-1,\beta} | \beta \in \mathbb{C}\}$, $g \in \{X^s | s < N\}$, $f \in \{\iota_p \circ \delta_0^{-1} | p^N = 1\}$ and $\gamma \in \{\pm 1\}$ generates a C_2-extension $\mathcal{C}_{(L,g,f,\gamma)}$ of the category of comodules of $T(q)$.*

We explicitly described the tensor structure. As an Abelian category

$$\mathcal{C}_{(T(q),X^s,\iota_p\delta_0^{-1},\gamma)} = Comod(T(q)) \oplus Comod(T(q)).$$

Since the category is C_2-graded, we denoted the objects as V_1 or V_u where $V \in Comod(T(q))$, $C_2 = \{1, u | u^2 = 1\}$. The tensor product is given

$$V_a \otimes W_b = \begin{cases} (V \otimes W \otimes \mathbb{C}_{X^s})_1 & a = b = u \\ (V \otimes W)_{ab} & \text{otherwise.} \end{cases} \tag{6}$$

The left comodule structure over $V \otimes W \otimes \mathbb{C}_{X^s}$ is $v \otimes w \otimes k \mapsto v_1 w_1 X^s \otimes v_0 \otimes w_0 \otimes k$. The associativity is trivial except $(V_u \otimes W_u) \otimes Z_u$, which is defined using ι_p and γ, see [5, Section 6.1].

As an Abelian category

$$\mathcal{C}_{(A_{-1,\beta},X^s,\iota_p\delta_0^{-1},\gamma)} = Comod(T(q)) \oplus Comod(T(q)).$$

The tensor product is given

$$V_a \otimes W_b = \begin{cases} (V \otimes (A_{-1,\beta}\square_{T(q)} W) \otimes \mathbb{C}_{X^s})_1 & a = b = u \\ (V \otimes (A_{-1,\beta}\square_{T(q)} W))_u & a = u, b = 1 \\ (V \otimes W)_{ab} & \text{otherwise.} \end{cases} \tag{7}$$

Next, we determine in which cases these collections generates monoidally equivalent categories, applying the second part of Theorem 2.2. Notice that L in Lemma 4.1 has two options, then we consider in the next propositions these three possible cases. Combining them we obtain Theorem 4.5.

(1) **Trivial biGalois objects in the tuples.**

Proposition 4.2. $\mathcal{C}_{(T(q),X^s,\iota_p\delta_0^{-1},\gamma)} \simeq \mathcal{C}_{(T(q),X^{s'},\iota_{p'}\delta_0^{-1},\gamma')}$ *as monoidal categories if, and only if,* $X^{s-s'} = X^{2t}$ *for any* $t < N$ *and* $p = p' = 1$.

Proof. Let $A = A_{\alpha,\beta}$ be a biGalois object and $h \in G(T(q))$, then

$$(A\square_{T(q)} T(q))^h \simeq A \simeq T(q)\square_{T(q)} A,$$

and we can define $\varphi = \overline{\lambda} \circ \overline{\rho}^{-1}$. Then Equation (1) is equivalent to $X^{s-s'} = X^{2t}$ for any $t < N$. Consider the following diagram ($R = T(q) \times A_{\alpha,\beta} \times T(q)$),

$$\begin{tikzcd}
A_{\alpha,\beta} \times T(q) \times T(q) \arrow[rr, "\varphi \times id"] & & R \arrow[rr, "id \times \varphi"] & & T(q) \times T(q) \times A_{\alpha,\beta} \\
& A_{\alpha,\beta} \times T(q) \arrow[ul, "\overline{\rho} \times id"'] \arrow[ur, "\overline{\lambda} \times id"] & & T(q) \times A_{\alpha,\beta} \arrow[ul, "id \times \overline{\rho}"'] \arrow[ur, "id \times \overline{\lambda}"] & \\
& & A_{\alpha,\beta} \arrow[ul, "\overline{\rho}"] \arrow[ur, "\overline{\lambda}"'] \arrow[dll, "\overline{\rho}"] \arrow[drr, "\overline{\lambda}"'] & & \\
A_{\alpha,\beta} \times T(q) \arrow[uuu, "id \times \delta_0"] \arrow[r, "id \times \iota_p"'] & A_{\alpha,\beta} \times T(q) & A_{\alpha,\beta} \arrow[l, "\overline{\rho}"] \arrow[r, "\overline{\lambda}"] & T(q) \times A_{\alpha,\beta} & T(q) \times A_{\alpha,\beta} \arrow[uuu, "\delta_0 \times id"'] \arrow[l, "\iota_{p'} \times id"]
\end{tikzcd} \tag{8}$$

then Equation (2) (exterior of (8)) is equivalent to

$$\overline{\rho}^{-1}(id \otimes \iota_p)\overline{\rho} = \overline{\lambda}^{-1}(\iota_{p'} \otimes id)\overline{\lambda} \tag{9}$$

(bottom triangle of (8)) since

(1) left and right triangles: $A_{\alpha,\beta}$ is a left and right comodule and $\delta_0 = \Delta$,
(2) central diamond: $A_{\alpha,\beta}$ is a bicomodule,
(3) left and right up triangles: φ definition.

Now,

$$\overline{\rho}^{-1}(id \otimes \iota_p)\overline{\rho}(x) = px \qquad \overline{\rho}^{-1}(id \otimes \iota_p)\overline{\rho}(y) = py$$
$$\overline{\lambda}^{-1}(\iota_{p'} \otimes id)\overline{\lambda}(x) = p'x \qquad \overline{\lambda}^{-1}(\iota_{p'} \otimes id)\overline{\lambda}(y) = y.$$

Then Equation (9) is valid if, and only if, $p = p' = 1$. □

For each $1 \neq p \in \mathbb{C}$ with $p^N = 1$, $s < N$ and $\gamma \in \{\pm 1\}$, we obtain a family of non-equivalent categories

$$\{\mathcal{C}_{(T(q), X^s, \iota_p \delta_0^{-1}, \gamma)}\}_{p,s,\gamma} \cup \{\mathcal{C}_{(T(q), X, \delta_0^{-1}, \gamma)}, \mathcal{C}_{(T(q), 1, \delta_0^{-1}, \gamma)}\}_{\gamma}, \tag{10}$$

the second set appears when $p = p' = 1$, then the possible values for s are $0, 1$.

(2) **Non-trivial biGalois objects in the tuples.**

Proposition 4.3. *$\mathcal{C}_{(A_{-1,\beta}, X^s, \iota_p \delta_0^{-1}, \gamma)} \simeq \mathcal{C}_{(A_{-1,\beta'}, X^{s'}, \iota_{p'} \delta_0^{-1}, \gamma')}$ as monoidal categories if, and only if, $X^{s-s'} = X^{2t}$ for any $t < N$ and $p = p' = 1$.*

Proof. Let $A = A_{\alpha,\beta''}$ be a biGalois object and $h \in G(T(q))$, then

$$(A \square_{T(q)} A_{-1,\beta})^h \simeq A_{-\alpha, -\beta'' + \beta}, \quad A_{-\alpha, \beta'\alpha + \beta''} \simeq A_{-1,\beta'} \square_{T(q)} A,$$

$\beta'' = \frac{\beta - \beta'\alpha}{2}$. Equation (1) is equivalent to $X^{s-s'} = X^{2t}$ for any $t < N$. Let $Q = A_{\alpha,\beta''} \times A_{-1,\beta} \times A_{-1,\beta}, W = A_{-1,\beta'} \times A_{\alpha,\beta''} \times A_{-1,\beta}, E = A_{-1,\beta'} \times A_{-1,\beta'} \times A_{\alpha,\beta''}$ and consider the following diagram.

$$
\begin{tikzcd}
Q \arrow[rrr, "\varphi\times id"] & & & W \arrow[rrr, "id\times\varphi"] & & & E \\
& & A_{-\alpha,-\beta''+\beta}\times A_{-1,\beta} \arrow[ull, "\delta_0\times id"'] \arrow[ur, "\delta_0\times id"] & & A_{-1,\beta'}\times A_{-\alpha,-\beta''+\beta} \arrow[ul, "id\times\delta_0"'] \arrow[urr, "id\times\delta_0"] & & \\
& & & A_{\alpha,\beta''} \arrow[ul, "\delta_0"] \arrow[ur, "\delta_0"] \arrow[dlll, "\delta_0"] \arrow[drrr, "\delta_0"] & & & \\
A_{\alpha,\beta''}\times T(q) \arrow[uuu, "\mathrm{id}\times\delta_0"] \arrow[r, "id\times\iota_s"'] & A_{\alpha,\beta''}\times T(q) & & A_{\alpha,\beta''} \arrow[ll, "\overline{\rho}"'] \arrow[rr, "\overline{\lambda}"] & & T(q)\times A_{\alpha,\beta''} & T(q)\times A_{\alpha,\beta''} \arrow[l, "\iota_{s'}\times id"'] \arrow[uuu, "\delta_0\times id"']
\end{tikzcd}
\tag{11}
$$

Notice that we use the same notation δ_0 to different morphisms, since they have the same definition but different domains and codomains. Then, Equation (2) (exterior of (11)) is equivalent to

$$\overline{\rho}^{-1}(id\otimes\iota_s)\delta_0=\overline{\lambda}^{-1}(\iota_{s'}\otimes id)\delta_0, \tag{12}$$

as in the previous proof, Equation (12) is valid if, and only if, $s=s'=1$. $\square$

For each $1\neq p\in\mathbb{C}$ with $p^N=1$, $s<N$, $\gamma\in\{\pm1\}$ and $\beta\in\mathbb{C}^\times$, we obtain a family of non-equivalent categories

$$\{\mathcal{C}_{(A_{-1,\beta},X^s,\iota_p\delta_0^{-1},\gamma)}\}_{p,s,\beta,\gamma}\cup\{\mathcal{C}_{(A_{-1,\beta},X,\delta_0^{-1},\gamma)},\mathcal{C}_{(A_{-1,\beta},1,\delta_0^{-1},\gamma)}\}_{\beta,\gamma}, \tag{13}$$

notice that the second set only depends on β, since for $p=p'=1$, as before, s is $0,1$. In (10) we calculate non-equivalent categories when associated biGalois objects are trivial, here in (13) when they are non-trivial.

(3) **Non-trivial and trivial biGalois objects in the tuples.**

As before, we obtain the following result.

Proposition 4.4. *$\mathcal{C}_{(T(q),X^s,\iota_p\delta_0^{-1},\gamma)}\simeq\mathcal{C}_{(A_{-1,\beta},X^{s'},\iota_{p'}\delta_0^{-1},\gamma')}$ as monoidal categories if, and only if, $X^{s-s'}=X^{2t}$ for any $t<N$ and $p=p'=1$.*

Therefore, for any $\beta\in\mathbb{C}^\times$ and $\gamma\in\{\pm1\}$

$$\mathcal{C}_{(A_{-1,\beta},X,\delta_0^{-1},\gamma)}\simeq\mathcal{C}_{(T(q),X,\delta_0^{-1},\gamma)},\mathcal{C}_{(T(q),1,\delta_0^{-1},\gamma)}\simeq\mathcal{C}_{(A_{-1,\beta},1,\delta_0^{-1},\gamma)}, \tag{14}$$

then two categories with associated biGalois objects trivial and non-trivial are equivalent only in the previous cases.

Finally, from (10), (13) and (14), we obtain the main theorem.

Theorem 4.5. *Let $1\neq p\in\mathbb{C}$ with $p^N=1$, $s<N$, $\gamma\in\{\pm1\}$ and $\beta\in\mathbb{C}^\times$. We obtain a family of non-equivalent tensor categories*

$$\{\mathcal{C}_{(A_{-1,\beta},X^s,\iota_p\delta_0^{-1},\gamma)}\}_{p,s,\beta,\gamma}\cup\{\mathcal{C}_{(T(q),X^s,\iota_p\delta_0^{-1},\gamma)}\}_{p,s,\beta}\cup\{\mathcal{C}_{(T(q),X,\delta_0^{-1},\gamma)},\mathcal{C}_{(T(q),1,\delta_0^{-1},\gamma)}\}.$$

Since $FPdim(T(q))$ is an integer, the Frobenius-Perron dimension of any of the categories listed before is $2FPdim(T(q))$, then they are the category of representations of a quasi-Hopf algebra.

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References

[1] P. Etingof and S. Gelaki, *Some properties of finite-dimensional semisimple Hopf algebras*, Math. Res. Lett., 5(1-2) (1998), 191-197.

[2] P. Etingof and S. Gelaki, *Quasisymmetric and unipotent tensor categories*, Math. Res. Lett., 15(5) (2008), 857-866.

[3] P. Etingof, D. Nikshych and V. Ostrik, *Fusion categories and homotopy theory*, Quantum Topol., 1(3) (2010), 209-273.

[4] C. Galindo, *Crossed product tensor categories*, J. Algebra, 337 (2011), 233-252.

[5] A. Mejía Castaño and M. Mombelli, *Crossed extensions of the corepresentation category of finite supergroup algebras*, Internat. J. Math., 26(9) (2015), 1550067 (26 pp).

[6] A. Mejía Castaño, *Example of (non)-braided tensor categories*, Int. Electron. J. Algebra, 27 (2020), 194-205.

[7] P. Schauenburg, *Bi-Galois objects over the Taft algebras*, Israel J. Math., 115 (2000), 101-123.

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