International Electronic Journal of Algebra
Volume 36 (2024) 215-224
DOI: 10.24330/ieja.1470687

# A UNIFORM CHARACTERIZATION OF THE OCTONIONS AND THE QUATERNIONS USING COMMUTATORS

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Received: 1 February 2024; Accepted: 8 April 2024
Communicated by Abdullah Harmancı

Abstract. Let $R$ be a ring which is not commutative. Assume that either $R$ is alternative, but not associative, or $R$ is associative and any commutator $v \in R$ satisfies: $v^2$ is in the center of $R$. We show (using commutators) that if $R$ contains no divisors of zero and $\text{char}(R) \neq 2$, then $R//C$, the localization of $R$ at its center $C$, is the octonions in the first case and the quaternions, in latter case. Our proof in both cases is essentially the same and it is elementary and rather self contained. We also give a short (uniform) proof that if a non-zero commutator in $R$ is not a zero divisor (with mild additional hypothesis when $R$ is alternative, but not associative (e.g. that $(R, +)$ contains no 3-torsion), then $R$ contains no divisors of zero.

**Mathematics Subject Classification (2020)**: 12E15, 17D05, 17A35, 11R52
**Keywords**: Quaternion algebra, Octonion algebra, division algebra, zero divisor, commutator

## 1. Introduction

The main point of this paper is to reprove theorems [5, Main Theorem] and [1, Theorem A] in a uniform way using commutators. So, the quaternions and the octonions are treated in a uniform way. This is done in Theorem A. In [1, Theorem A], Bruck and Kleinfeld proved part (b) of Theorem A using associators.

We also use commutators in Theorem B to show that $R$ contains no divisors of zero. So we assume that a non-zero commutator is not a divisor of zero in $R$.

In case (a) of Theorem B we assume (as in case (a) of Theorem A) that $(x, y)^2 \in C$, for all $x, y \in R$, where $C$ is the center of $R$. In case (b) of Theorem B we use Proposition 2.2(3(ii)) to *deduce* that $(x, y)^2 \in C$, however, we need some extra assumption to guarantee that $R$ contains no divisors of zero.

**Theorem A.** *Let $R$ be a ring which is not commutative. Suppose that $R$ contains no divisors of zero, and that the characteristic of $R$ is not* 2.

*Assume that either*

(a) (Main Theorem of [5]) *$R$ is associative and $(x,y)^2 \in C$, where $C$ is the center of $R$*

*or*

(b) (Theorem A of [1]) *$R$ is an alternative ring which is not associative.*

*Then $R//C$, the localization of $R$ at its center $C$, is a quaternion division algebra in case* (a) *and an octonion division algebra in case* (b)*.*

We note that if $x, y \in R$ are non-zero elements such that $xy = 0$, then we say that both $x$ and $y$ are zero divisors in $R$. Recall also that the commutator $(x,y) = xy - yx$.

**Theorem B.** *Let $R$ be a unital ring which is not commutative. Let $Z$ be the commutative center of $R$, and $C$ be the center of $R$. Assume that*

(i) *A non-zero commutator in $R$ is not a divisor of zero in $R$, and*

(ii) *one of the following holds:*

  (a) *$R$ is an associative ring such that $(x,y)^2 \in C$, for all $x, y \in R$.*

  (b) *$R$ is an alternative ring which is not associative, and $Z = C$.*

*Then*

(1) *$R$ contains no divisors of zero.*

(2) *Suppose, in addition, that the characteristic of $R$ is not* 2*, and let $R//C$ be the localization of $R$ at $C$. If $R$ is as in* (a)*, then $R//C$ is a quaternion division algebra, and if $R$ is as in* (b)*, then $R//C$ is an octonion division algebra.*

**Remarks 1.1.** (1) Note that in part (a) of Theorem A, it is *assumed* that the center $C$ of $R$ is nontrivial. The fact that the center of $R$ is nontrivial in part (b) of Theorem A follows from Proposition 2.2(3(ii)).

(2) The hypothesis that $Z = C$, in part ii(b) of Theorem B, can be replaced with any hypothesis that together with the assumption that a non-zero commutator in $R$ is not a divisor of zero in $R$, guarantees that $R$ contains no divisors of zero (this is needed only in the case where $R$ is alternative, but not associative). A number of such hypotheses are listed in [4].

Indeed part (2) of Theorem B, follows from Theorem A.

(3) In [4, Lemma 2.3(3)] we proved that for an alternative ring $R$:

$$\text{If } 3x = 0 \implies x = 0, \text{ for all } x \in R, \text{ then } Z = C.$$

So if $3x = 0 \implies x = 0$, for all $x \in R$, then Theorem B holds in the case where $R$ is alternative but not associative.

## 2. Preliminaries on alternative rings

Our main references for alternative rings are [2,3]. Let $R$ be a ring, not necessarily with **1** and not necessarily associative.

**Definitions 2.1.** Let $x, y, z \in R$.

(1) The **associator** $(x, y, z)$ is defined to be

$$(x, y, z) = (xy)z - x(yz).$$

(2) The **commutator** $(x, y)$ is defined to be

$$(x, y) = xy - yx.$$

(3) $R$ is an **alternative ring** if

$$(x, y, y) = 0 = (y, y, x),$$

for all $x, y \in R$. It is well known and is a theorem of E. Artin, that $R$ is an alternative ring if and only if any subring of $R$ generated by two elements is associative. This fact will be used throughout this paper.

(4) The **nucleus** of $R$ is denoted $N$ and defined

$$N = \{n \in R \mid (n, R, R) = 0\}.$$

Note that in an alternative ring the associator is skew symmetric in its 3 variables ([3, Lemma 1]). Hence $(R, n, R) = (R, R, n) = 0$, for $n \in N$.

(5) The **center** of $R$ is denoted $C$ and defined

$$C = \{c \in N \mid (c, R) = 0\}.$$

(6) The commutative center of $R$ is denoted here by $Z$ and defined by

$$Z = \{z \in R \mid (z, R) = 0\}.$$

In the remainder of this section $R$ **is an alternative ring** which is **not associative**. $N$ denotes the nucleus of $R$ and $C$ its center.

**Proposition 2.2.** *Let $R$ be an alternative ring which is not associative, and let $v \in R$ be a commutator, then*

(1) $v^4 \in N$.

(2) $(v^2, R, R)v = 0$.

(3) (i) *If $v$ is not a zero divisor in $R$, then $v^2 \in N$.*

(ii) *If $(t, s)$ is not a zero divisor, for all non-commuting $t, s \in R$, then $N = C$, so $(t, s)^2 \in C$, for all $t, s \in R$.*

**Proof.** For (1)&(2) see [2, Theorem 3.1]. Part (3(i)) is an immediate consequence of (2).

For part (3(ii)), let $w \in R$ and $n \in N$, and consider the commutator $(w, n)$. Let $x, y, z \in R$, with $(x, y, z) \neq 0$. By [4, Lemma 2.4(5)],

$$(w, n)(w, n)(x, y, z) = 0.$$

Since $(w, n)$ is not a zero divisor, we must have $(w, n) = 0$. As this holds for all $w \in R$, we see that $n \in C$, thus $N = C$. Then, by (3(i)), $(t, s)^2 \in C$, for all $t, s \in R$. $\square$

## 3. The proof of Theorems A and B

In this section $R$ is an alternative ring which is not commutative. We assume that either

(A) $R$ is one of the two possibilities of Theorem A.

(B) $R$ is unital, it satisfies hypothesis (i) of Theorem B, and it is one of the two possibilities of hypothesis (ii) of Theorem B.

We let $C$ denote the center of $R$, and $Z$ denote the commutative center of $R$. Note that by Remark 1.1(1), $C \neq 0$.

Notice that by hypothesis and by Lemma 2.2(3(ii)),

$$(x, y)^2 \in C, \text{ for all } x, y \in R. \tag{3.1}$$

**Lemma 3.1.** *Assume that $R$ is as in hypothesis B. Let $x \in R \setminus C$. Then there exists $y \in R$ with $v := (x, y) \neq 0$, and*

(1) *$v + vx$ and $vx$ are non-zero commutators.*

(2) *$ax^2 + bx + c = 0$, for some $a, b, c \in C$, with $a, c$ non-zero.*

**Proof.** If $R$ is associative, then of course $Z = C$, while if $R$ is not associative, then $Z = C$ by hypothesis. Hence $Z = C$, so $y$ exists.

(1) We have $v + vx = v(\mathbf{1} + x) = (x, y(\mathbf{1} + x))$, and $vx = (x, yx)$. By hypothesis (i) of Theorem B, these commutators are non-zero.

(2) Let $\alpha := (v + vx)^2 = v^2 + v^2x + vxv + (vx)^2$. Then $\alpha \in C$. We have $\alpha x = v^2x^2 + (v^2 + (vx)^2)x + (vx)^2$. Letting

$$a := v^2, b := v^2 + (vx)^2 - (v + vx)^2 \text{ and } c := (vx)^2,$$

we see that $a, b, c \in C$, and $ax^2 + bx + c = 0$, with $a \neq 0 \neq c$. $\square$

**Theorem 3.2.** *$R$ contains no divisors of zero, and every non-zero element $x \in R$ satisfies a quadratic equation $ax^2 + bx + c = 0$, for some $a, b, c \in C$, with $a \neq 0$.*

**Proof.** If $R$ is as in hypothesis A, this follows from [1, Theorem 4.1].

Suppose $R$ is as in hypothesis B. We first show that $R$ contains no divisors of zero. Assume first that $cr = 0$, for some non-zero $c \in C$. Let $v := (x, y)$ be a non-zero commutator. Then $(vc)r = v(cr) = 0$, but $vc = (x, yc) \neq 0$, hence $r = 0$, so $c$ is not a divisor of 0.

Suppose next that $x \in R \smallsetminus C$, and that $xs = 0$, for some non-zero $s \in R$. Then we immediately get from Lemma 3.1(2) that $cs = 0$, a contradiction.

Hence $R$ contains no divisors of zero, so the theorem follows from Lemma 3.1(2). $\square$

**Remark 3.3.** In view of Theorem 3.2, we can form the *localization of $R$ at $C$*, $R//C$. This is the set of all formal fractions $x/c$, $x \in R$, $c \in C$, $c \neq 0$, with the obvious definitions: (i) $x/c = y/d$ if and only if $dx = cy$; (ii) $(x/c)+(y/d) = (dx+cy)/(cd)$; (iii) $(x/c)(y/d) = (xy)/(cd)$. Then $R//C$ is an alternative ring, and it is easy to check that the center of $R//C$ is the fraction field of $C$. Since $(x/c, y/d) = (x, y)/cd$ in $R//C$, we may replace $R$ with $R//C$ in Theorem B.

Note also that if $R$ is as in Theorem A, then $R//C$ is unital. Indeed if $\mathbf{1}$ is the identity of $C//C$ (the fraction field of $C$), then $(x \cdot \mathbf{1} - x)\mathbf{1} = 0$, so $x \cdot \mathbf{1} = x$.

Thus **from now on we replace $R$ with $R//C$ and assume that $R$ has 1 and that $C$ is a field. We also assume that** $\mathrm{char}(C) \neq 2$.

The following technical lemma will be used to construct an octonion division algebra inside $R$, when $R$ is alternative, but not associative.

**Lemma 3.4.** *Suppose $R$ is alternative but not associative. Let $a, b \in R \smallsetminus \{0\}$ be a pair of anticommutative elements of $R$.*

(1) *If $c \in R \smallsetminus \{0\}$ anticommutes with $a$ and $b$, then for every permutation $\sigma$ of $a, b, c$*
   (i) $\sigma(a)\big(\sigma(b)\sigma(c)\big) = (\operatorname{sgn}\sigma)a(bc)$.
   (ii) $\big(\sigma(a)\sigma(b)\big)\sigma(c) = (\operatorname{sgn}\sigma)(ab)c$.

(2) *If $a^2 \in C$, then $a$ anticommutes with $ab$.*

(3) *Suppose that $c \in R$ anticommutes with $a, b$ and $ab$. Let $\{x, y, z\} = \{a, b, c\}$, then*
   (i) *$x$ anticommutes with $yz$.*
   (ii) *$(xy)z = -x(yz)$, hence $(xy)z, x(yz) \in \{a(bc), -a(bc)\}$.*
   (iii) *If, in addition, $a^2, b^2, c^2 \in C$, then $\{a, b, c, ab, ac, bc, a(bc)\}$ is a set of pairwise anticommutative elements.*

**Proof.** (1) This appears in [3, Lemma 7, p. 134]. Since associators in $R$ are skew symmetric, $(a,b,c)+(b,a,c)=0$. Hence

$$0=(a,b,c)+(b,a,c)=(ab)c-a(bc)+(ba)c-b(ac)=-a(bc)-b(ac),$$

so $b(ac)=-a(bc)$ this shows 1(i), and the proof of 1(ii) is similar.

(2) $a(ab)=a^2b=ba^2=(ba)a=-(ab)a$.

3(i) We show that $a$ anticommutes with $bc$, the proof that $b$ anticommutes with $ac$ is similar. Using (1), we have

$$a(bc)=-c(ba)=(ba)c=-(bc)a.$$

3(ii) By 3(i) and (1), $(xy)z=-z(xy)=z(yx)=-x(yz)$. The last part of 3(ii) follows from (1).

3(iii) First we show that

$$xy \text{ anticommutes with } xz. \tag{$\alpha$}$$

Indeed, by (1) and (2), $(xy)(xz)=-(x(xz))y=-x^2(zy)$, since $x^2\in C$. Similarly $(xz)(xy)=-x^2(yz)$.

Next note that

$$x \text{ anticommutes with } x(yz). \tag{$\beta$}$$

This follows from (2) and 3(i), since $x^2\in C$.

Finally we show that

$$xy \text{ anticommutes with } x(yz). \tag{$\gamma$}$$

Indeed, by 3(ii) $(xy)(x(yz))=-(xy)((xy)z)=-(xy)^2z$. And by 3(ii) and 3(i), $(x(yz))(xy)=-((xy)z)(xy)=(z(xy))(xy)=(xy)^2z$, because $(xy)^2\in C$.

Notice now that 3(iii) follows from (2), 3(i), ($\alpha$), ($\beta$) and ($\gamma$). $\square$

The next proposition is the main tool in this paper. It is used to construct a quaternion division algebra $Q$ inside $R$, when $R$ is associative, and to prove that $R=Q$. It is also used to construct an octonion division algebra $O$ inside $R$, when $R$ is alternative but not associative, and to prove that $R=O$.

**Proposition 3.5.** *Let $u_1,u_2,\dots,u_n\in R\smallsetminus C$, $n\geq 2$, such that $u_i^2\in C$, and $u_\ell u_s=-u_s u_\ell$, for all distinct $\ell,s$. Let*

$$V:=C+Cu_1+\cdots+Cu_n,$$

*be the subspace of $R$ spanned by $\mathbf{1},u_1,\dots,u_n$.*

(1) *If $p \in R$ satisfies*

$$pu_\ell + u_\ell p := d_\ell \in C, \quad \text{for all } \ell \in \{1, \dots, n\}, \tag{$*$}$$

*then the element*

$$m := p - \textstyle\sum_{i=1}^n (d_i/2u_i^2)u_i, \tag{3.2}$$

*satisfies $mu_\ell + u_\ell m = 0$, for all $\ell \in \{1, \dots, n\}$.*

(2) *If $R \neq V$, then there exists $p \in R \smallsetminus V$ such that the element $m$ of equation* (3.2) *satisfies $mu_\ell + u_\ell m = 0$, for all $\ell \in \{1, \dots, n\}$, and $m^2 \in C$.*

**Proof.** (1) We show that $mu_1 + u_1 m = 0$, the proof for $u_2, \dots, u_n$ is identical.

$$mu_1 + u_1 m = pu_1 + u_1 p - d_1 - \sum_{i=2}^n (d_i/2u_i^2)(u_i u_1 + u_1 u_i) = 0.$$

(2) We show that there exists $p \in R \smallsetminus V$, such that $p^2 \in C$, and such that $p$ satisfies $(*)$.

Let $x \in R \smallsetminus V$. By Theorem 3.2, $x$ satisfies a quadratic, and hence (since $C$ is a field) a monic quadratic equation $x^2 - bx + c = 0$, over $C$. Let $p := x - b/2$. Then $p \notin V$, and $p^2 \in C$. Let $u \in \{u_1, \dots u_n\}$. Then both $p + u$ and $p - u$ satisfy a monic quadratic equation over $C$. That is

$$(p+u)^2 = c_1(p+u) + c_2$$
$$(p-u)^2 = c_3(p-u) + c_4.$$

Adding we get

$$(c_1 + c_3)p + (c_1 - c_3)u + c_5 = 0, \quad \text{where } c_5 = c_2 + c_4 - 2p^2 - 2u^2 \in C.$$

Now $c_1 + c_3 = 0$, since $p \notin V$, and then $c_1 - c_3 = 0$, since $u \notin C$. We thus get that

$$pu + up = c_2 - p^2 - u^2 \in C.$$

Let now $m$ be as in equation (3.2). For $i \in \{1, \dots, n\}$, set $\alpha_i := (d_i/2u_i^2) \in C$. Note that

$$m^2 = p^2 + \sum_{i=1}^m \alpha_i^2 u_i^2 + \sum_{i=1}^m \alpha_i (pu_i + u_i p) \in C.$$ □

Now we construct the quaternions and the octonions inside $R$ in the respective cases.

**Proposition 3.6.**

(1) *$R$ contains a quaternion division algebra.*

$$Q = C\mathbf{1} + Ca + Cb + Cab, \qquad a, b \in R \smallsetminus \{0\}, a^2, b^2 \in C.$$

(2) *If $R$ is alternative, then there exists $c \in R \setminus \{0\}$ that anticommutes with $a, b$ and $ab$ above, and such that $c^2 \in C$. Hence $R$ contains an octonion division algebra*

$$O := F\mathbf{1} + Cu_1 + Cu_2 + Cu_3 + Cu_4 + Cu_5 + Cu_6 + Cu_7,$$

*where*

$$u_1 := a, u_2 := b, u_3 = ab, u_4 := c, u_5 = ac, u_6 := bc, u_7 := (bc)a,$$

*so $u_\ell \neq 0, u_\ell^2 \in C$, and $u_\ell u_s = -u_s u_\ell$, for all $\ell, s$.*

**Proof.** (1) Let $a \in R$ be a non-zero commutator. Thus $a^2 \in C \setminus \{0\}$. Of course $R \neq F\mathbf{1} + Fa$, because $R$ is not commutative. By Proposition 3.5(2), there exists $b \in R \setminus \{0\}$ such that $ab = -ba$ and $b^2 \in C$. Thus $Q := C\mathbf{1} + Ca + Cb + Cab$ is a quaternion algebra. Since, by Theorem 3.2, $R$ contains no divisors of zero, $Q$ is a division algebra.

(2) By (1) there exist non-zero $a, b \in R$ such that $ab = -ba$ and $a^2, b^2 \in C$. Since $R$ is not associative $R \neq Q$, where $Q$ is as in (1). Hence by Proposition 3.5(2), and since $a, b, ab$ pairwise anticommute, there exists $c \in R \setminus Q$, such that $c$ anticommute with $a, b, ab$, and $c^2 \in C$. By Lemma 3.4(3(iii)), $u_1, \dots, u_7$ satisfy the assertion of part (2).

Set

$$\alpha := a^2, \beta := b^2, \gamma = c^2.$$

It is easy to check now that $\{u_1, \dots, u_7\}$ satisfy the multiplication table on p. 137 of [3] (with $\mathbf{1} = u_0$). Hence $O$ is an octonion algebra. Since $R$ has no zero divisors, $O$ is a division algebra (see [6, section III]). □

Lemma 3.7 below will be used to show that $R = O$, when $R$ is alternative, where $O$ is as in Lemma 3.6(2) above.

**Lemma 3.7.** *Let $m, a, b, c \in R \setminus \{0\}$, and let $\{x, y, z\} = \{a, b, c\}$. Assume that*

(i) *$x$ anticommutes with $y$ and $yz$.*
(ii) *$m$ anticommutes with $x, xy, a(bc)$.*

*Then*

(1) *$mx$ anticommutes with $y, yz$; $m(xy)$ anticommutes with $z$, and $(mx)y$ anticommutes with $z$.*
(2) $2m(x(yz)) = 0$.

**Proof.** (1) First, by Lemma 3.4(3i), $mx$ anticommutes with $y$. Also, by Lemma 3.4 we get,

$$(mx)(yz) \stackrel{3.4(1ii)}{=} -((yz)x)m \stackrel{by\ (ii)}{=} m((yz)x) \stackrel{3.4(1i)}{=} -(yz)(mx),$$

and

$$(m(xy))z \stackrel{3.4(1ii)}{=} -(z(xy))m \stackrel{by\ (ii)}{=} m(z(xy)) \stackrel{3.4(1i)}{=} -z(m(xy)).$$

Also

$$((mx)y)z \stackrel{3.4(1ii)}{=} -(zy)(mx) = (mx)(zy) \stackrel{3.4(1i)}{=} -z((mx)y).$$

(2) We can now use Lemma 3.4(3(ii)) with $\{m, x, yz\}$, $\{mx, y, z\}$ and $\{m, xy, z\}$, in place of $\{a, b, c\}$. Thus we have

$$m(x(yz)) = -(mx)(yz) = ((mx)y)z,$$

and

$$m(x(yz)) = -m((xy)z) = (m(xy))z = -((mx)y)z. \qquad \square$$

**Proof of Theorems A and B.** We can now complete the proof of Theorems A and B. So suppose that $R$ satisfies one of hypotheses (A) or (B). By Theorem 3.2, $R$ contains no zero divisors. We now assume that the characteristic of $R$ is not 2, and we replace $R$ with $R//C$ as in Remark 3.3.

Let $a, b \in R$ and $Q$ be as in Proposition 3.6(1). Suppose that $R$ is associative and that $R \neq Q$. By Proposition 3.5(2), there exists $m \in R \smallsetminus Q$ that anticommutes with $a, b, ab$. But then

$$m(ab) = (ma)b = -(am)b = -a(mb) = (ab)m = -m(ab).$$

So $2m(ab) = 0$, hence $m(ab) = 0$. But $R$ has no divisors of 0, a contradiction.

Suppose now that $R$ is alternative but not associative. Let $O$ be as in Proposition 3.6(2). Assume that $R \neq O$. By Proposition 3.5(2), there exists $m \in R \smallsetminus O$ that anticommutes with $u_1, \ldots, u_7$. By Lemma 3.7, $2m(a(bc)) = 0$, again a contradiction. $\square$

**Acknowledgement.** We thank the referee for his/her review. The referee mentioned that Theorem A(a) can be proven using Polynomial Identity Theory. However one of the goals of this paper is to give a uniform proof for the Quaternions and the Octonions and, in addition, that the paper be self contained.

**Disclosure statement.** The authors report there are no competing interests to declare.

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