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PRODUCTS OF COMMUTATORS OF UNIPOTENT MATRICES OF INDEX 2 IN $\mathrm{GL}_n(\mathbb{H})$

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Abstract. The aim of this paper is to show that if $\mathbb{H}$ is the real quaternion division ring and n is an integer greater than 1, then every matrix in the special linear group $\mathrm{SL}_n(\mathbb{H})$ can be expressed as a product of at most three commutators of unipotent matrices of index 2.

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1. Introduction

Let G be a group and $G' = [G, G]$ the derived subgroup of G. For every $x, y \in G$, we denote by $[x, y] = xyx^{-1}y^{-1}$ the commutator of x and y. It is clear that each element in G' is a product of commutators in G. The exploration of decomposing elements within commutator subgroups into products of commutators associated with specific subgroups has a rich and extensive history across diverse categories of mathematical groups. To gain an in-depth understanding of this subject, we refer the reader to comprehensive surveys such as those presented in [11,19]. Recently, there has been interest in decomposing matrices into products of commutators of special matrices, such as involutions, skew-involutions, and reflections. Recall that, for a ring R, an $n \times n$ matrix A with coefficients in R is called an *involution* (respectively, *skew-involution* or *reflection*) if $A^2 = \mathrm{I}_n$ (resp., $A^2 = -\mathrm{I}_n$ or $A^2 = \mathrm{I}_n$ and the rank of $A - \mathrm{I}_n$ is 1). Interested readers can find recent results on this topic in [1,2,3,7,9,10,18,24,26].

Suppose D is a division ring, $D^* = D \setminus \{0\}$, and denote by $\mathrm{GL}_n(D)$ the group of invertible matrices with coefficients in D. The special linear group, which is the commutator subgroup of $\mathrm{GL}_n(D)$, is denoted by $\mathrm{SL}_n(D)$. Recall that a matrix $A \in \mathrm{GL}_n(D)$ is called a *unipotent matrix* of index 2 if $(\mathrm{I}_n - A)^2 = 0$. It is shown that if $D = \mathbb{C}$ is the field of complex numbers, then every matrix in $\mathrm{SL}_n(\mathbb{C})$ can be decomposed into a product of at most four unipotent matrices of index 2 (see [23, Theorem 3.5]). Recently, in [8], X. Hou has shown that every matrix in $\mathrm{SL}_n(\mathbb{C})$ can be written as a product of at most two commutators of unipotent matrices of index 2, and two is the smallest such number. Observe that if A is a unipotent matrix of index 2, then so are its inverse A^{-1} and conjugates BAB^{-1}, so a commutator $ABA^{-1}B^{-1}$ of unipotent matrices A, B of index 2 is a product of two unipotent matrices A and BAB^{-1} of index 2. Hence, the product of two commutators of unipotent matrices of index 2 is a product of at most four unipotent matrices of index 2. Thus, the result of X. Hou extends [23, Theorem 3.5]. In this paper, we are interested in the problem of decomposing matrices into products of commutators of unipotent matrices of index 2.

The techniques used in [8] can be applied to any algebraically closed field but cannot be applied to any general field. In this paper, we focus on the case of the real quaternion division ring. Throughout this paper, $\mathbb{H}, \mathbb{C}$ and $\mathbb{R}$ are respectively the real quaternion division ring, the field of complex numbers, and the field of real numbers. Recall that the *real quaternion division ring* $\mathbb{H}$ is the set of all elements of the form $a + bi + cj + dk$ in which $a, b, c, d \in \mathbb{R}$ and $i^2 = j^2 = k^2 = -1$, $ij = -ji = k$. Researching matrices on quaternion division rings plays a significant role and has attracted considerable attention. Some interesting results related to this topic can be found in [1,5,15,22]. For the theory of quaternion algebras, we refer to [12,13,16,20].

The first aim of this paper is the following result, which can be considered as a counter-example to show the result of X. Hou in [8] is no longer true in the division ring of real quaternions.

Theorem 1.1. *The matrix* $-\mathrm{I}_{2n+1}$ *in* $\mathrm{GL}_{2n+1}(\mathbb{H})$ *can be written as a product of exactly three commutators of unipotent matrices of index* 2.

The second is to present a division ring version of results in [8] for noncentral matrices over $\mathbb{H}$.

Theorem 1.2. *Let* $A \in \mathrm{SL}_n(\mathbb{H})$. *Then,* A *is a product of at most two commutators of unipotent matrices of index* 2 *in* $\mathrm{GL}_n(\mathbb{H})$ *unless* n *is odd and* $A = -\mathrm{I}_n$.

As a result of Theorem 1.2, we establish the following corollary.

Corollary 1.3. *Every matrix in* $\mathrm{SL}_n(\mathbb{H})$ *can be expressed as a product of at most three commutators of unipotent matrices of index* 2 *in* $\mathrm{GL}_n(\mathbb{H})$.

2. Proof of Theorem 1.1

In this section, two proofs of Theorem 1.1 will be presented. The first proof was originally conducted by the authors. During the peer review process of this paper, the journal's reviewer presented a concise and insightful alternative proof. With the reviewer's agreement, we intend to present both proofs.

2.1. The first proof of Theorem 1.1. We frequently use the following lemma, the proof of which is standard and will be omitted.

Lemma 2.1. *Let D be a division ring, m and n be positive integers. Suppose $A \in \mathrm{GL}_n(D)$ and $B \in \mathrm{GL}_m(D)$. Then,*

(1) *A is a unipotent matrix of index 2 if and only if $A + A^{-1} = 2\mathrm{I}_n$.*
(2) *A and B are unipotent matrices of index 2 if and only if $A \oplus B$ is also a unipotent matrix of index 2.*
(3) *Every matrix that is similar to a unipotent matrix of index 2 is also a unipotent matrix of index 2.*
(4) *Every matrix that is similar to a commutator of unipotent matrices of index 2 is also a commutator of unipotent matrices of index 2.*
(5) *If A and B are products of k and l commutators of unipotent matrices of index 2 respectively, then $A \oplus B$ is the product of at most $\max\{k, l\}$ commutators of unipotent matrices of index 2 in $\mathrm{GL}_{n+m}(D)$.*

For each $\lambda \in \mathbb{H}$, a Jordan block of size $m \times m$ is denoted as

$$J(m, \lambda) = \begin{pmatrix} \lambda & 1 & & & \\ & \lambda & 1 & & \\ & & & \ddots & \\ & & & \lambda & 1 \\ & & & & \lambda \end{pmatrix}.$$

The following lemma is a consequence of [14, Corollary 3.5].

Lemma 2.2. *Suppose $n \geq 1$ and $A \in \mathrm{GL}_n(\mathbb{H})$. Then, there exists $S \in \mathrm{GL}_n(\mathbb{H})$ such that*

$$S^{-1}AS = J(m_1, \lambda_1) \oplus \cdots \oplus J(m_k, \lambda_k) \qquad (*)$$

in which $\lambda_1, \ldots, \lambda_k \in \mathbb{C}$ and $m_1 + \cdots + m_k = n$. The right-hand side matrix of $(*)$ is called the Jordan form of A.

Assuming $A \in \mathrm{GL}_n(\mathbb{H})$, an element α is a (right) eigenvalue of A if there exists a nonzero $n \times 1$ matrix v such that $Av = v\alpha$. If α is an eigenvalue of A, then $\alpha\beta$

and α^{-1} are eigenvalues of βA and A^{-1} respectively for every $\beta \in \mathbb{R}$. This fact will be used in the following result.

Lemma 2.3. *Suppose D is a division ring and $n \geq 1$ is an integer. If $A \in \mathrm{GL}_{2n+1}(D)$ is a commutator of unipotent matrices of index 2, then A has an eigenvalue 1.*

Proof. Assume that B and C are unipotent matrices in $\mathrm{GL}_{2n+1}(D)$ such that $A = [B, C]$. By [4, Proposition 2.3], the matrix C can be chosen to be of a specific form. Without loss of generality, we can write C as follows:

$$C = \underbrace{\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}}_{r \text{ times}} \oplus \mathrm{I}_{2n+1-2r}$$

where r is the rank of the matrix $C - \mathrm{I}_{2n+1}$. Then, $AC = BCB^{-1}$, which implies that $AC + (AC)^{-1} = 2\mathrm{I}_{2n+1}$ by Lemma 2.1. We can deduce that $A^{-1} = 2C - CAC$, leading to $A^{-1}(A - \mathrm{I})^2 = 2C - CAC - 2\mathrm{I} + A$.

By direct calculation, we obtain that $[A^{-1}(A - \mathrm{I})^2]_{2l,2k-1} = 0$ for all $1 \leq l \leq n, 1 \leq k \leq n + 1$ and $[A^{-1}(A - \mathrm{I})^2]_{2n+1,2t-1} = 0$ for all $1 \leq t \leq n + 1$.

In the matrix $A^{-1}(A-\mathrm{I})^2$, there are $n+1$ rows, and each of these rows contains $n + 1$ zeros. Therefore, this matrix is equivalent to $\begin{pmatrix} X & 0 \\ Y & Z \end{pmatrix}$ where X, Y, Z are $(n + 1) \times n$, $n \times n$, and $n \times (n + 1)$ matrices, respectively. It is easy to see that the rank of $(X\ 0)$ and $(Y\ Z)$ is less than or equal to n. Therefore, by [21, Proposition 1.21], the rank of $A^{-1}(A - \mathrm{I})^2$ is less than $2n + 1$. Furthermore, the rank of A^{-1} is $2n + 1$, so the rank of $A - \mathrm{I}$ is less than $2n + 1$. This means that $A - \mathrm{I}$ has the eigenvalue 0, or equivalently, A has the eigenvalue 1. $\square$

Proposition 2.4. *Suppose D is a division ring, and $n \geq 1$ is an integer. If $A \in \mathrm{GL}_n(D)$ is a commutator of unipotent matrices of index 2, then A is not similar to a Jordan form containing $J(m, -1)$ where m is odd.*

Proof. Assume A is similar to $X \oplus J(m, -1)$, and $A = [B, C_1]$ in which B and C_1 are unipotent matrices. By Lemma 2.1, we can only consider the case $A = X \oplus J(m, -1)$.

Suppose $C_1 = \begin{pmatrix} * & * \\ * & C \end{pmatrix}$ where C is an $m \times m$ matrix. Because $AC_1 = BC_1B^{-1}$ is a unipotent matrix of index 2,

$$AC_1 + (2\mathrm{I}_m - C_1)A^{-1} = 2\mathrm{I}_m.$$

Therefore,

$$J(m,-1)C + (2\mathrm{I}_m - C)J(m,-1)^{-1} = 2\mathrm{I}_m.$$

Let $P = (p_{ij}) = J(m,-1)C + (2\mathrm{I}_m - C)J(m,-1)^{-1}$ and $C = (c_{ij})$. We shall show that $c_{ij} = 0$ for all $i \geq j+2$.

Indeed, considering row m, we have $p_{m,1} = p_{m,2} = \ldots = p_{m,m-2} = 0$ which corresponds to $c_{m,1}; c_{m,1}+c_{m,2}; \ldots; c_{m,1}+c_{m,2}+\cdots+c_{m,m-2}$. Therefore, $c_{m,t} = 0$ for all $t \leq m-2$. This leads to

$$\begin{aligned} &p_{m-1,1} = c_{m-1,1} \\ &p_{m-1,2} = c_{m-1,1} + c_{m-1,2} \\ &\ldots\ldots \\ &p_{m-1,m-3} = c_{m-1,1} + c_{m-1,2} + \cdots + c_{m-1,m-3}. \end{aligned}$$

Hence, $c_{m-1,t} = 0$ for all $t \leq m-3$. Using induction we have $c_{ij} = 0$ for all $i \geq j+2$.

Then,

$$\begin{aligned} &p_{11} = c_{21} - 2; \\ &p_{22} = c_{21} + c_{32} - 2; \\ &\ldots \\ &p_{m-1,m-1} = c_{m-1,m-2} + c_{m,m-1} - 2; \\ &p_{mm} = c_{m,m-1} - 2. \end{aligned}$$

and all these values equal 2. Thus, from p_{11} to $p_{m-1,m-1}$, we deduce that $c_{21} = 4, c_{32} = 0, c_{43} = 0, \ldots, c_{m,m-1} = 0$. However, this leads to $p_{mm} = -2$ (this is a contradiction). $\square$

Now we shall show the main results of this subsection.

The first proof of Theorem 1.1. Suppose $m = 2n+1$ and $-\mathrm{I}_m = AB$ where A and B are commutators of unipotent matrices of index 2. By Lemma 2.2, there exists a Jordan form matrix J such that $S^{-1}AS = J$, so $-\mathrm{I}_m = JS^{-1}BS$ where $S \in \mathrm{GL}_m(\mathbb{H})$. Without loss of generality, assume that A has a Jordan form. By Lemma 2.3, both A and B have eigenvalues 1. If 1 is an eigenvalue of A and appears an odd number of times, then B must have -1 as an eigenvalue that appears an odd number of times, because $A = -B^{-1}$, this is a contradiction by Proposition 2.4. Hence, A has 1 and -1 as eigenvalues, each with even multiplicity, which means A also contains other Jordan blocks.

Let $A = [E, D]$ and $A = A_1 \oplus A_2$, where A_1 is the direct sum of all Jordan blocks with eigenvalues 1 and -1 of A, and A_2 is the sum of the remaining blocks. Let $g \times g$ be the size of A_1, which is even. Assume $D = \begin{pmatrix} D_1 & D_2 \\ D_3 & D_4 \end{pmatrix}$ in which D_1, D_4 are $g \times g, (m-g) \times (m-g)$ matrices, respectively. Since AD is a unipotent matrix of index 2, by Lemma 2.1, $AD + (2\mathrm{I}_m - D)A^{-1} = 2\mathrm{I}_m$. We deduce $A_2D_3 - D_3A_1^{-1} = 0$ and $A_1D_2 - D_2A_2^{-1} = 0$. Let $D_3 = (d_{ij})$, we have $A_2D_3 = D_3A_1^{-1}$, where:

- The matrix A_2D_3 is obtained by multiplying each row of D_3 by an eigenvalue of A_2 and adding (or not adding) a row immediately below the corresponding row of D_3.
- Note that A_1^{-1} is an upper triangular matrix with the diagonal elements being 1 or -1, and all other entries are 0, 1, or -1. Therefore, $D_3A_1^{-1}$ is obtained by multiplying each column of D_3 by 1 or -1 and then adding or subtracting a finite number of columns to its left (possibly none).

Consider the last rows of both A_2D_3 and $D_3A_1^{-1}$. We have the corresponding entries as follows:

$$\lambda d_{m-g,1}; \lambda d_{m-g,2}; \ldots; \lambda d_{m-g,g}$$

$$\pm d_{m-g,1}; \pm d_{m-g,2}||d_{m-g,1}; \ldots; \pm d_{m-g,g}||d_{m-g,g-1}|| \ldots ||d_{m-g,1}.$$

Here, the notation $a||b$ can only take values $a, a+b$, or $a-b$. Since $\lambda \neq \pm 1$, we can conclude that the last row of D_3 is filled with zeros. The $(m-g-1)$-th row of A_2D_3 is determined by the product of an eigenvalue of A_2 and the $(m-g-1)$-th row of D_3. Thus, all coefficients on this row must also be equal to 0 by reasoning as above.

Therefore, we can show that each row of D_3 is also filled with zeros, by considering from the bottom row to the top. It follows that $D_3 = 0$ and we can show $D_2 = 0$, similarly; which implies that $D = D_1 \oplus D_4$.

Note that A^{-1} also has a Jordan form, and $A^{-1}E = DED^{-1}$ is a unipotent matrix of index 2. Suppose $E = \begin{pmatrix} E_1 & E_2 \\ E_3 & E_4 \end{pmatrix}$, we have $E_2 = E_3 = 0$ by similar argument. Hence, we also have $E = E_1 \oplus E_4$, where E_1 is a $g \times g$ matrix. This leads to $A_2 = [E_4, D_4]$. By Lemma 2.1, D_4 and E_4 are also unipotent matrices of index 2. Thus, $A_2 \in \mathrm{GL}_{m-g}(\mathbb{C})$ is a commutator of unipotent matrices of index 2. Note that $m - g$ is odd, and A_2 does not have eigenvalue -1, which contradicts Proposition 2.4. Therefore, $-\mathrm{I}_m$ can not be written as a product of 2 commutators of unipotent matrices of index 2.

Note that

$$\begin{pmatrix} \frac{i}{2} & 0 \\ 0 & 2i \end{pmatrix} = P^{-1} \begin{pmatrix} \frac{i}{2} & 0 \\ 0 & -2i \end{pmatrix} P$$

where $P = \mathrm{diag}(1, k)$ and $k \in \mathbb{C}$. By [8, Lemma 2.3], the matrix $\begin{pmatrix} \frac{i}{2} & 0 \\ 0 & -2i \end{pmatrix}$ is a commutator of unipotent matrices of index 2, hence so is $(1) \oplus \begin{pmatrix} \frac{i}{2} & 0 \\ 0 & 2i \end{pmatrix}$. Moreover,

$$-\mathrm{I}_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{i}{2} & 0 \\ 0 & 0 & 2i \end{pmatrix} \begin{pmatrix} \frac{i}{2} & 0 & 0 \\ 0 & 2i & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2i & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{i}{2} \end{pmatrix}.$$

Hence, $-\mathrm{I}_3$ can be written as a product of exactly three commutators of unipotent matrices of index 2. Furthermore, the matrix $-\mathrm{I}_2$ can be represented as a product of exactly two commutators of (complex) unipotent matrices of index 2 by [8, Lemma 2.5]. Thus,

$$-\mathrm{I}_{2n+1} = -\mathrm{I}_3 \oplus -\mathrm{I}_2 \oplus \cdots \oplus -\mathrm{I}_2$$

can be written as a product of at most three commutators of unipotent matrices of index 2 by Lemma 2.1. $\square$

2.2. The second proof of Theorem 1.1. By geometrically approaching, the reviewer has pointed out the following results to prove Theorem 1.1 in a general, short and concise way.

Throughout this subsection, we denote by D an arbitrary division ring with center C and by V a finite-dimensional right-D-vector space. Suppose that D does not have characteristic 2.

The first key is a classical lemma in the study of quadratic objects in algebras:

Lemma 2.5. (See e.g. [17, Lemma 1.3]) *Let a and b be unipotent elements of index 2 of a unital ring R. Then a and b commute with $ab + (ab)^{-1}$.*

The next key is the Fitting decomposition of an endomorphism u of V, consisting of the *nilspace* $\mathrm{Nil}(u) := \bigcup_{n\in\mathbb{N}} \mathrm{Ker}\, u^n$ of u, and of the *core* $\mathrm{Co}(u) = \bigcap_{n\in\mathbb{N}} \mathrm{Im}\, u^n$. One has $V = \mathrm{Nil}(u) \oplus \mathrm{Co}(u)$. For every central $\lambda \in C$, we can set $E_c^{\lambda}(u) := \mathrm{Nil}(u - \lambda \mathrm{id})$ (the characteristic subspace attached to λ), and in the special case $u \in \mathrm{GL}(V)$ we obtain a decomposition

$$E = \mathrm{Nil}(u - u^{-1}) \oplus \mathrm{Co}(u - u^{-1}) = E_c^1(u) \oplus E_c^{-1}(u) \oplus \mathrm{Co}(u - u^{-1}).$$

In this situation, note that every endomorphism in V that commutes with $u+u^{-1}$ must leave all three summands $E_c^1(u)$, $E_c^{-1}(u)$, and $\mathrm{Co}(u-u^{-1})$ invariant. Indeed, for the first two this follows from the observation that

$$\mathrm{Nil}(u-\varepsilon\mathrm{id}) = \mathrm{Nil}(u^{-1}(u-\varepsilon\mathrm{id})^2) = \mathrm{Nil}((u+u^{-1}) - 2\varepsilon\mathrm{id})$$

for all $\varepsilon = \pm 1$, and for the last one, this follows from $(u-u^{-1})^2 = (u+u^{-1})^2 - 4\mathrm{id}$. Hence, as a corollary to Lemma 2.5, we obtain:

Corollary 2.6. *Let a and b be unipotent elements of index 2 in $\mathrm{GL}(V)$. Set $u := ab$. Then a and b leave $\mathrm{E}_c^1(u)$, $\mathrm{E}_c^{-1}(u)$, and $\mathrm{Co}(u-u^{-1})$ invariant, and hence each one of the respective endomorphisms of these spaces induced by u is the product of two unipotent automorphisms of index 2.*

Note that the following lemma extends Lemma 2.3, with a simplified proof. And now the last key:

Lemma 2.7. *Let a and b be unipotent elements of index 2 in $\mathrm{GL}(V)$ such that 1 is not an eigenvalue of ab. Then $\dim V$ is even.*

Proof. Assume that $\dim V$ is odd. We have $\mathrm{Im}(a-\mathrm{id}) \subseteq \mathrm{Ker}(a-\mathrm{id})$ and $\dim(\mathrm{Im}(a-\mathrm{id})) + \dim(\mathrm{Ker}(a-\mathrm{id})) = \dim V$, whence $\dim \mathrm{Ker}(a-\mathrm{id}) > \frac{\dim V}{2}$. Likewise, $\dim \mathrm{Ker}(b-\mathrm{id}) > \frac{\dim V}{2}$, and it follows that there exists a nonzero vector $x \in \mathrm{Ker}(a-\mathrm{id}) \cap \mathrm{Ker}(b-\mathrm{id})$. As a consequence $ab(x) = x$, and 1 is an eigenvalue of ab. □

Combining the previous two results leads to:

Corollary 2.8. *Let a and b be unipotent elements of index 2 in $\mathrm{GL}(V)$. Set $u := ab$. Then $\mathrm{E}_c^{-1}(u)$ and $\mathrm{Co}(u-u^{-1})$ are even-dimensional.*

We are now able to finish the proof:

Theorem 2.9. *Assume that V is odd-dimensional. Then $-\mathrm{id}_V$ is not the product of four unipotent automorphisms of index 2 of V.*

Proof. Assume otherwise. Then $-\mathrm{id}_V = a_1a_2a_3a_4$ for unipotent automorphisms $a_1, \dots, a_4$ of index 2 of V. Hence, $u := a_1a_2 = -a_4^{-1}a_3^{-1} = -a'_1a'_2$ for $a'_1 := a_4^{-1}$ and $a'_2 := a_3^{-1}$, which are unipotent of index 2. Since V is odd-dimensional, we gather from Corollary 2.8 that $\mathrm{E}_c^1(u)$ is odd-dimensional, i.e., $\mathrm{E}_c^{-1}(-u)$ is odd-dimensional. But this is in contradiction with Corollary 2.8 applied to a'_1 and a'_2. □

Now we show Theorem 1.1 based on the results provided by the reviewer.

The second proof of Theorem 1.1. Note that a commutator in $\mathrm{SL}_{2n+1}(\mathbb{H})$ is a product of two unipotent matrices of index 2. Hence, by Theorem 2.9 the matrix $-\mathrm{I}_{2n+1}$ cannot be written as a product of two commutators of unipotent matrices of index 2.

In the first proof of Theorem 1.1, we have shown that $-\mathrm{I}_{2n+1}$ can be written as a product of three commutators of unipotent matrices of index 2. Thus, three is the smallest such number. □

3. Proof of Theorem 1.2

The notation $\mathbb{LT}_n(D)$ (resp. $\mathbb{UT}_n(D)$) represents the group of lower triangular (resp. upper triangular) matrices in $\mathrm{M}_n(D)$ with elements on the main diagonal equal to 1.

Lemma 3.1. *Let D be a division ring, n be an integer greater than 1 and A be a noncentral matrix in $\mathrm{GL}_n(D)$. For $h_1, h_2, \ldots, h_{n-1} \in D^*$, then there exist $P \in \mathrm{GL}_n(D)$ and $h_n \in D^*$ such that*

$$P^{-1}AP = XHY$$

where $X \in \mathbb{LT}_n(D), Y \in \mathbb{UT}_n(D)$ and $H = \mathrm{diag}(h_1, h_2, \ldots, h_n)$. Moreover, if $A \in \mathrm{SL}_n(D)$, then $h_1h_2 \ldots h_n \in [D^*, D^*]$. In particular, if D is finite dimensional over its center and A is a lower or upper triangular matrix with pairwise nonconjugate diagonal entries $a_{11}, \ldots, a_{nn} \in D$, then A is similar to the diagonal matrix $diag(a_{11}, \ldots, a_{nn})$.

Proof. The first part is from [1, Lemma 2.7] and the second one is from [2, Lemma 3.2]. □

The following result is very useful and is one of the distinctive properties of the real quaternion division ring.

Recall that the *norm* of α is $N(\alpha) = \sqrt{a^2 + b^2 + c^2 + d^2}$ for every $\alpha = a + bi + cj + dk \in \mathbb{H}$. By direct calculation, we obtain the following lemma.

Lemma 3.2. *Assume $\alpha \in \mathbb{H}^*$ and $\lambda \in \mathbb{R}$. Then, $\lambda\alpha$ and $(\lambda\alpha)^{-1}$ are similar if and only if $\lambda N(\alpha) = \pm 1$.*

Proof. Lemma 3.2 follows directly from [25, Theorem 2.2]. □

Next, we generalize [8, Lemma 2.3] for the real quaternion division ring.

Lemma 3.3. *If $a \in \mathbb{H}^*$ and $\lambda \in \mathbb{R}$ are such that $\lambda N(a) \neq \pm 1$, then $\begin{pmatrix} \lambda a & 0 \\ 0 & (\lambda a)^{-1} \end{pmatrix}$ is a commutator of unipotent matrices of index 2.*

Proof. Because $\lambda a \in \mathbb{H}^*$ and $\mathbb{H}$ is an algebraically closed division ring, there exists $b \in \mathbb{H}^*$ such that $\lambda a = b^2$. Let

$$A = \begin{pmatrix} 2c^{-1}(c+1) & (c+2)c^{-1} \\ -c^{-1}(c+2) & -2c^{-1} \end{pmatrix}; B = \begin{pmatrix} 2 & (c+1)^{-1} \\ -(c+1) & 0 \end{pmatrix}$$

in which $c = b - 1$. By direct calculation, we obtain that A and B are unipotent matrices of index 2 and $[A, B] = \begin{pmatrix} b^2 & 2b - 2b^{-1} \\ 0 & b^{-2} \end{pmatrix}$. Since $\lambda N(a) \neq \pm 1$, we have that b^2, b^{-2} are non-conjugated by Lemma 3.2. Therefore, $\begin{pmatrix} b^2 & 2b - 2b^{-1} \\ 0 & b^{-2} \end{pmatrix}$ is similar to $\begin{pmatrix} b^2 & 0 \\ 0 & b^{-2} \end{pmatrix}$. Thus, by Lemma 2.1, $\begin{pmatrix} b^2 & 0 \\ 0 & b^{-2} \end{pmatrix}$ is a commutator of unipotent matrices of index 2. $\square$

Now we are ready to show the second main result of this paper.

Proof of Theorem 1.2. We separate the proof into two cases.

Case 1. A is a noncentral matrix in $\mathrm{SL}_n(\mathbb{H})$. By Lemma 3.1, the matrix A is similar to XZY in which $X \in \mathbb{LT}_n(\mathbb{H})$, $Y \in \mathbb{UT}_n(\mathbb{H})$ and $Z = \mathrm{diag}(1, 1, \ldots, 1, s)$ with $s \in \mathbb{H}'$. Moreover, by [1, Lemma 2.5] there exist $a, b \in \mathbb{H}$ such that $s = aba^{-1}b^{-1}$. Choose λ that satisfies $\lambda N(a) \neq \pm 1$, then λa and $(\lambda a)^{-1}$ are not similar by Lemma 3.2 and $aba^{-1}b^{-1} = (\lambda a)b(\lambda a)^{-1}b^{-1}$. Then, A is similar to

$$\begin{pmatrix} 1 & & \\ & \ddots & & \\ & & (\lambda a)^{-1} & \\ & * & & (\lambda a) \end{pmatrix} \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 & \\ & & & b \end{pmatrix} \begin{pmatrix} 1 & & & \\ & \ddots & & * \\ & & \lambda a & \\ & & & (\lambda a)^{-1} \end{pmatrix} \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 & \\ & & & b^{-1} \end{pmatrix}.$$

Subcase 1.1. n is even, that is $n = 2k$ for some positive integer k. Since $\mathbb{H}$ is infinite, we can choose elements $x_1, x_2, \ldots, x_{k-1}$ such that

$$x_1, x_1^{-1}, x_2, x_2^{-1}, \ldots, x_{k-1}, x_{k-1}^{-1}, \lambda a, (\lambda a)^{-1}$$

are pairwise non-conjugate. Then, A is similar to $UP^{-1}VP$, which has the form of

$$\begin{pmatrix} x_1 & & & & & & \\ & x_1^{-1} & & & & & \\ & & \ddots & & & & \\ & & & x_{k-1} & & & \\ & * & & & x_{k-1}^{-1} & & \\ & & & & & (\lambda a)^{-1} & \\ & & & & & & (\lambda a) \end{pmatrix} P^{-1} \begin{pmatrix} x_1^{-1} & & & & & & \\ & x_1 & & & & & \\ & & \ddots & & & * & \\ & & & x_{k-1}^{-1} & & & \\ & & & & x_{k-1} & & \\ & & & & & (\lambda a) & \\ & & & & & & (\lambda a)^{-1} \end{pmatrix} P$$

in which

$$P = \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & b^{-1} \end{pmatrix}.$$

By Lemma 3.1, the matrix U is similar to $B_1 \oplus B_2 \oplus \cdots \oplus B_k$ where $B_i = \mathrm{diag}(x_i, x_i^{-1})$ for every $i = 1, \ldots, k-1$ and $B_k = \mathrm{diag}((\lambda a)^{-1}, \lambda a)$. For each $\lambda_i \in \mathbb{R}^*$, let $v_i = \lambda_i^{-1} x_i$. Then, $B_i = \mathrm{diag}(\lambda_i v_i, (\lambda_i v_i)^{-1})$ and B_k are commutators of unipotent matrices of index 2 by Lemma 3.3. Therefore, U is a commutator of unipotent matrices of index 2. Similarly, V is also a commutator of unipotent matrices of index 2. Then, $UP^{-1}VP$ is a product of two commutators of unipotent matrices of index 2. Thus, A is a product of two commutators of unipotent matrices of index 2.

Subcase 1.2. n is odd, that is $n = 2k+1$ for some positive integer k. By applying the same argument as in of the proof in Subcase 1.1, we can choose U that it is similar to $(1) \oplus B_1 \oplus B_2 \oplus \cdots \oplus B_k$. Then, A is a product of two commutators of unipotent of index 2.

Note that for every $n \geq 2$ the matrix $\mathrm{diag}(1, \ldots, 1, -1) = \mathrm{I}_{n-1} \oplus (-1)$ can not be written as a commutator of unipotent matrices of index 2 by Proposition 2.4. Therefore, A can be represented as a product of two commutators of unipotent of index 2 and 2 is the smallest such number.

Case 2. A is a central matrix in $\mathrm{SL}_n(\mathbb{H})$. By [6, Lemma 5.6], we have $A = \pm \mathrm{I}_n$. If $A = \mathrm{I}_n$, then A is a commutator of unipotent matrices of index 2. If $A = -\mathrm{I}_n$, then n must be even and we have that $-\mathrm{I}_n = -\mathrm{I}_2 \oplus \ldots \oplus -\mathrm{I}_2$ can be represented as a product of two commutators of unipotent matrices of index 2. □

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