International Electronic Journal of Algebra
Volume 35 (2024) 168-185
DOI: 10.24330/ieja.1402961

NORMALITY OF REES ALGEBRAS OF GENERALIZED MIXED PRODUCT IDEALS

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Received: 18 August 2023; Revised: 27 October 2023; Accepted: 30 October 2023
Communicated by Sait Halıcıoğlu

Abstract. Let K be a field and $K[x_1, x_2]$ the polynomial ring in two variables over K with each x_i of degree 1. Let L be the generalized mixed product ideal induced by a monomial ideal $I \subset K[x_1, x_2]$, where the ideals substituting the monomials in I are squarefree Veronese ideals. In this paper, we study the integral closure of L, and the normality of $\mathcal{R}(L)$, the Rees algebra of L. Furthermore, we give a geometric description of the integral closure of $\mathcal{R}(L)$.

Mathematics Subject Classification (2020): 13B22, 13A30, 13F55
Keywords: Integral closure, normality, Rees algebra, generalized mixed product ideal

1. Introduction

Let $S = K[x_1, \dots, x_n]$ be the polynomial ring over a field K in the variables $x_1, \dots, x_n$, and let $I \subset S$ be a monomial ideal with $I \neq S$ whose minimal set of generators is $G(I) = \{\mathbf{x}^{\mathbf{a}_1}, \dots, \mathbf{x}^{\mathbf{a}_m}\}$. We consider the polynomial ring T over K in the variables $x_{11}, \dots, x_{1m_1}, x_{21}, \dots, x_{2m_2}, \dots, x_{n1}, \dots, x_{nm_n}$. Notice that $T = T_1 \otimes_K T_2 \otimes_K \cdots \otimes_K T_n$, where $T_j = K[x_{j1}, x_{j2}, \dots, x_{jm_j}]$ for $j = 1, \dots, n$.

Restuccia and Villarreal [10] introduced the class of squarefree monomial ideals of mixed products and they gave a complete classification of normal mixed product ideals, as well as applications in graph theory.

Mixed product ideals are of the form

$$(I_q J_w + I_p J_s) K[x_{11}, \dots, x_{1m_1}, x_{21}, \dots, x_{2m_2}],$$

where for integers a and b, the ideal I_a (resp. J_b) is the ideal generated by all squarefree monomials of degree a in the polynomial ring $K[x_{11}, \dots, x_{1m_1}]$ (resp. of degree b in the polynomial ring $K[x_{21}, \dots, x_{2m_2}]$) and where $0 < p < q \leq m_1$, $0 < w < s \leq m_2$. Thus, the ideal $L = (I_q J_w + I_p J_s) K[x_{11}, \dots, x_{1m_1}, x_{21}, \dots, x_{2m_2}]$

The research that led to the present paper was partially supported by a grant of the group GNSAGA of INdAM, Italy.

is obtained from the monomial ideal $I = (x_1^q x_2^w, x_1^p x_2^s)$ by replacing x_1^q by I_q, x_1^p by I_p, x_2^w by J_w and x_2^s by J_s.

In the present paper, we consider generalized mixed product ideals which were introduced by Herzog and Yassemi [4] and which also include the so-called expansions of monomial ideals. A great deal of knowledge on the generalized mixed product ideal is accumulated in several papers [6,7,8,9].

The main objective of this paper is to study the normality of some algebras associated to generalized mixed product ideals. In our case, the normality of a generalized mixed product ideal L is equivalent to the normality of the Rees algebra $\mathcal{R}(L) = \bigoplus_{k=0}^{\infty} L^k t^k$. The integral closure $\overline{\mathcal{R}(L)}$ of the Rees algebra in its field of fractions is called normalization of L. It is well-known ([11]) that this graded algebra has the powers of the ideal $\overline{L}$ as components of the integral closure:

$$\overline{\mathcal{R}(L)} = T \oplus \overline{L}t \oplus \cdots \oplus \overline{L^k}t^k \oplus \cdots ,$$

where $\overline{L^k}$ is the integral closure of L^k.

The present paper is organized as follows. In Section 2 the combinatorics of the integral closure of generalized mixed product ideals is studied. In [7], the author studied how the generalized mixed product ideal commutes with the integral closure of a monomial ideal and proved that I is normal if and only if L is normal, provided the ideals substituting the monomials in I are all powers of the maximal ideals.

The squarefree Veronese ideal of S of degree d is the ideal of S which is generated by all squarefree monomials of S of degree d. This class of ideals is a special class of polymatroidal ideals, introduced in [13].

Our main result (Theorem 2.7) says that, if $I \subset K[x_1, x_2]$ is a Veronese type ideal and the ideals who substitute the generators of I are squarefree Veronese ideals, then L is normal.

Furthermore, let $L = (f_1, \ldots, f_r)$. The monomial subring spanned by $\{f_1, \ldots, f_r\}$ is the K-subalgebra $K[L] = K[f_1, \ldots, f_r]$. The integral closure of $K[L]$ in its field of fractions is called normalization of $K[L]$. If L is generated in the same degree and its Rees algebra is normal, then we obtain the normality of $K[L]$ ([12]).

In Section 3, the normality of $K[L]$ and $\mathcal{R}(L)$ is studied. Moreover, we give a geometric description of $\overline{\mathcal{R}(L)}$, see Proposition 3.3.

In Section 4, we focus on the Rees algebra of the edge ideal of a finite simple graph. The definition of expansion operator is motivated by constructions in various combinatorial contexts. Let G be a finite simple graph with vertex set $V(G) = \{x_1, \ldots, x_n\}$ and edge set $E(G)$, and let $I(G)$ be its edge ideals in S.

All graphs in this paper are simple finite undirected. We fix a vertex x_j of G. Thus, a new graph G' is defined by duplicating x_j, that is, $V(G') = V(G) \cup \{x_{j'}\}$ and

$$E(G') = E(G) \cup \{\{x_i, x_{j'}\} : \{x_i, x_j\} \in E(G)\}$$

where $x_{j'}$ is new vertex. Therefore, $I(G') = I(G) + (x_i x_{j'} : \{x_i, x_j\} \in E(G))$. This duplication can be iterated. The graph which is obtained from G by m_j duplications of x_j is denoted by $G^{(m_1,\dots,m_n)}$. Then edge ideal of $G^{(m_1,\dots,m_n)}$ can be described as follows: let P_j be the monomial prime ideal $(x_{j1}, \dots, x_{jm_j}) \subseteq T$. Hence,

$$I(G^{(m_1,\dots,m_n)}) = \sum_{\{x_i,x_j\} \in E(G)} P_i P_j.$$

Let $\mathcal{R}(I(G)) = \bigoplus_{k=0}^{\infty} I(G)^k t^k$ be the Rees algebra of the edge ideal $I(G)$. In Theorem 4.3 it is shown that the Rees algebra $\mathcal{R}(I(G))$ is normal if and only if $\mathcal{R}(I(G^{(m_1,\dots,m_n)}))$ is normal. The subring

$$K[G] = K[x_i x_j \mid \{x_i, x_j\} \quad \text{is an edge of } G] \subset S$$

is called the edge subring of G. In Proposition 4.4, we prove that $K[G^{(m_1,\dots,m_n)}]$ is normal if G is bipartite. We also give a formula to compute the dimension of $K[G^{(m_1,\dots,m_n)}]$, see Theorem 4.5.

2. Integral closure and normality of generalized mixed product ideals

Fix an integer $n > 0$ and set $[n] = \{1, 2, \dots, n\}$. Let $\mathbb{R}^n_+$ denote the set of those vectors $\mathbf{u} = (u_1, \dots, u_n) \in \mathbb{R}^n$ with each $u_i \geq 0$. Hence, in particular $\mathbf{u}(\{i\})$, or simply $\mathbf{u}(i)$, is the ith component u_i of $\mathbf{u}$.

Let $S = K[x_1, \dots, x_n]$ be the polynomial ring over a field K in the variables $x_1, \dots, x_n$, and let $I \subset S$ be a monomial ideal with $I \neq S$ whose minimal set of generators is $G(I) = \{\mathbf{x}^{\mathbf{a}_1}, \dots, \mathbf{x}^{\mathbf{a}_m}\}$. Here $\mathbf{x}^{\mathbf{a}} = x_1^{\mathbf{a}(1)} x_2^{\mathbf{a}(2)} \cdots x_n^{\mathbf{a}(n)}$ for $\mathbf{a} = (\mathbf{a}(1), \dots, \mathbf{a}(n)) \in \mathbb{N}^n$. For a subset $D \subseteq S$, we define the *exponent set* of D by $E(D) := \{\mathbf{d} : \mathbf{x}^{\mathbf{d}} \in D\} \subseteq \mathbb{N}^n$.

Next we consider the polynomial ring T over K in the variables

$$x_{11}, \dots, x_{1m_1}, x_{21}, \dots, x_{2m_2}, \dots, x_{n1}, \dots, x_{nm_n}.$$

In [4], the authors introduced the generalized mixed product ideals. For $i = 1, \dots, n$ and $j = 1, \dots, m$ let $L_{i,\mathbf{a}_j(i)}$ be a monomial ideal in the variables $x_{i1}, x_{i2}, \dots, x_{im_i}$ such that

$$L_{i,\mathbf{a}_j(i)} \subset L_{i,\mathbf{a}_k(i)} \quad \text{whenever} \quad \mathbf{a}_j(i) \geq \mathbf{a}_k(i). \tag{1}$$

Given these ideals we define for $j = 1, \dots, m$ the monomial ideals

$$L_j = \prod_{i=1}^{n} L_{i,\mathbf{a}_j(i)} \subset T, \tag{2}$$

and set $L = \sum_{j=1}^{m} L_j$. The ideal L is called a *generalized mixed product ideal* induced by I.

Example 2.1. Let $L = L_{1,q}L_{2,r} + L_{1,s}L_{2,t}$ be the generalized mixed product ideal induced by a monomial ideal $I = (x_1^q x_2^r, x_1^s x_2^t)$, where for integers a and b, the ideal $L_{1,a}$ (resp. $L_{2,b}$) is the ideal generated by all squarefree monomials of degree a in the polynomial ring $K[x_{11}, \dots, x_{1m_1}]$ (resp. of degree b in the polynomial ring $K[x_{21}, \dots, x_{2m_2}]$), and where $0 < s < q \leq m_1$, $0 < r < t \leq m_2$. Ideals of this type are called squarefree Veronese ideals.

Now we want to study the combinatorial structure of the integral closure of generalized mixed product ideals. Let I be a monomial ideal of S. The set of all elements that are integral over I is called *the integral closure* of I, and is denoted by $\overline{I}$. If $I = \overline{I}$, then I is called *integrally closed.* In addition, the integral closure of a monomial ideal is again a monomial ideal. In [12], it is given the following description for the integral closure of I:

$$\overline{I} = (f \mid f \quad \text{is a monomial in S and} \quad f^k \in I^k, \text{for some} \quad k \geq 1).$$

If all the powers I^k are integrally closed, hence I is called a *normal ideal.*

Let $\mathbf{u} \in \mathbb{Q}_+^n$, where $\mathbb{Q}_+$ is the set of nonnegative rational numbers. We define the *upper right corner* or *ceiling* of $\mathbf{u}$ as the vector $\lceil \mathbf{u} \rceil$ whose entries are given by $\lceil \mathbf{u} \rceil_i$, where

$$\lceil \mathbf{u} \rceil_i = \begin{cases} \mathbf{u}_i & \text{if} \quad \mathbf{u}_i \in \mathbb{N} \\ \lfloor \mathbf{u}_i \rfloor + 1 & \text{if} \quad \mathbf{u}_i \notin \mathbb{N} \end{cases}$$

and where $\lfloor \mathbf{u}_i \rfloor$ stands for the integer part of $\mathbf{u}_i$. Accordingly, we can define the *ceiling* of any vector in $\mathbb{R}^n$ or the *ceiling* of any real number. Let $\text{conv}(\mathbf{v}_1, \dots, \mathbf{v}_q)$ be the convex hull (over the rationals), that is,

$$\text{conv}(\mathbf{v}_1, \dots, \mathbf{v}_q) = \left\{ \sum_{i=1}^{q} \lambda_i \mathbf{v}_i \;\middle|\; \sum_{i=1}^{q} \lambda_i = 1, \lambda_i \in \mathbb{Q}_+ \right\}$$

is the set of all *convex combinations* of $\mathbf{v}_1, \dots, \mathbf{v}_q$. For more information refer to [5, Definition 1.4.3, Propositon 1.4.6, and Definition 1.4.7]. For a monomial ideal $I \subset S$ with $G(I) = \{\mathbf{x}^{\mathbf{a}_1}, \dots, \mathbf{x}^{\mathbf{a}_m}\}$, we set $L(I; \{L_{ij}\}) = \sum_{j=1}^{m} \prod_{i=1}^{n} L_{i,\mathbf{a}_j(i)}$. Notice that a generalized mixed product ideal depends not only on I but also on the family L_{ij}.

In the following, we prove that L is integrally closed if $I \subset K[x_1, x_2]$ is a Veronese type ideal and the ideals who substitute the generators of I are squarefree Veronese ideals.

Theorem 2.2. *Let*

$$L = \sum_{1 \le q_l \le m_l, \sum_{l=1}^{2} q_l = h} L_{1,q_1} L_{2,q_2} \subset K[x_{11}, \dots, x_{1m_1}, x_{21}, \dots, x_{2m_2}]$$

be the generalized mixed product ideal, where the ideals L_{1,q_1} in $K[x_{11}, x_{12}, \dots, x_{1m_1}]$ and the ideals L_{2,q_2} in $K[x_{21}, x_{22}, \dots, x_{2m_2}]$ are squarefree Veronese ideals of degree q_1 and q_2, respectively. Then L is integrally closed.

Proof. Let

$$L = \sum_{1 \le q_l \le m_l, \sum_{l=1}^{2} q_l = h} L_{1,q_1} L_{2,q_2},$$

where the ideals
L_{1,q_1} in $K[x_{11}, x_{12}, \dots, x_{1m_1}]$ and the ideals L_{2,q_2} in $K[x_{21}, x_{22}, \dots, x_{2m_2}]$ are squarefree Veronese ideals of degree q_1 and q_2, respectively.

We set $\mathbf{c}_i = (c_{i1}, \dots, c_{im_1}) \in \mathbb{N}^{m_1}$, $\mathbf{d}_i = (d_{i1}, \dots, d_{im_2}) \in \mathbb{N}^{m_2}$, and

$$(\mathbf{c}_i, \mathbf{d}_i) = (c_{i1}, \dots, c_{im_1}, d_{i1}, \dots, d_{im_2}) \in \mathbb{N}^{m_1+m_2}.$$

Let $\mathbf{X}_1^{\mathbf{c}_1}\mathbf{X}_2^{\mathbf{d}_1}, \dots, \mathbf{X}_1^{\mathbf{c}_r}\mathbf{X}_2^{\mathbf{d}_r}$ be the generators of L, where $\mathbf{X}_1^{\mathbf{c}_i}\mathbf{X}_2^{\mathbf{d}_i}$ stands for

$$x_{11}^{c_{i1}} \cdots x_{1m_1}^{c_{im_1}} x_{21}^{d_{i1}} \cdots x_{2m_2}^{d_{im_2}}$$

with $\sum_{j=1}^{m_1} c_{ij} + \sum_{j=1}^{m_2} d_{ij} = h$, $0 \le c_{ij} \le 1$, $0 \le d_{ij} \le 1$ for $i = 1, \dots, r$, and $h \ge 2$. Then [13, Proposition 12.1.4] implies that

$$\overline{L} = \left(\left\{ \mathbf{X}_1^{\lceil \mathbf{p} \rceil} \mathbf{X}_2^{\lceil \mathbf{q} \rceil} \mid (\mathbf{p}, \mathbf{q}) \in \operatorname{conv}((\mathbf{c}_1, \mathbf{d}_1), \dots, (\mathbf{c}_r, \mathbf{d}_r)) \right\} \right),$$

where

$$\operatorname{conv}((\mathbf{c}_1, \mathbf{d}_1), \dots, (\mathbf{c}_r, \mathbf{d}_r)) = \left\{ \sum_{i=1}^{r} \lambda_i (\mathbf{c}_i, \mathbf{d}_i) \,\middle|\, \sum_{i=1}^{r} \lambda_i = 1, \lambda_i \in \mathbb{Q}_+ \right\}.$$

This is a geometric description of the integral closure of L. Let $f = \mathbf{X}_1^{\lceil \mathbf{p} \rceil} \mathbf{X}_2^{\lceil \mathbf{q} \rceil}$ be a generator of $\overline{L}$. Therefore,

$$(\mathbf{p}, \mathbf{q}) = \left(\sum_{i=1}^{r} \lambda_i c_{i1}, \dots, \sum_{i=1}^{r} \lambda_i c_{im_1}, \sum_{i=1}^{r} \lambda_i d_{i1}, \dots, \sum_{i=1}^{r} \lambda_i d_{im_2} \right) \in \mathbb{Q}_+^{m_1+m_2}.$$

If $\lambda_i \in \mathbb{N}$, then $\lambda_i = 1$ and $\lambda_j = 0$ for all $j \neq i$. Thus, $\mathbf{X}_1^{\lceil \mathbf{p} \rceil} \mathbf{X}_2^{\lceil \mathbf{q} \rceil} = \mathbf{X}_1^{\mathbf{c}_i}\mathbf{X}_2^{\mathbf{d}_i}$ for some i with $1 \le i \le r$. If $\lambda_i \in \mathbb{Q}_+ \setminus \mathbb{N}$ with $\sum_{i=1}^{r} \lambda_i = 1$, then we have a monomial $\mathbf{X}_1^{\lceil \mathbf{p} \rceil} \mathbf{X}_2^{\lceil \mathbf{q} \rceil}$ with $\lceil \mathbf{p} \rceil \ge \mathbf{c}_i$ with respect to the order on $\mathbb{Q}_+^{m_1}$ and $\lceil \mathbf{q} \rceil \ge \mathbf{d}_i$ with respect to the order on $\mathbb{Q}_+^{m_2}$, where $\mathbf{p}_i \geq c_{ij}$ and $\mathbf{q}_i \geq d_{ij}$. Hence, the monomial $\mathbf{X}_1^{\lceil \mathbf{p} \rceil}\mathbf{X}_2^{\lceil \mathbf{q} \rceil}$ is divided by $\mathbf{X}_1^{\mathbf{c}_i}\mathbf{X}_2^{\mathbf{d}_i}$ for some $1 \leq i \leq r$. Thus, $\overline{L}$ is generated by $\mathbf{X}_1^{\mathbf{c}_i}\mathbf{X}_2^{\mathbf{d}_i}$ for all $1 \leq i \leq r$ and by $\mathbf{X}_1^{\lceil \mathbf{p} \rceil}\mathbf{X}_2^{\lceil \mathbf{q} \rceil}$ with $\lceil \mathbf{p} \rceil \geq \mathbf{c}_i$ and $\lceil \mathbf{q} \rceil \geq \mathbf{d}_i$. Therefore,

$$G(\overline{L}) = \left\{ \mathbf{X}_1^{\mathbf{c}_1}\mathbf{X}_2^{\mathbf{d}_1}, \ldots, \mathbf{X}_1^{\mathbf{c}_r}\mathbf{X}_2^{\mathbf{d}_r} \right\},$$

and hence $\overline{L} = L$. □

Example 2.3. Let $L = L_{1,1}L_{2,2} + L_{1,2}L_{2,1} \subset K[x_{11}, x_{12}, x_{21}, x_{22}]$ be a generalized mixed product ideal, where $L_{1,1} = (x_{11}, x_{12})$, $L_{1,2} = (x_{11}x_{12})$, $L_{2,1} = (x_{21}, x_{22})$ and $L_{2,2} = (x_{21}x_{22})$. Therefore, Theorem 2.2 implies that

$$\overline{L} \quad = \quad \left(\left\{ \mathbf{X}_1^{\lceil \mathbf{p} \rceil}\mathbf{X}_2^{\lceil \mathbf{q} \rceil} \;\middle|\; (\mathbf{p}, \mathbf{q}) \in \operatorname{conv}((\mathbf{c}_1, \mathbf{d}_1), (\mathbf{c}_2, \mathbf{d}_2), (\mathbf{c}_3, \mathbf{d}_3), (\mathbf{c}_4, \mathbf{d}_4) \right\} \right),$$

where $(\mathbf{c}_1, \mathbf{d}_1) = (1, 0, 1, 0) \in \mathbb{Z}_+^4$, $(\mathbf{c}_2, \mathbf{d}_2) = (1, 0, 0, 1) \in \mathbb{Z}_+^4$, $(\mathbf{c}_3, \mathbf{d}_3) = (0, 1, 1, 0) \in \mathbb{Z}_+^4$ and $(\mathbf{c}_4, \mathbf{d}_4) = (0, 1, 0, 1) \in \mathbb{Z}_+^4$. It follows that

$$L = \overline{L} = (x_{11}x_{21}, x_{11}x_{22}, x_{12}x_{21}, x_{12}x_{22}).$$

The *support* of a monomial $f = x_1^{a_1} \cdots x_n^{a_n}$, denoted by $\operatorname{supp}(f)$, is a subset of the set of variables given by

$$\operatorname{supp}(f) = \{x_i \mid a_i > 0\}.$$

For $\mathbf{a} = (\mathbf{a}(1), \ldots, \mathbf{a}(n)) \in \mathbb{Z}_+^n$, we set

$$GL(\mathbf{a}) = \left\{ \mathbf{b} \in \mathbb{Z}_+^{m_1 + \cdots + m_n} \mid \mathbf{X}^{\mathbf{b}} \in G(L(\mathbf{x}^{\mathbf{a}}; \{L_{ij}\})) \right\}.$$

In addition, for all $\mathbf{a} \in E(G(I))$, we define $\mathbf{X}^{GL(\mathbf{a})}$ for the set of monomials

$$\left\{ \mathbf{X}^{\mathbf{b}} \mid \mathbf{b} \in GL(\mathbf{a}) \right\},$$

where $E(G(I))$ denotes the exponent set of $G(I)$. Thus, $L(I; \{L_{ij}\})$ is a monomial ideal of T generated by the monomials $\mathbf{X}^{\mathbf{b}} = \prod_{i=1}^n x_{i1}^{b_{i1}} \cdots x_{im_i}^{b_{im_i}}$, where

$$\mathbf{b} = (b_{11}, \ldots, b_{1m_1}, b_{21}, \ldots, b_{2m_2}, \ldots, b_{n1}, \ldots, b_{nm_n}) \in GL(\mathbf{a})$$

for all $\mathbf{a} \in E(G(I))$.

Remark 2.4. (a) Let

$$L(I; \{L_{ij}\}) = \sum_{j=1}^{m} \prod_{i=1}^{n} L_{i, \mathbf{a}_j(i)} \subset T = K[x_{11}, \ldots, x_{1m_1}, \ldots, x_{n1}, \ldots, x_{nm_n}]$$

be a generalized mixed product ideal, induced by the monomial ideal I with $G(I) = \{\mathbf{x}^{\mathbf{a}_1}, \ldots, \mathbf{x}^{\mathbf{a}_m}\}$, where the ideals $L_{i, \mathbf{a}_j(i)} \subset T_i = K[x_{i1}, x_{i2}, \ldots, x_{im_i}]$ are squarefree Veronese ideals of degree $\mathbf{a}_j(i)$. Let $A = T[x_{iv}^{-1}]$ be the Laurent polynomial ring for some x_{iv}. We denote by $L'_{i,\mathbf{a}_j(i)}$ the ideal of $T'_i = K[x_{i1}, \dots, \widehat{x_{iv}}, \dots, x_{im_i}]$ generated by all the squarefree monomials of T'_i of degree $\mathbf{a}_j(i) - 1$. Hence, $L_{i,\mathbf{a}_j(i)}A = L'_{i,\mathbf{a}_j(i)}A$. In fact one has $L_{i,\mathbf{a}_j(i)} \subset L'_{i,\mathbf{a}_j(i)}$, hence $L_{i,\mathbf{a}_j(i)}A \subset L'_{i,\mathbf{a}_j(i)}A$. On the other hand consider a monomial f in $L'_{i,\mathbf{a}_j(i)}$, then $x_{iv}f \in L_{i,\mathbf{a}_j(i)}$ and $f \in L_{i,\mathbf{a}_j(i)}A$. Therefore, $L'_{i,\mathbf{a}_j(i)}A \subset L_{i,\mathbf{a}_j(i)}A$.

(b) If a variable x_{iv} is not in a prime ideal $\wp \subset T$, then the localization of $L_{i,\mathbf{a}_j(i)}$ at $\wp$ is the same as the localization of $L'_{i,\mathbf{a}_j(i)}$ at $\wp$.

In [7], the author studied the normality of L, where the ideals substituting the monomials in I are all powers of the maximal ideals.

Theorem 2.5. [7, Theorem 3.3] *Let $L(I;\{L_{ij}\}) = \sum_{j=1}^{m} \prod_{i=1}^{n} L_{i,\mathbf{a}_j(i)} \subset T$ be a generalized mixed product ideal, induced by the monomial ideal I with $G(I) = \{\mathbf{x}^{\mathbf{a}_1}, \dots, \mathbf{x}^{\mathbf{a}_m}\}$, where the ideals $L_{i,\mathbf{a}_j(i)} \subset K[x_{i1}, x_{i2}, \dots, x_{im_i}]$ are Veronese ideals of degree $\mathbf{a}_j(i)$. Then I is normal if and only if $L(I;\{L_{ij}\})$ is normal.*

Next we study the normality of L, provided the ideals substituting the monomials in I are squarefree Veronese. We set

$$L' = \sum_{1 \le q_l \le m_l, \sum_{l=1}^{2} q_l = h} L'_{1,q_1} L_{2,q_2},$$

where the ideals L_{1,q_1} in $K[x_{11}, x_{12}, \dots, x_{1m_1}]$ and the ideals L_{2,q_2} in $K[x_{21}, x_{22}, \dots, x_{2m_2}]$ are squarefree Veronese ideals of degree q_1 and q_2, respectively. Therefore, L' is a monomial ideal of the ring $K[x_{11}, \dots, \widehat{x_{1v}}, \dots, x_{1m_1}, x_{21} \dots, x_{2m_2}]$ generated by all the squarefree monomials

$$x_{11}^{a'_{11}} \cdots x_{1v-1}^{a'_{1v-1}} x_{1v+1}^{a'_{1v+1}} \cdots x_{1m_1}^{a'_{1m_1}} x_{21}^{a_{21}} \cdots x_{2m_2}^{a_{2m_2}}$$

of degree $h-1$. Similar considerations hold for

$$L'' = \sum_{1 \le q_l \le m_l, \sum_{l=1}^{2} q_l = h} L_{1,q_1} L''_{2,q_2} \subset K[x_{11}, \dots, x_{1m_1}, x_{21} \dots, \widehat{x_{2v'}}, \dots, x_{2m_2}].$$

Face ideals were introduced in [13, Definition 6.1.2]. A *face ideal* is an ideal $\wp$ of T generated by a subset of the set of variables.

Lemma 2.6. *Let $L = \sum_{1 \le q_l \le m_l, \sum_{l=1}^{2} q_l = h} L_{1,q_1} L_{2,q_2} \subset K[x_{11}, \dots, x_{1m_1}, x_{21}, \dots, x_{2m_2}]$, where the ideals L_{1,q_1} in $K[x_{11}, x_{12}, \dots, x_{1m_1}]$, the ideals L_{2,q_2} in $K[x_{21}, x_{22}, \dots, x_{2m_2}]$ are squarefree Veronese ideals of degree q_1 and q_2, respectively.*

Furthermore, let $\wp \subset K[x_{11}, \dots, x_{1m_1}, x_{21}, \dots, x_{2m_2}]$ be a face ideal such that $x_{1v} \notin \wp$ for some v (resp. $x_{2v'} \notin \wp$ for some v'). Then

$$LK[x_{11}, \dots, x_{1m_1}, x_{21}, \dots, x_{2m_2}]_\wp = L'K[x_{11}, \dots, x_{1m_1}, x_{21}, \dots, x_{2m_2}]_\wp$$

$(resp.\quad LK[x_{11},\dots,x_{1m_1},x_{21},\dots,x_{2m_2}]_\wp = L''K[x_{11},\dots,x_{1m_1},x_{21},\dots,x_{2m_2}]_\wp).$

Proof. For simplicity of notation we assume that $v=1$. Take a monomial

$$f = x_{12}^{a'_{12}} \cdots x_{1m_1}^{a'_{1m_1}} x_{21}^{a_{21}} \cdots x_{2m_2}^{a_{2m_2}}$$

of L' of degree $h-1$. As $a'_{12}+\cdots+a'_{1m_1}+a_{21}+\cdots+a_{2m_2}=h-1$ and $a'_{1t}\le 1$ for all $t\ge 2$, we have $x_{11}f\in L$ and $f\in LK[x_{11},\dots,x_{1m_1},x_{21},\dots,x_{2m_2}]_\wp$. Therefore,

$$L'K[x_{11},\dots,x_{1m_1},x_{21},\dots,x_{2m_2}]_\wp \subseteq LK[x_{11},\dots,x_{1m_1},x_{21},\dots,x_{2m_2}]_\wp.$$

Conversely, take a squarefree monomial g of L of degree h. By Remark 2.4, we have $g\in L'K[x_{11},\dots,x_{1m_1},x_{21},\dots,x_{2m_2}]_\wp$. It follows that

$$LK[x_{11},\dots,x_{1m_1},x_{21},\dots,x_{2m_2}]_\wp \subseteq L'K[x_{11},\dots,x_{1m_1},x_{21},\dots,x_{2m_2}]_\wp,$$

as desired. □

We now come to the main result of the present paper.

Theorem 2.7. *Let*

$$L = \sum_{1\le q_l\le m_l,\sum_{l=1}^{2} q_l=h} L_{1,q_1}L_{2,q_2}$$

be the generalized mixed product ideal, where the ideals L_{1,q_1} in $K[x_{11},x_{12},\dots,x_{1m_1}]$ and the ideals L_{2,q_2} in $K[x_{21},x_{22},\dots,x_{2m_2}]$ are squarefree Veronese ideals of degree q_1 and q_2, respectively. Then L is normal.

Proof. Let L be the generalized mixed product ideal induced by a Veronese type ideal I generated by the monomials $x_1^{q_1}x_2^{q_2}$ with $1\le q_l\le m_l$ and $\sum_{l=1}^{2} q_l = h$. By induction on h we show that L is normal. If $h=2$, then [10, Theorem 2.9] implies that $L=L_{1,1}L_{2,1}$ is normal.

Assume that $h>2$ and the result holds for the generalized mixed product ideal of degree less than h. Take any prime $\wp\neq\mathfrak{n}$ and pick $x_{1v}\notin\wp$, where $\mathfrak{n}$ is the maximal ideal $\mathfrak{n}=(x_{11},\dots,x_{1m_1},x_{21},\dots,x_{2m_2})$. By Lemma 2.6 we obtain

$$LK[x_{11},\dots,x_{1m_1},x_{21},\dots,x_{2m_2}]_\wp = L'K[x_{11},\dots,x_{1m_1},x_{21},\dots,x_{2m_2}]_\wp.$$

Thus, by the induction hypothesis, we conclude that L' is normal. According to [3, Proposition 4.2], we obtain $L'K[x_{11},\dots,x_{1m_1},x_{21},\dots,x_{2m_2}]_\wp$ is normal. Therefore,

$$LK[x_{11},\dots,x_{1m_1},x_{21},\dots,x_{2m_2}]_\wp$$

is normal for any prime ideal $\wp\neq\mathfrak{n}$.

Now we proceed by contradiction assuming $\overline{L^k} \cap (L^k : \mathfrak{n}) \neq L^k$ for some $k \geq 1$. This means we can choose a monomial f in $\overline{L^k} \setminus L^k$ such that $x_{1v}f \in L^k$ for all v. Hence, there are monomials $f_1, \ldots, f_k$ of degree h in L and satisfying the equality:

$$x_{11}f = gf_1 \cdots f_k,$$

where g is a monomial with $\deg(g) > 0$, because $f \in \overline{L^k}$. Notice that $x_{11} \notin \operatorname{supp}(g)$ because $f \notin L^k$. Hence, we may assume that $x_{11} \in \operatorname{supp}(f_1)$ and $g = x_{12}^{c_{i2}} x_{13}^{c_{i3}} \cdots x_{1q'}^{c_{iq'}} x_{21}^{d_{s1}} \cdots x_{2v}^{d_{sv}}$ and $c_{it}, d_{sz} > 0$. Observe that x_{1t} divides f_1 for all $2 \leq t \leq q'$. Otherwise, we can write

$$f = ((f_1 x_{1t})/x_{11}) f_2 \cdots f_k (g/x_{1t})$$

to derive $f \in L^k$, a contradiction. We distinguish two cases:

Case (I): x_{1t} does not divide f_w for some $2 \leq t \leq q'$ and $2 \leq w \leq k$. Then for each $x_{1v} \in \operatorname{supp}(f_w)$ with $v \neq t$, we have x_{1v} divides f_1. Otherwise, if x_{1v} does not divide f_1, then we have the equality

$$f = ((x_{1v} f_1)/x_{11}) f_2 \cdots f_{w-1} ((x_{1t} f_w)/x_{1v}) f_{w+1} \cdots f_k (g/x_{1t}),$$

where $(x_{1v}f_1)/x_{11} \in L$ and $(x_{1t}f_w)/x_{1v} \in L$. Therefore, $f \in L^k$, a contradiction. Hence, since we have already seen that also x_{1t} divides f_1, we obtain x_{1t} divides f_w, which is a contradiction.

Case (II): x_{1t} divides f_w for all $2 \leq t \leq q'$ and $2 \leq w \leq k$. Since x_{1t} divides f_1, it follows that $\deg_{x_{12}}(f) \geq k+1$, where $\deg_{x_{12}}(f)$ denotes the degree of f in the variable x_{12}. Recall that $x_{12}f \in L^k$, which by degree considerations readily implies $f \in L^k$, a contradiction.

Altogether we see that in both cases the equality $\overline{L^k} \cap (L^k : \mathfrak{n}) \neq L^k$ leads to a contradiction. Therefore, [13, Proposition 12.2.1] implies that L is normal, as desired. □

Example 2.8. Let $L = L_{1,1}L_{2,3} + L_{1,2}L_{2,2} + L_{1,3}L_{2,1} \subset K[x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}]$ be the generalized mixed product ideal induced by a monomial ideal

$$I = (x_1 x_2^3, x_1^2 x_2^2, x_1^3 x_2),$$

where for integers a and b, the ideal $L_{1,a}$ (resp. $L_{2,b}$) is the ideal generated by all squarefree monomials of degree a in the polynomial ring $K[x_{11}, x_{12}, x_{13}]$ (resp. of degree b in the polynomial ring $K[x_{21}, x_{22}, x_{23}]$). Therefore,

$$\begin{aligned} L \;=\; & (x_{11}x_{21}x_{22}x_{23}, x_{12}x_{21}x_{22}x_{23}, x_{13}x_{21}x_{22}x_{23}, x_{11}x_{12}x_{21}x_{22}, x_{11}x_{12}x_{21}x_{23},\\ & x_{11}x_{12}x_{22}x_{23}, x_{11}x_{13}x_{21}x_{22}, x_{11}x_{13}x_{21}x_{23}, x_{11}x_{13}x_{22}x_{23}, x_{12}x_{13}x_{21}x_{22},\\ & x_{12}x_{13}x_{21}x_{23}, x_{12}x_{13}x_{22}x_{23}, x_{11}x_{12}x_{13}x_{21}, x_{11}x_{12}x_{13}x_{22}, x_{11}x_{12}x_{13}x_{23}). \end{aligned}$$

Hence, Theorem 2.7 implies that $\overline{L^k} = L^k$ for all $k \geq 1$.

3. Normalization of generalized mixed product ideals

In this section, we want to study the normality of Rees algebras of generalized mixed product ideals. Let I be a graded ideal of $S = K[x_1, \dots, x_n]$ generated by homogeneous polynomials $f_1, \dots, f_r$ with $\deg f_1 = \deg f_2 = \cdots = \deg f_r$. Let t be a variable over S. The graded subalgebra

$$\mathcal{R}(I) := \bigoplus_{k=0}^{\infty} I^k t^k = S[f_1 t, \dots, f_r t]$$

of $S[t]$ is called the *Rees algebra* of I.

The integral closure $\overline{\mathcal{R}(I)}$ of the Rees algebra in its field of fractions is called *normalization* of I. It is well-known ([11]) that is the graded algebra:

$$\overline{\mathcal{R}(I)} = S \oplus \overline{I}t \oplus \cdots \oplus \overline{I^k}t^k \oplus \cdots ,$$

where $\overline{I^k}$ is the integral closure of I^k. The ring $\mathcal{R}(I)$ is said to be normal if $\mathcal{R}(I)$ is equal to its integral closure. Therefore, $\mathcal{R}(I)$ is normal if and only if I is normal.

Proposition 3.1. *Let L be the generalized mixed product ideal induced by a monomial ideal I with $G(I) = \{\mathbf{x}^{\mathbf{a}_1}, \dots, \mathbf{x}^{\mathbf{a}_m}\}$, where the ideals $L_{i,\mathbf{a}_j(i)}$ are Veronese ideals of degree $\mathbf{a}_j(i)$. Then I is normal if and only if $\mathcal{R}(L)$ is normal.*

Proof. Let $L(I; \{L_{ij}\}) = (f_1, \dots, f_r)$, and let $\mathcal{R}(L(I; \{L_{ij}\}))$ be the subring of $T[t]$ given by $T[f_1 t, \dots, f_r t]$, where t is a new variable. Notice that

$$\mathcal{R}(L(I; \{L_{ij}\})) = T \oplus Lt \oplus \cdots \oplus L^k t^k \oplus \cdots \subset T[t]$$

is a graded algebra. By [13, Theorem 4.3.17] the normality of an ideal $L(I; \{L_{ij}\})$ of the polynomial ring T is equivalent to the normality of its Rees algebra. Therefore, using [13, Theorem 4.3.17] and Theorem 2.5, the assertion follows. □

Now we consider the case that all L_{ij} are squarefree Veronese ideals.

Proposition 3.2. *Let $L = \sum_{1 \leq q_l \leq m_l, \sum_{l=1}^{2} q_l = h} L_{1,q_1} L_{2,q_2}$ be the generalized mixed product ideal, where the ideals L_{1,q_1} in $K[x_{11}, x_{12}, \dots, x_{1m_1}]$ and the ideals L_{2,q_2} in*

$K[x_{21}, x_{22}, \dots, x_{2m_2}]$ *are squarefree Veronese ideals of degree* q_1 *and* q_2*, respectively. Then* $\mathcal{R}(L)$ *is normal.*

Proof. The assertion follows by Theorem 2.7 and [13, Theorem 4.3.17]. □

Next we study the combinatorics of the normalization of generalized mixed product ideals. Let $\mathcal{V} = \{v_1, \dots, v_q\}$ be a set of vectors in $\mathbb{N}^n \backslash \{0\}$. The *integral closure* or *normalization* of the affine semigroup

$$\mathbb{N}\mathcal{V} := \mathbb{N}v_1 + \cdots + \mathbb{N}v_q \subset \mathbb{N}^n,$$

is defined as $\overline{\mathbb{N}\mathcal{V}} := \mathbb{Z}\mathcal{V} \cap \mathbb{R}_+\mathcal{V}$, where $\mathbb{Z}\mathcal{V}$ is the subgroup of $\mathbb{Z}^n$ generated by $\mathcal{V}$. The semigroup $\mathbb{N}\mathcal{V}$ is called *normal* or *integrally closed* if $\overline{\mathbb{N}\mathcal{V}} = \mathbb{N}\mathcal{V}$.

Let I be a monomial ideal of S minimally generated by the set

$$G(I) = \{\mathbf{x}^{\mathbf{a}_1}, \dots, \mathbf{x}^{\mathbf{a}_m}\}.$$

The *Rees cone* of I is the rational polyhedral cone on $\mathbb{R}^{n+1}$, denoted by $\mathbb{R}_+E(G(I))'$ or $\mathbb{R}_+(I)$, generated by

$$E(G(I))' := \{e_1, \dots, e_n, (\mathbf{a}_1, 1), \dots, (\mathbf{a}_m, 1)\} \subset \mathbb{R}^{n+1},$$

where e_i is the ith unit vector.

Now let L be the generalized mixed product ideal induced by a monomial ideal I. More precisely let $G(L) = \left\{\mathbf{X}^{\mathbf{b}_1}, \dots, \mathbf{X}^{\mathbf{b}_r}\right\}$ and $E(G(L))$ be the set of exponent vectors of the generators of L. As usual we use $\mathbf{X}^{\mathbf{u}}$ as an abbreviation for

$$\mathbf{X}^{\mathbf{u}} = \prod_{i=1}^{n} x_{i1}^{u_{i1}} \cdots x_{im_i}^{u_{im_i}},$$

where $\mathbf{u} = (u_{11}, \dots, u_{1m_1}, u_{21}, \dots, u_{2m_2}, \dots, u_{n1}, \dots, u_{nm_n})$ is in $\mathbb{Z}_+^{m_1+\cdots+m_n}$. We set

$$E(G(L))' = \{e_1, \dots, e_{m_1+\cdots+m_n}, (\mathbf{b}_1, 1), \dots, (\mathbf{b}_r, 1)\},$$

where e_i is the ith unit vector of $\mathbb{R}^{m_1+\cdots+m_n+1}$. On the other hand according to [11, Theorem 7.2.28] one has

$$\mathcal{R}(L) = K[\{\mathbf{X}^{\mathbf{u}}t^z \mid (\mathbf{u}, z) \in \mathbb{N}E(G(L))'\}],$$

where $\mathbb{N}E(G(L))'$ is the subsemigroup of $\mathbb{N}^{m_1+\cdots+m_n+1}$ generated by $E(G(L))'$, consisting of the linear combinations of $E(G(L))'$ with non-negative integer coefficients, and the integral closure of $\mathcal{R}(L)$ in its field of fractions can be expressed as

$$\overline{\mathcal{R}(L)} = K[\{\mathbf{X}^{\mathbf{u}}t^z \mid (\mathbf{u}, z) \in \mathbb{Z}^{m_1+\cdots+m_n+1} \cap \mathbb{R}_+E(G(L))'\}],$$

where $\mathbb{R}_+E(G(L))' \subseteq \mathbb{R}^{m_1+\cdots+m_n+1}$ consists of the linear combinations of $E(G(L))'$ with real coefficients. Therefore, $\mathcal{R}(L)$ is normal if and only if any of the following equivalent conditions hold:

(1) $\mathbb{N}E(G(L))' = \mathbb{Z}^{m_1+\cdots+m_n+1} \cap \mathbb{R}_+E(G(L))'$;
(2) $L^k = \overline{L^k}$ for all $k \geq 1$.

In the following, we give a description of the normalization of $\overline{\mathcal{R}(L)}$.

Proposition 3.3. *Let*

$$L = \sum_{1\leq q_l\leq m_l, \sum_{l=1}^{2} q_l=h} L_{1,q_1}L_{2,q_2},$$

where the ideals L_{1,q_1} in $K[x_{11}, x_{12}, \ldots, x_{1m_1}]$, the ideals L_{2,q_2} in $K[x_{21}, x_{22}, \ldots, x_{2m_2}]$ are squarefree Veronese ideals of degree q_1 and q_2, respectively. Then

$$\overline{\mathcal{R}(L)} = K[\{\mathbf{X}_1^{\mathbf{c}}\mathbf{X}_2^{\mathbf{d}}t^q \mid \mathbf{c} \in \mathbb{N}E(G(L_{1,q_1})), \mathbf{d} \in \mathbb{N}E(G(L_{2,q_2})), q \in \mathbb{N}\}],$$

where $E(G(L_{1,q_1}))$ (resp. $E(G(L_{2,q_2}))$) is the set of the exponent vectors of the monomials of L_{1,q_1} (resp. L_{2,q_2}) in the variables $x_{11}, \ldots, x_{1m_1}$ (resp. $x_{21}, \ldots, x_{2m_2}$).

Proof. According to Proposition 3.2 one has $\mathcal{R}(L)$ is normal. Hence, $\overline{\mathcal{R}(L)} = \mathcal{R}(L)$. We show that $\mathcal{R}(L) = K[\{\mathbf{X}_1^{\mathbf{c}}\mathbf{X}_2^{\mathbf{d}}t^q \mid \mathbf{c} \in \mathbb{N}E(G(L_{1,q_1})), \mathbf{d} \in \mathbb{N}E(G(L_{2,q_2})), q \in \mathbb{N}\}]$.

We assume that $\mathfrak{B} = K[\{\mathbf{X}_1^{\mathbf{c}}\mathbf{X}_2^{\mathbf{d}}t^q \mid \mathbf{c} \in \mathbb{N}E(G(L_{1,q_1})), \mathbf{d} \in \mathbb{N}E(G(L_{2,q_2})), q \in \mathbb{N}\}]$, where $E(G(L_{1,q_1})) = \{\mathbf{u}_1, \ldots, \mathbf{u}_r\}$ (resp. $E(G(L_{2,q_2})) = \{\mathbf{v}_1, \ldots, \mathbf{v}_s\}$) is the set of the exponent vectors of the monomials of L_{1,q_1} (resp. L_{2,q_2}) in the variables $x_{11}, \ldots, x_{1m_1}$ (resp. $x_{21}, \ldots, x_{2m_2}$).

By hypotheses $\mathbf{c} = \sum_{i=1}^{r} \alpha_i\mathbf{u}_i$ with $\alpha_i \in \mathbb{N}$, $\mathbf{u}_i \in E(G(L_{1,q_1}))$, $\mathbf{d} = \sum_{i=1}^{s} \beta_i\mathbf{v}_i$ with $\beta_i \in \mathbb{N}$, $\mathbf{v}_i \in E(G(L_{2,q_2}))$, $q \in \mathbb{N}$. Then $\mathbf{X}_1^{\mathbf{c}} = \mathbf{X}_1^{\mathbf{u}_i}$ for all $1 \leq i \leq r$, and $\mathbf{X}_1^{\mathbf{c}} = f\mathbf{X}_1^{\mathbf{u}_i}$, where f is a monomial in the variables $x_{11}, \ldots, x_{1m_1}$, and $\mathbf{X}_2^{\mathbf{d}} = \mathbf{X}_2^{\mathbf{v}_i}$ for all $1 \leq i \leq s$, and $\mathbf{X}_2^{\mathbf{d}} = w\mathbf{X}_2^{\mathbf{v}_i}$, where w is a monomial in the variables $x_{21}, \ldots, x_{2m_2}$. Hence, the monomials $\mathbf{X}_1^{\mathbf{c}}\mathbf{X}_2^{\mathbf{d}}t^q$ of minimal degree are the generators of $\mathcal{R}(L)$, as desired. □

Suppose that $L(I; \{L_{ij}\}) = (f_1, \ldots, f_r)$. The monomial subring spanned by $\{f_1, \ldots, f_r\}$ is the K-subalgebra $K[L(I; \{L_{ij}\})] = K[f_1, \ldots, f_r]$.

The integral closure of $K[L(I; \{L_{ij}\})]$ in its field of fractions is called *normalization* of $K[L(I; \{L_{ij}\})]$. In addition, we denote $\overline{K[L(I; \{L_{ij}\})]}$ for the integral closure of $K[L(I; \{L_{ij}\})]$. The toric ring $K[L(I; \{L_{ij}\})]$ is said to be normal if

$$\overline{K[L(I; \{L_{ij}\})]} = K[L(I; \{L_{ij}\})].$$

Proposition 3.4. *Let* $L = \sum_{1\leq q_l\leq m_l, \sum_{l=1}^{2} q_l=h} L_{1,q_1}L_{2,q_2}$, *where for integers* q_1 *and* q_2, *the ideal* L_{1,q_1} *(resp.* L_{2,q_2}*) is the ideal generated by all squarefree monomials of degree* q_1 *in the polynomial ring* $K[x_{11},\ldots,x_{1m_1}]$ *(resp. of degree* q_2 *in the polynomial ring* $K[x_{21},\ldots,x_{2m_2}]$*). Then* $K[L]$ *is normal.*

Proof. Suppose that $L = (f_1,\ldots,f_r)$ be the generalized mixed product ideal induced by a monomial ideal I. Assume further that $q_1 + q_2 = h$. The monomial subring $K[f_1,\ldots,f_r]$ is a graded subring of $K[x_{11},\ldots,x_{1m_1},x_{21},\ldots,x_{2m_2}]$ with grading

$$K[f_1,\ldots,f_r]_k = K[f_1,\ldots,f_r] \cap K[x_{11},\ldots,x_{1m_1},x_{21},\ldots,x_{2m_2}]_k.$$

Since L is generated in the same degree h, by Proposition 3.2 together with [12, Proposition 7.4.1], we have $K[L]$ is normal. □

Proposition 3.5. *Let* L *be the generalized mixed product ideal induced by a monomial ideal* I *with* $G(I) = \{\mathbf{x}^{\mathbf{a}_1},\ldots,\mathbf{x}^{\mathbf{a}_m}\}$, *where the ideals* $L_{i,\mathbf{a}_j(i)}$ *are Veronese ideals of degree* $\mathbf{a}_j(i)$. *Assume that* I *is generated in the same degree* d. *Then* $K[L]$ *is normal if* I *is normal.*

Proof. Suppose that

$$\mathbb{F}\; 0 \to F_p \to F_{p-1} \to \cdots \to F_2 \to F_1 \to F_0 \to S/I \to 0$$

be the $\mathbb{Z}^n$-graded minimal free S-resolution of S/I.
We assume that $F_i = \bigoplus_{j=1}^{\beta_i} S(-\mathbf{a}_{ij})$ with $\mathbf{a}_{ij} \in \mathbb{N}^n$ for $i = 1,\ldots,n$. Thus, $F_i = \bigoplus_{j=1}^{\beta_i} Sf_{ij}$ where f_{ij} is a basis element of the free S-module F_i of $\mathbb{Z}^n$-degree $\mathbf{a}_{ij}$. Let ∂ denote the chain map of $\mathbb{F}$. Then

$$\partial(f_{ij}) = \sum_k \lambda_{kj}^{(i)} \mathbf{x}^{\mathbf{a}_{ij}-\mathbf{a}_{i-1,k}} f_{i-1,k}.$$

Here $\lambda_{kj}^{(i)} = 0$ if $\mathbf{a}_{ij} = \mathbf{a}_{i-1,k}$ or $\mathbf{a}_{ij} - \mathbf{a}_{i-1,k} \notin \mathbb{N}^n$. The matrices $(\lambda_{kj}^{(i)})_{\substack{k=1,\ldots,\beta_{i-1}\\ j=1,\ldots,\beta_i}}$ are scalar matrices of the resolution $\mathbb{F}$. Now we choose for each of the generators $\mathbf{x}^{\mathbf{a}_j}$ of I a monomial ideal L_j in T (not necessary of the form (2)). The multi-graded free resolution $\mathbb{F}$ of I are used to construct an acyclic complex $\mathbb{F}^*$ of direct sums of ideals. We set $F_0^* = T$ and $F_i^* = \bigoplus_{j=1}^{\beta_i} L_{ij}$ where the monomial ideals L_{ij} are inductively defined as follows: we assume that $L_{1j} = L_j$ for all j. Suppose that $L_{i-1,j}$ is already defined for all j. For a given number j with $1 \leq j \leq \beta_i$, let $k_1, k_2, \ldots, k_r$ be the numbers for which $\lambda_{k_t j}^{(i)} \neq 0$. In addition, we set $L_{ij} = \bigcap_{t=1}^{r} L_{i-1,k_t}$. The chain map ∂^* of $\mathbb{F}^*$ is given by

$$\partial^* \bigoplus_{j=1}^{\beta_i} L_{ij} \longrightarrow \bigoplus_{j=1}^{\beta_{i-1}} L_{i-1,j}, \quad u \mapsto \lambda^{(i)} u,$$

where

$$u = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_{\beta_i} \end{pmatrix} \quad \text{with} \quad u_j \in L_{ij}.$$

Hence, $\partial^*(\bigoplus_{j=1}^{\beta_i} L_{ij}) \subset \bigoplus_{j=1}^{\beta_{i-1}} L_{i-1,j}$. Let $v \in \bigoplus_{j=1}^{\beta_i} L_{ij}$ be a column vector. Suppose that $v_\ell = 0$ for $\ell \neq j$. Thus,

$$\partial^*(v) = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_{\beta_{i-1}} \end{pmatrix},$$

where $u_k = \lambda_{kj}^{(i)} v_j$ for $k = 1, \ldots, \beta_{i-1}$.

Next we show that L is generated in degree d if and only if I is generated in degree d. By [4, Lemma 2.4], $\partial^*(F_2^*) \subset \mathfrak{n} F_1^*$ where $\mathfrak{n}$ is the graded maximal ideal of T. This then implies $\bigoplus_j L_j/\mathfrak{n}L_j \cong L/\mathfrak{n}L$. Our assumptions on the ideals $L_{i,\mathbf{a}_{j(i)}}$ imply that L_j is minimally generated in degree $|\mathbf{a}_j|$. Hence, it follows that L has generators exactly in the same degrees as I. Proposition 3.1 with [12, Proposition 7.4.1] guarantees that $K[L]$ is normal. Thus, the desired conclusion follows. □

4. Rees algebra of an edge ideal

The main goal of this section is to study monomial subrings associated to graphs. Let G be a finite simple graph with vertex set $V(G) = \{x_1, \ldots, x_n\}$ and edge set $E(G)$, and let $I(G)$ be its edge ideals in $S = K[x_1, \ldots, x_n]$. As usual we denote the Rees algebra of $I(G)$ by $\mathcal{R}(I(G))$.

In [2] Bayati and Herzog introduced the expansion functor in the category of finitely generated multigraded S-module. We assume that $S^{(m_1,\ldots,m_n)}$ be the polynomial ring over a field K in the variables

$$x_{11}, \ldots, x_{1m_1}, x_{21}, \ldots, x_{2m_2}, \ldots, x_{n1}, \ldots, x_{nm_n}.$$

Let $I \subset S$ be a monomial ideal minimally generated by $\mathbf{x}^{\mathbf{a}_1}, \ldots, \mathbf{x}^{\mathbf{a}_m}$, the *expansion* of I with respect to the n-tuple $(m_1, \ldots, m_n)$, is defined by $I^{(m_1,\ldots,m_n)} =$

$\sum_{j=1}^{m} \prod_{i=1}^{n} P_i^{\mathbf{a}_j(i)} \subset S^{(m_1,\ldots,m_n)}$ where P_i is the monomial prime ideal $(x_{i1},\ldots,x_{im_i}) \subseteq S^{(m_1,\ldots,m_n)}$ and $\mathbf{a}_j(i)$ is the i-th component of the vector $\mathbf{a}_j$.

Theorem 4.1. [1, Theorem 2.1] *Let I be a monomial ideal of a polynomial ring $S = K[x_1,\ldots,x_n]$. Then I is normal if and only if $I^{(m_1,\ldots,m_n)}$ is normal, where $I^{(m_1,\ldots,m_n)}$ denotes the expansion of I.*

For the n-tuple $(m_1,\ldots,m_n) \in \mathbb{N}^n$, with positive integer entries, the *expansion* of the graph G is denoted by $G^{(m_1,\ldots,m_n)}$. We consider the monomial prime ideal $P_j = (x_{j1},\ldots,x_{jm_j})$ in $S^{(m_1,\ldots,m_n)}$. Hence,

$$I(G^{(m_1,\ldots,m_n)}) = \sum_{\{x_i,x_j\}\in E(G)} P_iP_j.$$

It follows from [2, Lemma 1.1] that

$$I(G^{(m_1,\ldots,m_n)}) = \sum_{\{x_i,x_j\}\in E(G)} x_i^{(m_1,\ldots,m_n)} x_j^{(m_1,\ldots,m_n)} = I(G)^{(m_1,\ldots,m_n)}.$$

Example 4.2. Let G be a graph on the vertex set $V(G) = \{x_1,x_2,x_3,x_4\}$ and edge set $E(G) = \{\{x_1,x_3\},\{x_1,x_4\},\{x_2,x_3\},\{x_2,x_4\}\}$. We consider the polynomial ring T over K in the variables $x_{11},x_{21},x_{22},x_{31},x_{41},x_{42}$, and the order 4-tuple $(1,2,1,2)$. Hence, $G^{(1,2,1,2)}$ is a graph with vertex set $V\left(G^{(1,2,1,2)}\right) = \{x_{11},x_{21},x_{22},x_{31},x_{41},x_{42}\}$ and edge set

$$\begin{aligned} E\left(G^{(1,2,1,2)}\right) = \; & \{\{x_{11},x_{31}\},\{x_{11},x_{41}\},\{x_{11},x_{42}\},\{x_{21},x_{31}\},\{x_{22},x_{31}\},\{x_{21},x_{41}\},\\ & \{x_{21},x_{42}\},\{x_{22},x_{41}\},\{x_{22},x_{42}\}\}. \end{aligned}$$

Then we have $P_1 = (x_{11})$, $P_2 = (x_{21},x_{22})$, $P_3 = (x_{31})$ and $P_4 = (x_{41},x_{42})$. Therefore,

$$\begin{aligned} I(G^{(1,2,1,2)}) &= P_1P_3 + P_1P_4 + P_2P_3 + P_2P_4 \\ &= (x_{11}x_{31}, x_{11}x_{41}, x_{11}x_{42}, x_{21}x_{31}, x_{22}x_{31}, x_{21}x_{41}, x_{21}x_{42}, x_{22}x_{41}, x_{22}x_{42}). \end{aligned}$$

The ideal $I(G^{(1,2,1,2)}) \subset T$ is obtained from $I(G)$ by expansion with respect to the 4-tuple $(1,2,1,2)$ with positive integer entries.

Theorem 4.3. *Let G be a graph on the vertex set $V(G) = \{x_1,\ldots,x_n\}$. Fix an order n-tuple $(m_1,\ldots,m_n)$ of positive integers. Then the Rees algebra $\mathcal{R}(I(G))$ is normal if and only if $\mathcal{R}(I(G^{(m_1,\ldots,m_n)}))$ is normal.*

Proof. We assume that $\mathcal{R}(I(G)) = K[\{x_1,\ldots,x_n,tf_i \mid 1 \le i \le r\}]$ be the Rees algebra of $I(G) = (f_1,\ldots,f_r)$, where $f_1,\ldots,f_r$ are the monomials corresponding to the edges of G. Let k be a positive integer. If $\mathcal{R}(I(G))$ is normal, then by [13, Theorem 4.3.17] we obtain $I(G)$ is normal. Hence, $I(G)^k = \overline{I(G)^k}$. It is known [9, Lemma 2.2] that $I(G^{(m_1,\ldots,m_n)})^k$ is the expansion of $I(G)^k$ with respect to the n-tuple $(m_1, \ldots, m_n)$. Hence, [7, Lemma 2.3] together with [9, Lemma 2.2] now yields

$$I(G^{(m_1,\ldots,m_n)})^k = \overline{I(G^{(m_1,\ldots,m_n)})^k}.$$

Therefore, [13, Theorem 4.3.17] yields $\mathcal{R}(I(G^{(m_1,\ldots,m_n)})^k)$ is normal. Necessity follows in a similar way and the proof is complete. □

The *edge subring* of the graph G, denoted by $K[G]$, is the K-subalgebra of S given by:

$$K[G] = K[\{x_i x_j \mid x_i \quad \text{is adjacent to} \quad x_j\}] \subset S.$$

To obtain a presentation of the edge subring of G note that $K[G]$ is a standard K-algebra with the normalized grading $K[G]_i = K[G] \cap S_{2i}$.

Let I be a monomial ideal of S and $P_1, \ldots, P_r$ the minimal primes of I. Given an integer $k \geq 1$, the kth *symbolic power* of I is defined to be the ideal $I^{(k)} = Q_1 \cap \cdots \cap Q_r$, where Q_i is the primary component of I^k corresponding to P_i. The reader can find more information in [13, Definition 4.3.22].

Proposition 4.4. *Let G be a connected bipartite graph. Fix an order n-tuple $(m_1, \ldots, m_n)$ of positive integers. Then $K[G^{(m_1,\ldots,m_n)}]$ is normal.*

Proof. Let G be a connected bipartite graph and let $I(G)$ be its edge ideal. Let k be a positive integer. Thus, [13, Corollary 13.3.6] yields $I(G)^{(k)} = I(G)^k$. According to [9, Theorem 2.3] we have $I(G^{(m_1,\ldots,m_n)})^k$ is the expansion of $I(G)^k$ with respect to the n-tuple $(m_1, \ldots, m_n)$ and $I(G^{(m_1,\ldots,m_n)})^k = (I(G)^k)^{(m_1,\ldots,m_n)}$. Then [2, Corollary 1.5] implies that $I(G^{(m_1,\ldots,m_n)})^{(k)} = I(G^{(m_1,\ldots,m_n)})^k$. Now the result follows from [13, Corollary 13.3.6] and [13, Corollary 10.5.6]. □

Now we give a formula to compute the dimension of $K[G^{(m_1,\ldots,m_n)}]$.

Theorem 4.5. *If G is a connected graph with n vertices and $K[G]$ is its edge subring, then*

$$\dim(K[G^{(m_1,\ldots,m_n)}]) = \begin{cases} m_1 + \cdots + m_n & \textit{if } G \textit{ is not bipartite, and} \\ m_1 + \cdots + m_n - 1 & \textit{otherwise.} \end{cases}$$

Proof. We assume that G is a connected graph with r edges and n vertices. Let $I(G)$ be minimally generated by monomials $f_1, \ldots, f_r$. Then there is a spanning tree T of G so that $I(T) = (f_1, \ldots, f_{n-1})$ ([14]). Hence,

$$\dim(K[G]) \geq n - 1.$$

If G is bipartite, then by [13, Corollary 10.1.21] one has that $\dim(K[G]) = n-1$. Fix an order n-tuple $(m_1, \ldots, m_n)$ of positive integers. Let k be a positive integer. Thus, [2, Corollary 1.5] yields $I(G^{(m_1,\ldots,m_n)})^{(k)}$ is the expansion of $I(G)^{(k)}$ with respect to the n-tuple $(m_1, \ldots, m_n)$. Therefore, by [13, Corollary 13.3.6] together with [9, Theorem 2.3] we conclude that

$$I(G^{(m_1,\ldots,m_n)})^{(k)} = I(G^{(m_1,\ldots,m_n)})^k.$$

By [13, Corollary 10.1.21] we have

$$\dim(K[G^{(m_1,\ldots,m_n)}]) = m_1 + \cdots + m_n - 1.$$

If G is not bipartite, thus by [13, Corollary 10.1.21] we obtain $\dim(K[G]) = n$. Then [13, Corollary 13.3.6] implies that $I(G)^{(k)} \neq I(G)^k$. From [9, Lemma 2.2], and [2, Corollary 1.5], together with [13, Corollary 13.3.6] we have

$$I(G^{(m_1,\ldots,m_n)})^{(k)} \neq I(G^{(m_1,\ldots,m_n)})^k.$$

It follows from [13, Corollary 10.1.21] that

$$\dim(K[G^{(m_1,\ldots,m_n)}]) = m_1 + \cdots + m_n. \qquad \square$$

Acknowledgement. The authors would like to thank the referee for the valuable suggestions and comments.

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