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SAGBI BASES OVER THE PRODUCT OF RINGS

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Abstract. Let R be a commutative ring such that $R = R_1 \times \cdots \times R_n$. In this paper, we give a method to compute (strong) Sagbi bases for subalgebras of a polynomial ring over R provided that these bases are computable in a polynomial ring over R_i for $1 \leq i \leq n$. As an application, we prove the existence of strong Sagbi bases for subalgebras in a polynomial ring over a principal ideal ring.

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1. Introduction and preliminaries

Gröbner bases were first introduced by Bruno Buchberger to solve polynomial ideal theory problems over fields ([3]). Later, this theory was extended in a polynomial ring over noetherian commutative rings ([2]). Gröbner bases are essential tools in computational algebraic geometry and polynomial algebra ([4]). Moreover, over commutative noetherian rings, strong Gröbner bases have been introduced (see [7] and [8]), which are Gröbner bases with additional properties that can augment the efficiency and efficacy of various algebraic computations.

Sagbi bases, a generalization of Gröbner bases to the context of subalgebras, were introduced by Robianno and Sweedler over fields ([5], [9]). Miller extended this concept for polynomial rings over noetherian commutative rings ([6]). Unlike Gröbner bases, Sagbi bases could be infinite for even finitely generated subalgebras. They are used to study subalgebras of polynomial rings arising in geometric contexts, such as invariant, toric, and coordinate rings of algebraic varieties ([10]). In [1], the concept of strong Sagbi bases was introduced, showing that they exist in a polynomial ring over principal ideal domains.

Let R be a commutative ring with unity and a finite direct product of commutative rings R_i with unities. In [8], a method is developed to construct (strong) Gröbner bases of non-zero ideals in polynomial rings over R. It is achieved by joining (strong) Gröbner bases of the projected ideals in polynomial rings over R_i. This work aims to develop an analogous method for (strong) Sagbi bases for subalgebras of a polynomial ring over R. As an application of our work, we prove the existence of a strong Sagbi basis in a polynomial ring over a principal ideal ring (which is a commutative ring such that all its ideals are principal and can be viewed as a direct product of principal ideal domains and finite chain rings, see Theorem 33, Section 15, Chapter 4 of [11]).

The article is structured as follows: The polynomial ring in t variables x_1, x_2, $\cdots$, x_t over the ring R is denoted by $R[\underline{x}]$. In Section 2, the S-join is defined for the subsets of $R_i[\underline{x}]$ for $i = 1, \cdots, n$ (see Definition 2.2). We also prove that the Sagbi bases for subalgebras of $R[\underline{x}]$ can be obtained by S-joining of Sagbi bases for projected subalgebras of $R_i[\underline{x}]$ (see Proposition 2.3). Moreover, we define the concept of strong S-join for the subsets of $R_1[\underline{x}]$ and $R_2[\underline{x}]$, respectively (see Definition 2.10). We also present a key result (Theorem 2.11) which demonstrates that the strong Sagbi bases for subalgebras of $R[\underline{x}]$ can be obtained by strong S-joining of the strong Sagbi bases for the projected subalgebras of $R_1[\underline{x}]$ and $R_2[\underline{x}]$, respectively. Furthermore, we provide Proposition 2.12 which enables us to generalize Theorem 2.11 for $R = R_1 \times \cdots \times R_n$. We also present Algorithm 2.14 to compute strong Sagbi bases for the subalgebras in $R[\underline{x}]$. In Section 3, we prove the existence of a strong Sagbi basis in a polynomial ring over a principal ideal ring (see Theorem 3.5).

Let $>$ be an admissible monomial ordering on the set of monomials (power product of indeterminates x_i) of $R[\underline{x}]$. If $f \in R[\underline{x}]$, then we can write it as $f = \sum_k c_k t_k$ where $0 \neq c_k \in R$ and t_k is the monomial. We denote the leading monomial t_l which is $max_k\{t_k\}$ by $lm(f)$, the leading coefficient c_l which is the coefficient of t_l by $lc(f)$ and leading term $c_l t_l$ by $lt(f)$.

Let $G = \{g_1, g_2, \cdots, g_s\}$ be a subset of $R[\underline{x}]$ and $>$ be a monomial ordering on $R[\underline{x}]$. We denote by $R[G]$, a subalgebra of $R[\underline{x}]$ finitely generated by G whose elements could be viewed as a polynomial in terms of elements of G. For $\beta = (\beta_1, \beta_2, \cdots, \beta_s) \in \mathbb{N}^{|G|}$, let $G^\beta = g_1^{\beta_1} g_2^{\beta_2} \cdots g_s^{\beta_s}$ be a G-monomial. We say that f in $R[G]$ has a subalgebra standard representation with respect to G if $f = \sum_j c_j G^{\alpha_j}$, where $c_i \in R \setminus \{0\}$ and $\alpha_j \in \mathbb{N}^{|G|}$ such that $lm(G^{\alpha_j}) \leq lm(f)$. We write $SStd(G)$ for the polynomials of $R[G]$ which have a subalgebra standard representation with respect to G. Now, we can define Sagbi and strong Sagbi bases.

Definition 1.1. Let B be a subalgebra of $R[\underline{x}]$, and let G be a finite set of $R[\underline{x}] \setminus \{0\}$. Then G is called a Sagbi basis for a subalgebra B if $B \setminus \{0\} = SStd(G)$.

Definition 1.2. For a finite set G of $R[\underline{x}]\setminus\{0\}$, G is called a strong Sagbi basis for a subalgebra B of $R[\underline{x}]$ if for any $f \in B\setminus\{0\}$, $lt(f) = lt(c_jG^{\alpha_j})$ for some $c_j \in R\setminus\{0\}$ and $\alpha_j \in \mathbb{N}^{|G|}$. Also, a strong Sagbi basis G is called minimal if no proper subset of G is a strong Sagbi basis for B.

Throughout the article, we assume that $R = R_1 \times \cdots \times R_n$, where R_i is also a commutative ring with unity for $1 \leq i \leq n$. Note that all the above notions also hold for $R_i[\underline{x}]$. Moreover, for computational feasibility, it is assumed that the subalgebras admit finite (strong) Sagbi bases.

2. S-join and strong S-join

In this section, we introduce the term S-join (strong S-join) and compute a Sagbi basis (strong Sagbi basis) for subalgebras of $R[\underline{x}]$.

Let $\pi_i : R \to R_i$ be projections which induce maps $\pi : R[\underline{x}] \to R_i[\underline{x}]$. Moreover, these maps induce a map $\pi : R[\underline{x}] \to R_1[\underline{x}] \times \cdots \times R_m[\underline{x}]$ given by $\pi(f) = (\pi_1(f), \cdots, \pi_m(f))$. Let $\kappa : R_1[\underline{x}] \times \cdots \times R_m[\underline{x}] \to R[\underline{x}]$ be a map which collects coefficient of like terms. We can see that π and κ are mutually inverse ring homomorphisms.

Now, we see a result which shows that a projection of a Sagbi basis for a subalgebra is a Sagbi basis for the projected subalgebra.

Proposition 2.1. *If G is a finite Sagbi basis for non-zero finitely generated subalgebra $B \subset R[\underline{x}]$, then $\pi_i(G)\setminus\{0\}$ is a Sagbi basis for $\pi_i(B)$ in $R_i[\underline{x}]$ for $i = 1, \cdots, m$.*

Proof. We can assume that $i = 1$. For $f_1 \in \pi_1(B) \setminus \{0\} \subset R_1[\underline{x}]$, we need to show that $f_1 \in SStd(\pi_1(G))$. Let $f = \kappa(f_1, 0, \cdots, 0) \in B$, therefore, $lt(f) = (lc(f_1), 0, \cdots, 0)lm(f_1)$ such that $lm(f) = lm(f_1)$. Since G is a Sagbi basis for B which implies $f = \sum_j c_jG^{\alpha_j}$ for some $c_j \in R \setminus \{0\}$ and $\alpha_j \in \mathbb{N}^{|G|}$ with $lm(G^{\alpha_j}) \leq lm(f) = lm(f_1)$. After taking projection with respect to π_1, we have $f_1 = \sum_j \pi_1(c_j)\pi_1(G^{\alpha_j})$=$\sum_j \pi_1(c_j)\pi_1(G)^{\alpha_j}$ such that $lm(\pi_1(G^{\alpha_j})) \leq lm(G^{\alpha_j}) \leq lm(f) = lm(f_1)$, where $\pi_1(c_j) \in R_1$ and $\pi_1(G)^{\alpha_j}$ is a $\pi_1(G)$-monomial. Therefore, $f_1 \in SStd(\pi_1(G))$ and $\pi_1(G)$ is a Sagbi basis for $\pi_1(B)$ in $R_1[\underline{x}]$. □

Now, we define the S-join of n sets to find a Sagbi basis for a subalgebra of $R[\underline{x}]$ with the help of Sagbi bases of the projected subalgebras in $R_i[\underline{x}]$ for $i = 1, \cdots, n$.

Definition 2.2. Let $G_i \subset R_i[\underline{x}] \setminus \{0\}$ for $i = 1, 2, \cdots, n$. Then $G_1 \sqcup \cdots \sqcup G_n$, the S-join of G_i is the subset $G_1 \times \{0\}^{n-1} \cup \cdots \cup \{0\}^{n-1} \times G_n$ of $R_1[\underline{x}] \times \cdots \times R_n[\underline{x}]$.

Proposition 2.3. *Let B be a non-zero subalgebra of $R[\underline{x}]$ and $G_i \subset \pi_i(B)$ for $i = 1, \cdots, m$. Then $\kappa(G_1 \sqcup \cdots \sqcup G_m)$ is a Sagbi basis for B if and only if G_i is a Sagbi basis for $\pi_i(B)$ for all $i = 1, \cdots, m$.*

Proof. $\Longleftarrow$ Let $H = \kappa(G_1 \sqcup \cdots \sqcup G_m) \subset B$. Since $SStd(H) \subset B$, we only need to show that $B \subset SStd(H)$ if each G_i is a Sagbi basis for $\pi_i(B)$. Let $f \in B \setminus \{0\}$ which implies that $\pi_i(f) \in \pi_i(B) = SStd(G_i)$, therefore, we can write $\pi_i(f) = \sum_{j_i} c_{j_i} G_i^{\alpha_{j_i}}$ for some $c_{j_i} \in R_i \setminus \{0\}$, $\alpha_{j_i} \in \mathbb{N}^{|G_i|}$ with $lm(G_i^{\alpha_{j_i}}) \leq lm(\pi_i(f)) = lm(f)$, where $1 \leq i \leq n$. Then

$$\begin{aligned}
f &= \kappa(\pi_1(f), \cdots, \pi_m(f)) \\
&= \kappa(\pi_1(f), 0, \cdots, 0) + \cdots + \kappa(0, \cdots, 0, \pi_m(f)) \\
&= \kappa(\textstyle\sum_{j_1} c_{j_1} G_1^{\alpha_{j_1}}, 0, \cdots, 0) + \cdots + \kappa(0, \cdots, 0, \sum_{j_m} c_{j_m} G_m^{\alpha_{j_m}}) \\
&= \textstyle\sum_{j_1} (c_{j_1}, 0, \cdots, 0)\kappa(G_1^{\alpha_{j_1}}, 0, \cdots, 0) + \cdots + \sum_{j_m} (0, \cdots, 0, c_{j_m})\kappa(0, \cdots, 0, G_m^{\alpha_{j_m}}).
\end{aligned}$$

Since $\kappa(0, \cdots, 0, G_i^{\alpha_{j_i}}, 0, \cdots, 0) \in R[H]$ and $lm(\kappa(0, \cdots, 0, G_i^{\alpha_{j_i}}, 0, \cdots, 0)) = lm(G_i^{\alpha_{j_i}}) \leq lm(f)$ for all i and j, we have $f \in SStd(H)$. It follows from Proposition 2.1. $\square$

Example 2.4. Let $G = \{g_1 = 2x^2 + 3x + 1, g_2 = 2x + 1\} \subset \mathbb{Z}_6[x]$ and $B = \mathbb{Z}_6[G]$. Note that G is not a Sagbi basis for subalgebra B since $3x \in B$ but $3x \notin SStd(G)$ as $3x = 3(2x^2+3x+1)+3(2x+1)$ and $lm(2x^2+3x+1) > lm(3x)$. The isomorphism $\chi : \mathbb{Z}_6 \to \mathbb{Z}_2 \times \mathbb{Z}_3$ induces an isomorphism $\chi : \mathbb{Z}_6[x] \to (\mathbb{Z}_2 \times \mathbb{Z}_3)[x]$. We have $\chi(g_1) = (0, 2)x^2 + (1, 0)x + (1, 1)$, $\chi(g_2) = (0, 2)x + (1, 1)$ and $\chi(B) = (\mathbb{Z}_2 \times \mathbb{Z}_3)[\chi(g_1), \chi(g_2)]$. On applying the map π induced by the projection maps $\pi_1 : (\mathbb{Z}_2 \times \mathbb{Z}_3)[x] \longrightarrow \mathbb{Z}_2[x]$ and $\pi_2 : (\mathbb{Z}_2 \times \mathbb{Z}_3)[x] \longrightarrow \mathbb{Z}_3[x]$, we have $\pi(\chi(g_1)) = (x+1, 2x^2+1)$ and $\pi(\chi(g_2)) = (1, 2x+1)$. The projected subalgebras $\pi_1(\chi(B)) = \mathbb{Z}_2[x+1, 1]$, $\pi_2(\chi(B)) = \mathbb{Z}_3[2x^2+1, 2x+1]$ have Sagbi bases $G_1 = \{x\}$ and $G_2 = \{x^2 + 2, x + 2\}$, respectively. Now, by Proposition 2.3, $\kappa(G_1 \sqcup G_2) = \{(1, 0)x, (1, 0), (0, 1)x^2 + (0, 2), (0, 1)x + (0, 2)\}$ is a Sagbi basis for $\chi(B)$ which implies that $H = \chi^{-1}(\kappa(G_1 \sqcup G_2)) = \{3x, 4x^2+2, 4x+2\}$ is a Sagbi basis for B.

Remark 2.5. In Example 2.4, G_1 and G_2 are strong Sagbi bases of $\mathbb{Z}_2[x+1, 1]$ and $\mathbb{Z}_3[2x^2+1, 2x+1]$, respectively but the Sagbi basis H obtained after S-join is not a strong Sagbi basis since $(4x^2+2) - (3x)^2 + 3 = x^2 + 5 \in B$, therefore $lt(x^2+5) = x^2$. However, the set of leading terms of H is $lt(H) = \{3x, 4x^2, 4x\}$ and also observe that 3 and 4 are not units in $\mathbb{Z}_6$, so there does not exist any H-monomial $\mathcal{H}$ and element $c \in \mathbb{Z}_6$ such that $lt(c\mathcal{H}) = x^2$.

The following result shows that we can use a strong Sagbi basis for a subalgebra B in $R[\underline{x}]$ to obtain strong Sagbi bases in the projected subalgebras $\pi_i(B) \subset R_i[\underline{x}]$.

Proposition 2.6. *Let B be a non-zero finitely generated subalgebra of $R[\underline{x}]$. If G is a finite strong Sagbi basis for B, then $\pi_i(G)\setminus\{0\}$ is a strong Sagbi basis for $\pi_i(B)$ in $R_i[\underline{x}]$ for $i = 1, \cdots, m$.*

Proof. We can assume that $i = 1$ and $f_1 \in \pi_1(B) \setminus \{0\} \subset R_1[\underline{x}]$. Put $f = \kappa(f_1, 0, \cdots, 0) \in B$. Since G is a strong Sagbi basis for B, there exists $\alpha \in \mathbb{N}^{|G|}$, $c \in R \setminus \{0\}$ such that $lt(f) = lt(cG^\alpha)$ and $\pi_1(lt(cG^\alpha)) = lt(f_1)$. This means that $\pi_1(lt(cG^\alpha)) \neq 0$, so $\pi_1(cG^\alpha) \neq 0$ and $\pi_1(lt(cG^\alpha)) = lt(\pi_1(cG^\alpha))$. Now, we have $lt(\pi_1(cG^\alpha)) = lt(f_1)$, where $\pi_1(cG^\alpha) = \pi_1(c)\pi_1(G^\alpha) = \pi_1(c)(\pi_1(G))^\alpha$, therefore $lt(f_1) = lt(\pi_1(c)(\pi_1(G))^\alpha)$, where $\pi_1(c) \in R_1$ and $(\pi_1(G))^\alpha$ is a $\pi_1(G)$-monomial. Thus $\pi_1(G)$ is a strong Sagbi basis for $\pi_1(B)$. □

Theorem 2.3 illustrates that a Sagbi basis could be computed with the help of S-join of Sagbi bases of the projected subalgebras. Remark 2.5 shows that a Sagbi basis obtained after the S-join of strong Sagbi bases need not to be a strong Sagbi basis. Therefore, to find out strong Sagbi basis, we need a concept of strong S-join which could be achieved by adding some elements in the S-join.

Definition 2.7. Let $G \subset R_1[\underline{x}]$ and $H \subset R_2[\underline{x}]$. An element $(g, h) \in R_1[G] \times R_2[H]$ is said to be a critical element of $R_1[G] \times R_2[H]$ if $lm(g) = lm(h)$. If this element cannot be expressed in the form $g = g_1 \cdots g_s$, $h = h_1 \cdots h_s$ such that $lm(g_i) = lm(h_i)$, $g_i \in R_1[G]$, $h_i \in R_2[H]$, $deg(g_i) \geq 1$ and $deg(h_i) \geq 1$, for all i, then it is referred to as a necessary critical element.

Remark 2.8. Note that if (g, h) is a necessary critical element of $R_1[G] \times R_2[H]$, then for some $r \in R_1$ and $s \in R_2$, (rg, sh) may also be a necessary critical element of $R_1[G] \times R_2[H]$.

Remark 2.9. If a critical element (g, h) of $R_1[G] \times R_2[H]$ is not a necessary critical element of $R_1[G] \times R_2[H]$, then it can be factored into a product as $g = g_1 \cdots g_s$, $h = h_1 \cdots h_s$ such that $lm(g_i) = lm(h_i)$, i.e., (g_i, h_i) is a necessary critical element of $R_1[G] \times R_2[H]$ for all i. Additionally, we only consider necessary critical elements (g, h) that cannot be factored as $g = rg'$ and $h = sh'$ for some $1 \neq r \in R_1$, $1 \neq s \in R_2$ and a necessary critical element (g', h').

For $G \subset R_1[\underline{x}]$ and $H \subset R_2[\underline{x}]$, let $\Lambda_{G,H}$=$\{\kappa(g, h) \mid (g, h)$ *is a required necessary critical element of* $R_1[G] \times R_2[H]\}$. Now, we define the strong S-join of G and H.

Definition 2.10. Let $G = \{g_1, \cdots, g_s\}$ and $H = \{h_1, \cdots, h_t\}$ be subsets of $R_1[\underline{x}]$ and $R_2[\underline{x}]$, respectively. Then the strong S-join of G and H, denoted by $G \underline{\sqcup} H$, is defined as:

$$G \underline{\sqcup} H = G \sqcup H \cup \pi(\Lambda_{G,H})$$

Now, we find a strong Sagbi basis for a subalgebra in $R[\underline{x}]$ for $R = R_1 \times R_2$ by using the strong S-join of the strong Sagbi bases for the projected subalgebras in $R_1[\underline{x}]$ and $R_2[\underline{x}]$.

Theorem 2.11. *Let B be a non-zero subalgebra of $R[\underline{x}]$ which admits a finite strong Sagbi basis. Let G_i be a subset of $\pi_i(B)$ for $i = 1, 2$. Then $\kappa(G_1 \underline{\sqcup} G_2)$ is a strong Sagbi basis for B if and only if G_i is a strong Sagbi basis for $\pi_i(B)$ for all $i = 1, 2$.*

Proof. $\Longleftarrow$ Assume that G_i is a strong Sagbi basis of $\pi_i(B)$ for $i = 1, 2$ and $G = \kappa(G_1 \underline{\sqcup} G_2)$. We need to show that for any $f \in B \setminus \{0\}$, there are a G-monomial τ and $c \in R \setminus \{0\}$ such that $lt(f) = lt(c\tau)$. Let $\pi_i(f) = f_i$ for $i = 1, 2$. We have the following cases:
(i) $f_1 \neq 0$ and $f_2 = 0$: Then $lt(f) = (lc(f_1), 0)lm(f_1)$. Since G_1 is a strong Sagbi basis for $\pi_1(B)$, there exists a G_1-monomial $G_1^{\alpha_1}$ (where $\alpha_1 \in \mathbb{N}^{|G_1|}$) and $c_1 \in R_1$ such that $lt(f_1) = lt(c_1 G_1^{\alpha_1})$. Therefore, $lc(f_1) = c_1 lc(G_1^{\alpha_1})$ and $lm(f_1) = lm(G_1^{\alpha_1})$. Define $\tau = \kappa(G_1^{\alpha_1}, 0)$, a G-monomial, and $c = (c_1, 0) \in R$. Note that $lc(\tau) = lc(G_1^{\alpha}, 0)$, therefore, $lt(c\tau) = c\ lc(\tau)lm(\tau) = (c_1, 0)(lc(G_1^{\alpha_1}), 0)lm(G_1^{\alpha_1}) = (c_1 lc(G_1^{\alpha_1}), 0)lm(G_1^{\alpha_1}) = (lc(f_1), 0)lm(f_1) = lt(f)$. Thus $lt(f) = lt(c\tau)$.
(ii) $f_1 = 0$ and $f_2 \neq 0$: Same as (i).
(iii) $f_1 \neq 0$, $f_2 \neq 0$ and $lm(f_1) > lm(f_2)$: Similar to (i) since $lt(f) = (lc(f_1), 0)lm(f_1)$.
(iv) $f_1 \neq 0$, $f_2 \neq 0$ and $lm(f_1) < lm(f_2)$: Similar to (ii) since $lt(f) = (0, lc(f_2))lm(f_2)$.
(v) $f_1 \neq 0$, $f_2 \neq 0$ and $lm(f_1) = lm(f_2) = lm(f)$: Then $lt(f) = (lc(f_1), lc(f_2))lm(f_1)$ $=(lc(f_1), lc(f_2))lm(f_2)$. Since $lm(f_1) = lm(f_2)$, we have that (f_1, f_2) is a critical element of $\pi_1(B) \times \pi_2(B)$. By Remark 2.9, we can write it as $f_1 = g_1 \cdots g_s = \sqcap_{i=1}^{s} g_i$ and $f_2 = g_1^{'} \cdots g_s^{'} = \sqcap_{i=1}^{s} g_i^{'}$ such that $lm(g_i) = lm(g_i^{'})$ for all i, and $(g_i, g_i^{'})$ are necessary critical elements which implies that $\kappa(g_i, g_i^{'}) \in \Lambda_{G_1, G_2}$. Note that $(f_1, f_2) = (g_1, \cdots g_s, g_1^{'} \cdots g_s^{'})$ which is denoted by $\sqcap_{i=1}^{s}(g_i, g_i^{'})$. On applying κ, we have $f = \kappa(f_1, f_2) = \kappa(\sqcap_{i=1}^{s}(g_i, g_i^{'})) = \sqcap_{i=1}^{s}\kappa(g_i, g_i^{'})$. Moreover, $lm(\sqcap_{i=1}^{s}\kappa(g_i, g_i^{'})) = lm(\sqcap_{i=1}^{s} g_i) = lm(\sqcap_{i=1}^{s} g_i^{'}) = lm(f_2) = lm(f_1) = lm(f)$ and $lc(\sqcap_{i=1}^{s}\kappa(g_i, g_i^{'})) = lc(\kappa(f_1, f_2)) = lc(f)$. Since $\kappa(g_i, g_i^{'}) \in \Lambda_{G_1, G_2} \subset G$, $\sqcap_{i=1}^{s}\kappa(g_i, g_i^{'})$ could be considered as a G-monomial denoted by τ. By setting $c = (1, 1)$, we have, $lt(c\tau) = (1, 1)lc(\tau)lm(\tau) = (1, 1)lc(\sqcap_{i=1}^{s}\kappa(g_i, g_i^{'}))lm(\sqcap_{i=1}^{s}\kappa(g_i, g_i^{'})) = lc(f)lm(f) = lt(f)$. It follows from Proposition 2.6. $\square$

The following proposition plays an important role in generalizing Theorem 2.11 inductively for n-sets.

Proposition 2.12. *Let G_i be a subset of $R_i[\underline{x}]$ for $i = 1, 2, 3$, then*

$$\kappa(\kappa(G_1 \sqcup G_2) \sqcup G_3) = \kappa(G_1 \sqcup \kappa(G_2 \sqcup G_3)).$$

Proof. First, we prove the result for the elements obtained after the S-join of G_i for $i = 1, 2, 3$. Let $h \in \kappa(\kappa(G_1 \sqcup G_2) \sqcup G_3)$, we have three cases: $h = \kappa(\kappa(g_1, 0), 0)$, $h = \kappa(\kappa(0, g_2), 0)$, $h = \kappa(0, g_3)$, where $g_i \in G_i$ for $i = 1, 2, 3$. Note that, according to the cases, we can write h as: $h = \kappa(\kappa(g_1, 0), 0) = \kappa(g_1, 0)$, $h = \kappa(\kappa(0, g_2), 0) = \kappa(0, \kappa(g_2, 0))$, $h = \kappa(0, g_3) = \kappa(0, \kappa(0, g_3))$, respectively. In all the cases, h can be viewed as an element of $\kappa(G_1 \sqcup \kappa(G_2 \sqcup G_3))$. The reverse inclusion could be proved in the same way.

Now, we prove the result for the elements of $\pi(\Lambda_{G_i, G_j})$. Let $h \in \kappa(\kappa(G_1 \sqcup G_2) \sqcup G_3)$ such that $lm(h) = lm(\kappa(h', h_3))$ with $h' = \kappa(h_1, h_2) \in \kappa(G_1 \sqcup G_2))$, where $h_i \in G_i$ for $i = 1, 2, 3$. Note that $lm(h) = lm(h') = lm(h_3)$. Moreover, $lm(h') = lm(h_1) = lm(h_2)$. We can see an element $(h_2, h_3) \in \pi(\Lambda_{G_2, G_3})$ since $lm(h_2) = lm(h_3)$, therefore $\kappa(h_2, h_3) \in \kappa(G_2 \sqcup G_3)$. Thus $h = \kappa(h_1, \kappa(h_2, h_3)) \in \kappa(G_1 \sqcup \kappa(G_2 \sqcup G_3))$. Similarly, we can show the reverse inclusion. □

Remark 2.13. We can compute the necessary critical elements algorithmically: Let $e = (e_1, \cdots, e_t) \in \mathbb{N}^t$ be the exponent vector of the monomial $x_1^{e_1} \cdots x_t^{e_t}$, $G \subset R_1[\underline{x}]$ and $H \subset R_2[\underline{x}]$. For each $g \in G$ and $h \in H$, let e_g and e_h be the leading exponents (exponent of leading monomial) of g and h, respectively. We need to find non-zero vectors $\vec{a} \in \mathbb{N}^{|G|}$ and $\vec{b} \in \mathbb{N}^{|H|}$ such that $lm(G^{\vec{a}}) = lm(H^{\vec{b}})$. Let $\vec{y} = \{y_g\}_{g \in G}$ and $\vec{z} = \{z_h\}_{h \in H}$ be variable vectors whose entries can take non-negative integer values such that $lm(G^{\vec{y}}) = lm(H^{\vec{z}})$. This leads to a system of t linear equations given by:

$$\sum_{g \in G} e_g y_g - \sum_{h \in H} e_h z_h = \mathbf{0} \tag{1}$$

where $\mathbf{0}$ is the zero vector in $\mathbb{N}^t$. The vectors $\vec{a} = \{a_g\}_{g \in G}$ and $\vec{b} = \{b_h\}_{h \in H}$ are solutions to System 1 if and only if $\sum_{g \in G} e_g a_g = \sum_{h \in H} e_h b_h$, which implies that $lm(G^{\vec{a}}) = lm(H^{\vec{b}})$, i.e., $(G^{\vec{a}}, H^{\vec{b}})$ is a critical element. By using a basis of the solution set of System 1, we have all necessary critical elements of $R_1[G] \times R_2[H]$.

Based on Theorem 2.11 and Proposition 2.12, we develop the following algorithm which computes a strong Sagbi basis for a subalgebra in $R[\underline{x}]$ with the help of strong S-join of the strong Sagbi bases for the projected subalgebras in $R_i[\underline{x}]$ for $i = 1, \cdots, n$.

Algorithm 2.14. *Input: a monomial ordering $>$ on $R[\underline{x}]$, a finite subset F of $R[\underline{x}]$, an algorithm SB_i to compute strong Sagbi basis for every $\pi_i(F) \subseteq R_i[\underline{x}]$ ($i = 1, \cdots, n$).*
Output: A strong Sagbi basis G for subalgebra $R[F]$.
Instructions:

- *For i=1 to n;*
  $G_i = SB_i(\pi_i(F))$;
  minimize G_i
- $G = G_1$;
- *For i=2 to n;*
  Compute all the necessary critical elements of $R_1[G] \times R[G_i]$ *by using Remark 2.13;*
  Compute $\Lambda(G, G_i)$;
  $G \underline{\sqcup} G_i = G \sqcup G_i \cup \pi(\Lambda(G, G_i))$;
  $G = \kappa(G \underline{\sqcup} G_i)$;
  minimize G;
- *return G.*

Example 2.15. Let $G = \{2x^2 + 3x + 1, 2x + 1\} \subset \mathbb{Z}_6[x]$ and $B = \mathbb{Z}_6[G]$ as in Example 2.4. The projected subalgebras $\pi_1(\chi(B)) = \mathbb{Z}_2[x + 1, 1]$ and $\pi_2(\chi(B)) = \mathbb{Z}_3[2x^2 + 1, 2x + 1]$ have minimal strong Sagbi bases $G_1 = \{g_1 = x\}$ and $G_2 = \{g_1^{'} = x^2 + 2, g_2^{'} = x + 2\}$, respectively. From Remark 2.5, $\chi^{-1}(\kappa(G_1 \sqcup G_2)) = \{3x, 2+4x^2, 4x+2\}$ is not a strong Sagbi basis of B, therefore, we need a strong join $G_1 \underline{\sqcup} G_2 = G_1 \sqcup G_2 \cup \pi(\Lambda_{G_1,G_2})$. Note that by using the approach of finding necessary critical elements described in Remark 2.13, the corresponding system is a single linear equation $y_{g_1} - 2z_{g_1^{'}} - z_{g_2^{'}}$. A basis {(2,1,0), (1,0,1), (3,1,1)} to the solution set of this equation provides us the necessary critical elements: $(g_1^2, g_1^{'})$, $(g_1, g_2^{'})$ and $(g_1^3, g_1^{'} g_2^{'})$. Therefore, on applying κ on these elements, we have $\Lambda_{G_1,G_2} = \{(1,1)x^2 + (0,2), (1,1)x + (0,2), (1,1)x^2 + (0,1)x + (0,1)\}$. From Theorem 2.11, $\chi^{-1}(\kappa(G_1 \underline{\sqcup} G_2)) = \{3x, 4x^2 + 2, 4x + 2, x^2 + 2, x + 2, x^3 + 2x^2 + 2x + 4\}$ is a strong Sagbi basis for B and $G^{'} = \{3x, x^2 + 2, x + 2\}$ is a minimal strong Sagbi basis for B.

3. Existence of strong Sagbi basis over the principal ideal rings

In this section, as an application of our main results in Section 2, we prove the existence of a strong Sagbi basis in the case of principal ideal rings. We restrict R to be a principal ideal ring onwards in this section unless specified.

Definition 3.1. Let $G \subset R[\underline{x}] \setminus \{0\}$ be a finite set and $f, h \in R[\underline{x}]$.

(i) We say that f reduces to h with respect to G in one step (and that f is reducible with respect to G) if $h = f - \sum_i c_i G^{\alpha_i}$ where $c_i \in R$, $\alpha_i \in \mathbb{N}^{|G|}$, such that $lt(f) = \sum_i c_i lt(G^{\alpha_i})$, $c_i \neq 0$, $lc(f) = \sum_i c_i lc(G^{\alpha_i})$ and $lm(f) = lm(G^{\alpha_i})$, for all i (see [6, Definition 3.2]). We write it as $f \longrightarrow^G h$.

(ii) We say that f strongly reduces to h with respect to G in one step (and that f is strongly reducible with respect to G) if $h = f - cG^{\alpha}$ where $c \in R$, $\alpha \in \mathbb{N}^{|G|}$ such that $lt(f) = lt(cG^{\alpha})$.

Remark 3.2. A set S is a Sagbi basis for a subalgebra B if and only if for all $f \in B$, $f \longrightarrow_*^S 0$, i.e., there exists a chain of one-step reductions and $h_1, h_2, \cdots, h_k$ in B such that $f \longrightarrow^S h_1 \longrightarrow^S h_2 \longrightarrow^S \cdots h_k \longrightarrow^S 0$ (see [6, Proposition 3.3]) which is equivalent to the condition that f has a subalgebra standard representation with respect to S.

Proposition 3.3. *Let R be a finite chain ring*[1] *and $G \subset R[\underline{x}] \setminus \{0\}$ be a finite set and $f, h \in R[\underline{x}]$. Then f is one step reducible with respect to G if and only if f is strongly one step reducible with respect to G.*

Proof. Let f be one step reducible to h with respect to G, then $h = f - \sum_i c_i G^{\alpha_i}$ where $c_i \in R$, $\alpha_i \in \mathbb{N}^{|G|}$, such that $lt(f) = \sum_i c_i lt(G^{\alpha_i})$, $c_i \neq 0$, $lc(f) = \sum_i c_i lc(G^{\alpha_i})$ and $lm(f) = lm(G^{\alpha_i})$, for all i. The collection of the leading coefficients $lc(G^{\alpha_i})$ is a subset of R which is a finite chain ring, therefore, by [7, Proposition 2.2], there exist some j and $\alpha_i \in R$ such that $lc(G^{\alpha_i}) = \alpha_i lc(G^{\alpha_j})$ for every i. Then $lt(f) = lt(cG^{\alpha_j})$ for some $c \in R$. Hence f is strongly one-step reducible with respect to G. □

Proposition 3.3 and Remark 3.2 yield the following result.

Corollary 3.4. *Let B be a subalgebra over a finite-chain ring R. Then G is a Sagbi basis for B if and only if G is a strong Sagbi basis for B.*

Proof. Let G be a Sagbi basis for B. For any $f \in B$, $f \longrightarrow_*^S 0$. At the first one step of the reduction of f, we can assume $f \longrightarrow^S h_1$, so $h_1 = f - \sum_i c_i G^{\alpha_i}$ such that $lm(f) = lm(G^{\alpha_i})$ for all i, and $lc(f) = \sum_i c_i lc(G^{\alpha_i})$. Since the one-step reduction is also a strong one-step reduction in R by Proposition 3.3, there exists some $i = i_0$ such that $lm(f) = lm(G^{\alpha_{i_0}})$, and $lc(f) = clc(G^{\alpha_{i_0}})$ for some $c \in R$, i.e., $lt(f) = lt(cG^{\alpha_{i_0}})$. It implies that G is a strong Sagbi basis for B. The converse holds trivially. □

[1]Finite chain rings are precisely finite local rings whose maximal ideals are principal.

Now, we show the existence of a strong Sagbi basis of a subalgebra over a principal ideal ring provided that it has a finite Sagbi basis.

Proposition 3.5. *Let R be a principal ideal ring and B be a subalgebra of $R[\underline{x}]$ with a finite Sagbi basis. Then B has a strong Sagbi basis.*

Proof. We have $R \cong R_1 \times \cdots \times R_m$, where each R_i is a principal ideal domain or a finite chain ring (see Theorem 33, Section 15, Chapter 4 of [11]).

In the case when R_i is a principal ideal domain, we can obtain a strong Sagbi basis of $\pi_i(B)$ over R_i (see [1, Proposition 5.1 and Theorem 5.2]).

If R_i is a finite-chain ring, from assumption and Proposition 2.1, $\pi_i(B)$ has a finite Sagbi basis. From Corollary 3.4, it is a strong Sagbi basis for $\pi_i(B)$.

Each $\pi_i(B) \subseteq R_i[\underline{x}]$ admits a strong Sagbi basis. Therefore, with the help of Theorem 2.11, we can construct a strong Sagbi basis for B in $R[\underline{x}]$. □

Example 3.6. Let $H = \{4xy+x, 3x^2+y\} \subset \mathbb{Z}_{20}[x]$ and a subalgebra $B = \mathbb{Z}_{20}[H]$, where $\mathbb{Z}_{20}$ is a principal ideal ring isomorphic to $\mathbb{Z}_4 \times \mathbb{Z}_5$ which is a product of principal ideal domain $\mathbb{Z}_5$ and a finite chain ring $\mathbb{Z}_4$. The isomorphism $\chi : \mathbb{Z}_{20} \to \mathbb{Z}_4 \times \mathbb{Z}_5$ induces an isomorphism $\chi : \mathbb{Z}_{20}[x] \to (\mathbb{Z}_4 \times \mathbb{Z}_5)[x]$ and $\chi(B) = (\mathbb{Z}_4 \times \mathbb{Z}_5)[\chi(H)]$. Note that $G_1 = \{x, y\}$ and $G_2 = \{4xy + x, x^2 + 2y\}$ are Sagbi bases for $\pi_1(\chi(B)) = \mathbb{Z}_4[H_1]$ and $\pi_2(\chi(B)) = \mathbb{Z}_5[H_2]$, respectively, where $H_i = \pi_i(H)$. With the help of Theorem 2.11, we obtain a strong Sagbi basis $\kappa(G_1 \sqcup G_2)$ for $\chi(B)$. Therefore, $\chi^{-1}(\kappa(G_1 \sqcup G_2)) = \{5x, 5y, 4x + 16xy, 16x^2 + 12y, xy + 4x, x^2 + 12y\}$ is a strong Sagbi basis for $\mathbb{Z}_{20}[H]$. Moreover, $G = \{5x, 5y, xy + 4x, x^2 + 12y\}$ is a minimal strong Sagbi basis for $\mathbb{Z}_{20}[H]$.

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References

[1] W. W. Adams, S. Hosten, P. Loustaunau and J. L. Miller, *Sagbi and Sagbi-Gröbner bases over principal ideal domains*, J. Symbolic Comput., 27(1) (1999), 31-47.

[2] W. W. Adams and P. Loustaunau, An Introduction to Gröbner Bases, Graduate Studies in Mathematics, 3, American Mathematical Society, Providence, RI, 1994.

[3] B. Buchberger, An Algorithm for Finding the Bases of the Residue Class Ring Modulo a Zero Dimensional Polynomial Ideal, Ph.D. thesis, University of Innsbruck, Austria, 1965.

[4] B. Buchberger and F. Winkler, Gröbner Bases and Applications, London Mathematical Society Lecture Note Series, 251, Cambridge University Press, Cambridge, 1998.

[5] M. Kreuzer and L. Robbiano, Computational Commutative Algebra 1, Springer-Verlag, Berlin, 2000.

[6] J. L. Miller, *Analogs of Gröbner bases in polynomial rings over a ring*, J. Symbolic Comput., 21(2) (1996), 139-153.

[7] G. H. Norton and A. Salagean, *Strong Gröbner bases for polynomials over a principal ideal ring*, Bull. Austral. Math. Soc., 64(3) (2001), 505-528.

[8] G. H. Norton and A. Salagean, *Gröbner bases and products of coefficient rings*, Bull. Austral. Math. Soc., 65(1) (2002), 145-152.

[9] L. Robbiano and M. Sweedler, *Subalgebra bases*, Commutative algebra (Salvador, 1988), Lecture Notes in Math., Springer, Berlin, 1430 (1990), 61-87.

[10] M. Stillman and H. Tsai, *Using Sagbi bases to compute invariants*, J. Pure Appl. Algebra, 139(1-3) (1999), 285-302.

[11] O. Zariski and P. Samuel, Commutative Algebra, Vol. 1, Graduate Texts in Mathematics, 28, Springer-Verlag, New York-Heidelberg-Berlin, 1975.

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