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ON SEMI-PROJECTIVE MODULAR LATTICES

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Abstract. A. Haghany and M. Vedadi, as well as M. K. Patel, explored the relationship between a semi-projective and retractable module and its endomorphism ring. In this work, we study the lattice-theoretic counterparts of these results. To this end, we consider the category of linear modular lattices. Specifically, we show a relation between a retractable and semi-projective complete modular lattice and its monoid of endomorphisms.

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1. Introduction

For each right module M_R, the set $L(M)$ of submodules of M, ordered by inclusion, is a complete modular lattice. Recall the correspondence theorem: if $f : M_R \to N_R$ is a morphism of right modules, then there is a function $f_* : [\ker f, M] \to [0_R, \mathrm{Im}(f)]$ such that $f_*(A) := \{f(x) \mid x \in A\}$. f_* is a lattice isomorphism, whose inverse is f^* defined by $f^*(B) := \{x \in M \mid f(x) \in B\}$.

For a bounded lattice L, write 0_L (resp., 1_L) for the least (resp., greatest) element of L. Albu and Iosif, in [1], define a linear morphism f between two bounded modular lattices L and M as a function $f : L \to M$ for which there exists $k_f \in L$ such that $f \mid: [k_f, 1_L] \to [0_M, f(1_L)]$ is an isomorphism of lattices, and such that $f(x \vee k_f) = f(x)\ \forall x \in L$. In this definition, k_f is called the kernel of f. We will denote an interval $[a, b]$ in the lattice L by b/a. Special cases are the *initial interval* $a/0_L$, where $a \in L$, and the *quotient interval* $1_L/b$, where $b \in L$.

There is a category $\mathcal{L}_{\mathcal{M}}$ whose objects are the bounded modular lattices, and whose morphisms are linear morphisms.

From these definitions, it is easy to see that for an R-morphism $f : M_R \to N_R$, the function $f_* : [0_R, M] \to [0_R, N]$ is a linear morphism whose kernel k_{f_*} coincides with $\ker f$ because $f_* \mid [\ker f, M] \to [0_R, \mathrm{Im}(f)]$ is a lattice isomorphism, by the correspondence theorem.

In fact, the assignments

$$M \longmapsto L(M); \qquad \begin{array}{c} M \\ \big\downarrow f \\ N \end{array} \quad \longmapsto \quad \begin{array}{c} L(M) \\ \big\downarrow f_* \\ L(N) \end{array}$$

define a functor from R-Mod to $\mathcal{L}_{\mathcal{M}}$.

In [5] A. Haghany and M. Vedadi studied the relation between a semi-projective and retractable module and its endomorphism ring. In particular, they found necessary and sufficient conditions on a module M for its endomorphism ring S to be semiprime, right non-singular, finitely cogenerated, cocyclic, or weakly co-Hopfian. Furthermore, they give descriptions of the right singular ideal of S and of the socle of M.

In [9] M. K. Patel studies how the Hopfian, co-Hopfian, or directly finite properties of a semi-projective module are reflected in its endomorphism ring. He further proves that for a pseudo-semi-injective module to be Hopfian, co-Hopfian, and directly finite are equivalent conditions.

In this work, we consider the full subcategory of $\mathcal{L}_{\mathcal{M}}$ whose objects are complete modular lattices. It is in the context of this category that we aim to extend concepts and results of module theory, such as the aforementioned ones, to their lattice-theoretic counterparts. Previous research on this line has been done in [7] and [8]. Here, we translate the notion of semi-projective modules to linear modular lattices.

The rest of the paper is organized as follows: Section 2 gives some preliminary definitions and introduces the notation for the category $\mathcal{L}_{\mathcal{M}}$. In Section 3, we describe semi-projective lattices and their relation with their corresponding monoids of endomorphisms.

2. Preliminaries

This section provides basic notions and definitions of linear modular lattices and linear morphisms. We refer the reader to [1] for a concise description of the category $\mathcal{L}_{\mathcal{M}}$.

We denote by $\mathcal{L}$ the class of all bounded modular lattices. $\mathcal{L}$ is the class of objects of a category where the morphisms are the usual lattice morphisms, this is, functions that respect the lattice operations of infimum and supremum. However, these lattice morphisms fail to express important module properties. In contrast, linear lattice morphisms, or linear morphisms for short, come with two elements that evoke the notion of kernel and image, so the First Isomorphism Theorem for modules holds now for lattices. These linear morphisms, in turn, define a suitable category that we will use to extend module-related properties to their lattice-theoretic counterparts.

Definition 2.1. [1, Definition 1.1] Let $L, L' \in \mathcal{L}$. The mapping $f : L \longrightarrow L'$ is called a *linear morphism* if there exists $k \in L$, called the kernel of f, and $a' \in L'$ such that the following two conditions hold:

(1) $f(x) = f(x \vee k)$ for all $x \in L$.
(2) The function f induces a lattice isomorphism $\overline{f} : 1_L/k \longrightarrow a'/0_{L'}$ such that $\overline{f}(x) = f(x)$ for all $x \in 1_L/k$.

In [1, Proposition 2.2(1)], the authors denote by $\mathcal{L}_{\mathcal{M}}$ the category of linear modular lattices, whose objects are bounded modular lattices, and whose morphisms are linear morphisms. Throughout this work, the objects of $\mathcal{L}_{\mathcal{M}}$ will be *complete* modular lattices.

In the category $\mathcal{L}_{\mathcal{M}}$, the isomorphisms are precisely the lattice isomorphisms, the monomorphisms are precisely the injective linear morphisms (which coincide with those linear morphisms with kernel zero), and the epimorphisms are precisely the surjective linear morphisms. These results can be found in [1, Proposition 2.2(2)-(4) and Corollary 1.6]. We shall make use of them freely.

Remark 2.2. The category $\mathcal{L}_{\mathcal{M}}$ has a zero object and this is the lattice with a single element, called the *zero lattice* and denoted by 0, because for any $L \in \mathcal{L}_{\mathcal{M}}$ the only linear morphisms $0 \longrightarrow L$ and $L \longrightarrow 0$ are in both cases the zero constants.

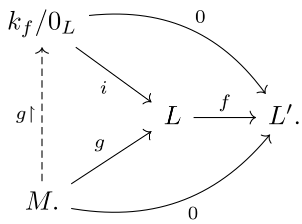
The *zero morphism* $0_{L,M} : L \longrightarrow M$ is the only morphism that factors through the zero object: $0_{L,M} : L \longrightarrow 0 \longrightarrow M$. Thus the zero morphism between two lattices must be the constant zero. If there is no risk of confusion, we shall simply write $0 : L \to M$.

Theorem 2.3. *In the category $\mathcal{L}_{\mathcal{M}}$, every linear morphism $L \xrightarrow{f} L'$ has a kernel*[1]*, and it is given by the inclusion mapping $k_f/0_L \xrightarrow{i} L$, where k_f is the kernel of f as a linear morphism.*

Proof. By [1, Corollary 1.4], f is an increasing map, so for every $x \in k_f/0_L$, $f(x) \leq f(k_f) = 0_{L'}$, that is, $f(x) = 0_{L'}$, and therefore the composite $f \circ i$ is the zero linear morphism.

[1]In the sense of category theory.

On the other hand, if we assume that $M \in \mathcal{L}_{\mathcal{M}}$ and $M \xrightarrow{g} L$ is a linear morphism such that $f \circ g = 0$, then by [1, Proposition 1.3(2)], $g(1_M) \leq k_f$ and therefore the corestriction $M \xrightarrow{g\restriction} k_f/0_L$ is the only linear morphism that makes the following diagram commute:



□

Lemma 2.4. *If $L, L' \in \mathcal{L}$ and $f, g : L \longrightarrow L'$ are linear morphisms with respective kernels k_f, k_g, such that $k_f = k_g$ and that the induced lattice isomorphisms $\overline{f}$ and $\overline{g}$ coincide, then $f = g$.*

Proof. If $x \in L$, then $f(x) = f(x \bigvee k_f) = \overline{f}(x \bigvee k_f) = \overline{g}(x \bigvee k_g) = g(x \bigvee k_g) = g(x)$. □

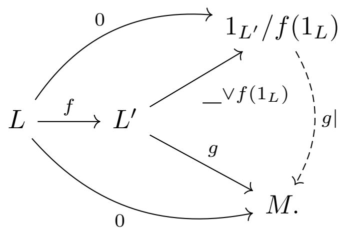
Theorem 2.5. *In the category $\mathcal{L}_{\mathcal{M}}$, every linear morphism $L \xrightarrow{f} L'$ has a cokernel, and it is given by the linear morphism $L' \xrightarrow{\_\vee f(1_L)} 1_{L'}/f(1_L)$.*

Proof. For every element $x \in L$, $f(x) \vee f(1_L) = f(1_L)$, so the composite

$$L \xrightarrow{f} L' \xrightarrow{\_\vee f(1_L)} 1_{L'}/f(1_L)$$

is the zero linear morphism.

On the other hand, if we assume that $L' \xrightarrow{g} M$ is a linear morphism such that the composite $L \xrightarrow{f} L' \xrightarrow{g} M$ is the zero linear morphism, then $g(f(x)) = 0_M$ for every element $x \in L$, and then by [1, Proposition 1.3(2)], $f(1_L) \leq k_g$, so that the restriction of g given by $1_{L'}/f(1_L) \xrightarrow{g|} M$ is a linear morphism that makes the following diagram commute:



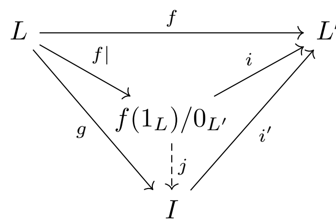
In addition, for any other linear morphism $1_{L'}/f(1_L) \xrightarrow{h} M$ such that $h \circ (\_ \vee f(1_L)) = g$, we have, for $x \in 1_{L'}/f(1_L)$, that

$$h(x) = h(x \vee f(1_L)) = g(x) = g|(x)$$

so that $h = g|$. □

Theorem 2.6. *In the category $\mathcal{L}_{\mathcal{M}}$, every linear morphism $L \xrightarrow{f} L'$ has an image in the categorical sense, and it is given by the inclusion mapping $f(1_L)/0_{L'} \xrightarrow{i} L'$.*

Proof. Let $f| : L \longrightarrow f(1_L)/0_{L'}$ be the corestriction of f, then $f = i \circ f|$.
On the other hand, if we assume that $I = a/0_{L'}$ is an initial interval of L', and that there is a linear morphism $g : L \longrightarrow I$ such that $f = i' \circ g$ where $I \xrightarrow{i'} L'$ is the inclusion mapping, then $f(1_L) = g(1_L) \leq a$, therefore the inclusion mapping $f(1_L)/0_{L'} \xrightarrow{j} I$ is the only linear morphism that makes the following diagram commutative:



□

Definition 2.7. We will say that a sequence of linear morphisms

$$\dots \xrightarrow{f_{i-2}} L_{i-1} \xrightarrow{f_{i-1}} L_i \xrightarrow{f_i} L_{i+1} \xrightarrow{f_{i+1}} \dots$$

is *exact* if for any pair of consecutive linear morphisms f_{i-1} and f_i, the image of f_{i-1} is the kernel of f_i. By Theorems 2.3 and 2.6, this happens if and only if $f_{i-1}(1_{L_{i-1}}) = k_{f_i}$.

Remark 2.8. The sequence of linear morphisms $L \xrightarrow{f} M \xrightarrow{g} N$ is exact if and only if $g \circ f = 0$ and for all $m \in M$ such that $g(m) = 0_N$, there exists $a \in L$ such that $f(a) = m$.

Also, note that the sequence $0 \to L \xrightarrow{f} M$ (resp., $L \xrightarrow{f} M \to 0$) in $\mathcal{L}_{\mathcal{M}}$ is exact if and only if f is a monomorphism (resp., an epimorphism).

Although the category $\mathcal{L}_{\mathcal{M}}$ is not abelian, we have at this point built enough structure for it to satisfy, except for the non-existence of subtraction, all the so-called elementary rules for chasing elements in diagrams stated by Mac Lane in [6, VIII-4 Theorem 3].

Remark 2.9. If $L \xrightarrow{f} L'$ is a linear morphism, we can decompose it as $f = m_f \circ e_f$ where m_f is an injective linear morphism and e_f is a surjective linear morphism; for this we only need to consider the following diagram:

$$\underbrace{L \xrightarrow{f\restriction} f(1_L)/0_{L'} \xrightarrow{i} L'}_{f}.$$

Let us note that the inclusion $f(1_L)/0_{L'} \xrightarrow{i} L'$ is the kernel of the cokernel of f, while the corestriction $L \xrightarrow{f\restriction} f(1_L)/0_{L'} \cong 1_L/k_f$ is the cokernel of the kernel of f. Moreover, if $M \xrightarrow{f'} M'$ is another linear morphism and we have the commutative diagram, for linear morphisms g, h,

$$\begin{CD} L @>{f}>> L' \\ @V{h}VV @VV{g}V \\ M @>{f'}>> M' \end{CD}$$

then $(g \circ f)(1_L) = (f' \circ h)(1_L) \leq f'(1_M)$, so that by restricting and correcting g, the linear morphism $f(1_L)/0_{L'} \xrightarrow{g|} f'(1_M)/0_{M'}$ is well defined. Thus, by decomposing f and f' as in Remark 2.9, we obtain the following diagram

$$\begin{CD} L @>{f\restriction}>> f(1_L)/0_{L'} @>{i}>> L' \\ @V{h}VV @V{g|}VV @VV{g}V \\ M @>{f'\restriction}>> f'(1_M)/0_{M'} @>{i}>> M' \end{CD}$$

where $g|$ is the only linear morphism that makes it commutative.

3. Semi-projective lattices

We start this section by translating well-known definitions of some module properties into their lattice-theoretic counterpart.

Definition 3.1. Let $L \in \mathcal{L}_{\mathcal{M}}$ and $a \in L$. We call the initial interval $a/0_L$ *L-cyclic* if $a/0_L$ is isomorphic to a quotient interval of L.

Definition 3.2. Let $L \in \mathcal{L}_{\mathcal{M}}$. We call L *retractable* if for every non-trivial initial interval $a/0_L$ of L, one has that

$$Hom_{\mathcal{L}_{\mathcal{M}}}\big(L, a/0_L\big) \neq 0.$$

In other words, L is retractable if every non-trivial initial interval of L has a non-trivial L-cyclic initial interval. The following two examples display a retractable and a non-retractable lattice, respectively.

Example 3.3. Every complemented lattice $L \in \mathcal{L}_{\mathcal{M}}$ is retractable. Indeed, let $a/0_L$ be a non-trivial initial interval of L, and let $b \in L$ be a complement of a. Then, the function defined by $L \xrightarrow{\_\vee b} 1_L/b$ is a linear morphism with kernel b.

Furthermore, we have, by modularity, the lattice isomorphism

$$1_L/b = {}^{a\vee b}/_{b} \xrightarrow{\ \_\wedge a\ } {}^{a}/_{a\wedge b} = a/0_L.$$

Therefore, the non-trivial linear morphism $(\_\wedge a)\circ(\_\vee b)$ lies in $Hom_{\mathcal{L}_\mathcal{M}}\big(L, a/0_L\big)$, showing thus that L is retractable.

Example 3.4. Consider the totally ordered set $L=\mathbb{N}\cup\{\infty\}$. Since no finite initial interval of $\mathbb{N}\cup\{\infty\}$ is L-cyclic, $\mathbb{N}\cup\{\infty\}$ is not retractable.

Definition 3.5. A lattice $L\in\mathcal{L}_\mathcal{M}$ is *semi-projective* if for any initial interval $a/0_L$ of L, and any diagram

$$\begin{array}{ccccc} & & L & & \\ & & \big\downarrow{\scriptstyle g} & & \\ L & \xrightarrow{\ f\ } & a/0_L & \longrightarrow & 0 \end{array}$$

with the bottom row exact, there exists a linear morphism $h: L\longrightarrow L$ that makes

$$\begin{array}{ccc} & & L \\ & {}^{h}\swarrow & \big\downarrow{\scriptstyle g} \\ L & \xrightarrow{\ f\ } & a/0_L \end{array}$$

a commutative diagram; that is, $f\circ h = g$.

Proposition 3.6. *$L\in\mathcal{L}_\mathcal{M}$ is semi-projective if and only if $g\circ End_{\mathcal{L}_\mathcal{M}}(L)\subseteq f\circ End_{\mathcal{L}_\mathcal{M}}(L)$ for any pair of linear endomorphisms $f,g\in End_{\mathcal{L}_\mathcal{M}}(L)$ such that $g(1_L)\le f(1_L)$.*

Proof. $(\Longrightarrow)$ If f and g are corestricted to $f(1_L)/0_L$, we obtain the following diagram, whose bottom row is exact.

$$\begin{array}{ccccc} & & L & & \\ & & \big\downarrow{\scriptstyle g} & & \\ L & \xrightarrow{\ f\ } & f(1_L)/0_L & \longrightarrow & 0 \end{array}$$

As L is semi-projective, there is a linear morphism $h: L\longrightarrow L$ such that $f\circ h = g$, so that $g\circ End_{\mathcal{L}_\mathcal{M}}(L) = (f\circ h)\circ End_{\mathcal{L}_\mathcal{M}}(L)\subseteq f\circ End_{\mathcal{L}_\mathcal{M}}(L)$.
$(\Longleftarrow)$ Suppose $a/0_L$ is an initial interval and we have the following diagram with exact bottom row.

$$\begin{array}{ccccc} & & L & & \\ & & \big\downarrow{\scriptstyle g} & & \\ L & \xrightarrow{\ f\ } & a/0_L & \longrightarrow & 0 \end{array}$$

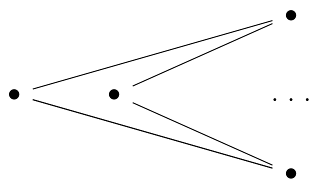
Extending the codomain of f and g to all L, we have $f, g \in End_{\mathcal{L}_\mathcal{M}}(L)$ and $g(1_L) \leq a = f(1_L)$, so $g \circ End_{\mathcal{L}_\mathcal{M}}(L) \subseteq f \circ End_{\mathcal{L}_\mathcal{M}}(L)$ and thus there is a linear morphism $h : L \longrightarrow L$ such that $f \circ h = g$. $\square$

Next, we show two examples of lattices which are both retractable and semi-projective.

Example 3.7. The simple lattice $\{0, 1\}$ is retractable and semi-projective, as it is a complemented lattice whose only non-trivial initial interval is the lattice $\{0, 1\}$.

Recall that the *length* of a chain C is $|C| - 1$, and that the length of a lattice L is the greatest length of a chain in L.

Example 3.8. Any lattice $L \in \mathcal{L}_\mathcal{M}$ of length 2 is retractable and semi-projective. Indeed, any such of these lattices has the form



Clearly, if L has only three elements, then it is retractable, and if L has more than three elements, then it is complemented. Thus, these lattices are all retractable.

Let us now take any $L \in \mathcal{L}_\mathcal{M}$ with the above shape and let $a/0_L$ be an initial interval of L. By considering the following diagram

$$\begin{array}{ccccc} & & L & & \\ & & \big\downarrow{\scriptstyle g} & & \\ L & \xrightarrow{f} & a/0_L & \longrightarrow & 0, \end{array}$$

we see that if $g = 0$, then the linear morphism $0_{L,L} : L \longrightarrow L$ satisfies $f \circ 0_{L,L} = g$ (seeing as all linear morphisms send zero to zero). That is, it makes the diagram

$$\begin{array}{ccccc} & {\scriptstyle 0_{L,L}} \swarrow & L & & \\ & & \big\downarrow{\scriptstyle g} & & \\ L & \xrightarrow{f} & a/0_L & \longrightarrow & 0 \end{array}$$

commutative. Therefore, we can assume that $g \neq 0$.

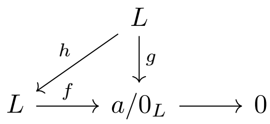
If $a = 1_L$, the exactness of the bottom row implies that f is a lattice isomorphism. Thus, $f^{-1} \circ g$ is a linear morphism that makes the diagram commutative.

Let us now assume that a is an atom of L, so that the interval $a/0_L$ is a simple lattice. This last implies that the kernel of the linear morphisms g and f, denoted as k_g and k_f, respectively, are both atoms of L. Further, by the definition of linear morphisms, it follows that, for any atom $b \neq k_g$ of L,

$$g(b) = g(k_g \vee b) = g(1_L) = a,$$

showing thus that its kernel uniquely determines every non-trivial linear morphism $L \longrightarrow a/0_L$.

Considering the above, let $h : L \longrightarrow L$ be the lattice isomorphism such that $h(k_g) = k_f$, $h(k_f) = k_g$, and $h(x) = x$ whenever $k_g \neq x \neq k_f$. Note that h is a linear morphism since h is a lattice isomorphism. Furthermore, as $(f \circ h)(1_L) = f(1_L) = a$, the composite $f \circ h$ is a non-trivial linear morphism such that $(f \circ h)(k_g) = f(k_f) = 0_L$. Therefore, $k_{f \circ h} \geq k_g$, and thus, given that $k_{f \circ h} < 1_L$ and that k_g is a coatom of L, it follows that $k_{f \circ h} = k_g$. Hence, $f \circ h = g$, that is, h makes the diagram

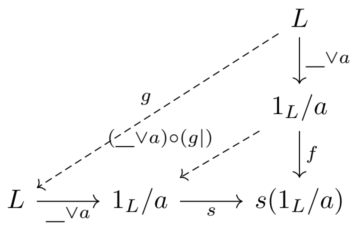


commute.

Definition 3.9. For a lattice $L \in \mathcal{L}_{\mathcal{M}}$ and an element $a \in L$, we call the element a *strongly invariant* (in L) if $f(a) \leq a$ for any linear endomorphism $f \in End_{\mathcal{L}_{\mathcal{M}}}(L)$.

Proposition 3.10. *If L is semi-projective and $a \in L$ is strongly invariant, then $1_L/a$ is semi-projective.*

Proof. Let $s \in End_{\mathcal{L}_{\mathcal{M}}}(1_L/a)$ and $f : 1_L/a \longrightarrow s(1_L/a)$ be a linear morphism. We have the solid part of the following diagram:



Since L is semi-projective, there exists some linear morphism g which makes the outermost triangle commutative. Let us verify that $g| : 1_L/a \longrightarrow {}^{1_L}/_{g(a)}$ (the restriction and corestriction of g) is a linear morphism with kernel $k_g \vee a$. Indeed, for $x \in 1_L/a$,

$$g|((k_g \vee a) \vee x) = g|(a \vee x) = g|(x).$$

Also, $\overline{g|} : {}^{1_L}/_{k_g \vee a} \longrightarrow {}^{g(1_L)}/_{g(a)}$ is a lattice isomorphism because it is the restriction and corestriction of $\overline{g}$.

Since a is strongly invariant in L, $g(a) \leq a$, so that $(\_ \vee a) : {}^{1_L}/_{g(a)} \to 1_L/a$ is a linear morphism with kernel a. Therefore, the composite $(\_ \vee a) \circ (g|) \in End_{\mathcal{L_M}}(1_L/a)$.

Let us verify that the smaller triangle is commutative. For $x \in 1_L/a$, we have that

$$f(x) = f(x \vee a) = (f \circ (\_ \vee a))(x) = (s \circ (\_ \vee a) \circ g)(x) = (s \circ (\_ \vee a) \circ (g|))(x). \square$$

Theorem 3.11. *Let $L \in \mathcal{L_M}$ and let $S = End_{\mathcal{L_M}}(L)$. Then, L is semi-projective if and only if*

$$(f \circ S) = Hom_{\mathcal{L_M}}\big(L, f(L)\big)$$

holds for every endomorphism $f \in S$.

Proof. ($\Longrightarrow$) Let $f : L \longrightarrow L$ be an endomorphism in $\mathcal{L_M}$. For $g \in S$, it follows that $(f \circ g)(L) \subseteq f(L)$, and thus, $f \circ g \in Hom_{\mathcal{L_M}}(L, f(L))$. And, given $g \in Hom_{\mathcal{L_M}}(L, f(L))$, we have the diagram

$$\begin{array}{ccccc} & & L & & \\ & & \big\downarrow{\scriptstyle g} & & \\ L & \xrightarrow{f} & {}^{f(1_L)}/_{0_L} & \longrightarrow & 0, \end{array}$$

where ${}^{f(1_L)}/_{0_L} = f(L)$ and whose bottom row is exact. As by hypothesis, L is a semi-projective lattice, there exists $h \in S$ such that $f \circ h = g$. Therefore, $g \in f \circ S$.

($\Longleftarrow$) Let $L \in \mathcal{L_M}$, and let $a/0_L$ be an initial interval of L. Consider the diagram

$$\begin{array}{ccccc} & & L & & \\ & & \big\downarrow{\scriptstyle g} & & \\ L & \xrightarrow{f} & a/0_L & \longrightarrow & 0 \end{array}$$

with the bottom row exact. Then, as $g \in Hom_{\mathcal{L_M}}(L, f(L)) = f \circ S$, there exists $h \in S$ such that $f \circ h = g$. Hence, the diagram

$$\begin{array}{ccccc} & & L & & \\ & \swarrow{\scriptstyle h} & \big\downarrow{\scriptstyle g} & & \\ L & \xrightarrow{f} & a/0_L & \longrightarrow & 0 \end{array}$$

commutes, and thus, L is semi-projective. $\square$

Remark 3.12. For each lattice $L \in \mathcal{L_M}$, the set $S = End_{\mathcal{L_M}}(L)$ becomes a monoid whose binary operation is the composition of linear morphisms. The identity element for this operation corresponds to the linear morphism given by the identity function on L. Moreover, this is a monoid with zero: $0_{L,L}$.

(A similar observation holds for any category with a zero object.)

Definition 3.13. Let $L \in \mathcal{L}_{\mathcal{M}}$ and let $S = End_{\mathcal{L}_{\mathcal{M}}}(L)$. We say that $H \subseteq S$ is a *right ideal of* S if H is non-empty and closed under right composition with elements of S; this is, for $h \in H$ and $f \in S$, we have that $(h \circ f) \in H$. Note that H is a right ideal of S if and only if $0_{L,L} \in H$ and H is closed under right composition with elements of S.

Assume now that $L \in \mathcal{L}_{\mathcal{M}}$, and let $m, n \in L$ be such that $m \leq n$. Then, the inclusion mapping $m/0_L \xrightarrow{\iota} n/0_L$ is a linear morphism. Moreover, ι induces an injective mapping $Hom_{\mathcal{L}_{\mathcal{M}}}\big(L, m/0_L\big) \xrightarrow{\iota\circ\_} Hom_{\mathcal{L}_{\mathcal{M}}}\big(L, n/0_L\big)$. This way, we can think of $Hom_{\mathcal{L}_{\mathcal{M}}}\big(L, m/0_L\big)$ as a subset of $Hom_{\mathcal{L}_{\mathcal{M}}}\big(L, n/0_L\big)$. Furthermore, if $n = 1_L$, the image of $\iota \circ \_$ is a right ideal of S. Thus, we may think of any linear morphism $L \xrightarrow{f} m/0_L$ as an element of S.

Theorem 3.14. *Let $L \in \mathcal{L}_{\mathcal{M}}$ be a semi-projective lattice. Then, a bijection exists between the set of L-cyclic initial intervals of L and the set of principal right ideals of $S = End_{\mathcal{L}_{\mathcal{M}}}(L)$.*

Proof. We start the proof by noting that each L-cyclic initial interval $a/0_L$ of L comes with a linear epimorphism $f : L \longrightarrow a/0_L$. (Indeed, there is some $u \in L$ such that $a/0_L \cong 1_L/u$, so we may set as f the composite $L \xrightarrow{-\vee u} 1_L/u \cong a/0_L$.) With this in mind, we define the mapping $\mathcal{F}$ with the correspondence rule $a/0_L \longmapsto f \circ S$. We claim that $\mathcal{F}$ is well-defined. Indeed, if $g : L \longrightarrow a/0_L$ is another linear epimorphism, then we have the diagram

$$\begin{array}{ccccc} & & L & & \\ & & \downarrow{\scriptstyle g} & & \\ L & \xrightarrow{f} & a/0_L & \longrightarrow & 0 \\ & & \downarrow & & \\ & & 0, & & \end{array}$$

where the row and the column are exact. Furthermore, since L is semi-projective, we obtain two linear morphisms $h, h' : L \longrightarrow L$ such that $f \circ h = g$ and $g \circ h' = f$. From these equalities it follows that $g \circ S \subseteq f \circ S$ and $f \circ S \subseteq g \circ S$, respectively. Thus, $f \circ S = g \circ S$.

We shall now show that the mapping $\mathcal{F}$ is a bijection. For surjectivity, note that each endomorphism $f : L \longrightarrow L$ induces the L-cyclic initial interval $f(L) = \big(f(1_L)/0_L\big)$, as $f(L)$ is isomorphic to the quotient interval $1_L/k_f$ of L. Hence, the principal right ideal generated by f is obtained by evaluating $\mathcal{F}$ in the L-cyclic initial interval $f(L)$. Therefore, $\mathcal{F}$ is surjective. For injectivity, let $a/0_L$ and $b/0_L$

be two L-cyclic initial intervals of L. If $f : L \longrightarrow a/0_L$ and $g : L \longrightarrow b/0_L$ are linear morphisms such that $f \circ S = g \circ S$, then there exist $h, t \in S$ such that $f \circ h = g$ and $g \circ t = f$. We then have that

$$a = f(1_L) = (g \circ t)(1_L) = g(t(1_L)) \in g(1_L/0_L) = b/0_L.$$

Thus, $a \leq b$. Similarly, one shows that $b \leq a$, wherewith we obtain that $a = b$. Therefore, $\mathcal{F}$ is injective. □

Recall that an element a of a lattice L with zero is said to be *essential* (in L) if for every $0_L \neq b \in L$, it happens that $a \wedge b \neq 0_L$. Also, for $m, n \in L$ such that $m \leq n$, we say that m is *essential* in n if m is essential in $n/0_L$.

Definition 3.15. Let $L \in \mathcal{L}_{\mathcal{M}}$. We say that $a \in L$ is *uniform* (in L) if every nonzero $b \in L$ such that $b \leq a$ is essential in a. Furthermore, we say that the lattice L is uniform if the element 1_L is uniform in L.

Definition 3.16. Let $L \in \mathcal{L}_{\mathcal{M}}$, and let $a/0_L$ be an initial interval of L. We say that L *generates* $a/0_L$ if there exists a family of linear morphisms $\{f_t\}_{t \in T} \subseteq Hom_{\mathcal{L}_{\mathcal{M}}}(L, a/0_L)$ such that

$$a = \bigvee_{t \in T} f_t(1_L).$$

Let $L \in \mathcal{L}_{\mathcal{M}}$ and $S = End_{\mathcal{L}_{\mathcal{M}}}(L)$. Write $\mathcal{R}(S)$ for the set of right ideals of S. Note that $\mathcal{R}(S)$ is partially ordered by inclusion. Furthermore, $(\mathcal{R}(S), \subseteq)$ is a lattice, whose meet and join operations are intersection and union of sets, respectively. The least (resp., greatest) element of $\mathcal{R}(S)$ is $\{0_{L,L}\}$ (resp., S). Since every distributive lattice is modular, $\mathcal{R}(S) \in \mathcal{L}_{\mathcal{M}}$.

Recall that, for $I, J \in \mathcal{R}(S)$ such that $I \subseteq J$, we say that I is essential in J if and only if I is an essential element of the initial interval ${}^{J}/_{\{0\}}$ of $(\mathcal{R}(S), \subseteq)$. Particularly, for $m \leq n$, we say that $Hom_{\mathcal{L}_{\mathcal{M}}}\big(L, m/0_L\big)$ is essential in $Hom_{\mathcal{L}_{\mathcal{M}}}\big(L, n/0_L\big)$ if and only if $Hom_{\mathcal{L}_{\mathcal{M}}}\big(L, m/0_L\big)$ is an essential element of the initial interval ${}^{Hom_{\mathcal{L}_{\mathcal{M}}}(L, n/0_L)}/_{\{0\}}$ of $(\mathcal{R}(S), \subseteq)$.

Theorem 3.17. *Let $L \in \mathcal{L}_{\mathcal{M}}$ be a retractable lattice, and let $S = End_{\mathcal{L}_{\mathcal{M}}}(L)$. If $I, J \in \mathcal{R}(S)$ such that $I \subseteq J$ and $m, n \in L$ such that $m \leq n$, then the following statements hold:*

(a) *If $Hom_{\mathcal{L}_{\mathcal{M}}}\big(L, m/0_L\big)$ is essential in $Hom_{\mathcal{L}_{\mathcal{M}}}\big(L, n/0_L\big)$, then m is essential in n.*

(b) *If L is semi-projective and m is essential in n, then $Hom_{\mathcal{L_M}}\big(L, m/0_L\big)$ is essential in $Hom_{\mathcal{L_M}}\big(L, n/0_L\big)$.*

(c) *Suppose that $Hom_{\mathcal{L_M}}\big(L, \bigvee_{j\in J} j(1_L)/0_L\big) = J$. If I is essential in J, then $\bigvee_{i\in I} i(1_L)$ is essential in $\bigvee_{j\in J} j(1_L)$.*

(d) *Suppose that L is semi-projective and that $Hom_{\mathcal{L_M}}\big(L, \bigvee_{i\in I} i(1_L)/0_L\big) = I$. If $\bigvee_{i\in I} i(1_L)$ is essential in $\bigvee_{j\in J} j(1_L)$, then I is essential in J.*

(e) *If $Hom_{\mathcal{L_M}}\big(L, m/0_L\big)$ is uniform as a right ideal of S, then $m \in L$ is uniform. For semi-projective L, the converse holds.*

(f) *Suppose that L is semi-projective. If $\bigvee_{i\in I} i(1_L)$ is a uniform element in L, then I is a uniform right ideal of S.*

(g) *Consider the following statements:*

  (i) *$Hom_{\mathcal{L_M}}\big(L, m/0_L\big)$ is simple (that is, minimal in $\mathcal{R}(S)\backslash\{\{0\}\}$).*

  (ii) *$m/0_L$ is the simple lattice.*

If L generates $m/0_L$, then (i) implies (ii). If L is semi-projective, then (ii) implies (i).

(h) *If $I = Hom_{\mathcal{L_M}}\big(L, \bigvee_{i\in I} i(1_L)/0_L\big)$ and I is simple, then $\bigvee_{i\in I} i(1_L)/0_L$ is the simple lattice. If L is semi-projective, then the converse holds.*

Proof. (a) Let $k \in n/0_L$ such that $m \wedge k = 0_L$, and let us assume that $k \neq 0_L$. Then, as L is a retractable lattice, we have that $Hom_{\mathcal{L_M}}\big(L, k/0_L\big) \neq \{0\}$. Further, by hypothesis, $Hom_{\mathcal{L_M}}\big(L, m/0_L\big)$ is essential in $Hom_{\mathcal{L_M}}\big(L, n/0_L\big)$, so that

$$Hom_{\mathcal{L_M}}\big(L, m/0_L\big) \cap Hom_{\mathcal{L_M}}\big(L, k/0_L\big) \neq \{0\}.$$

However, for any $f \in Hom_{\mathcal{L_M}}\big(L, m/0_L\big) \cap Hom_{\mathcal{L_M}}\big(L, k/0_L\big)$, we have that

$$f(1_L) \leq m \wedge k = 0_L,$$

so that $f = 0$. This contradiction establishes that m is essential in n.

(b) Let $X \in Hom_{\mathcal{L_M}}(L, n/0_L)/\{0\}$ such that $X \cap Hom_{\mathcal{L_M}}\big(L, m/0_L\big) = \{0\}$, and take $f \in X$. Note that $f \circ S \subseteq X$, so that

$$Hom_{\mathcal{L_M}}\big(L, m/0_L\big) \cap (f \circ S) = \{0\}.$$

Since L is semi-projective by hypothesis, Theorem 3.11 provides that $f \circ S = Hom_{\mathcal{L_M}}\big(L, f(L)\big)$. It follows that

$$Hom_{\mathcal{L_M}}\Big(L, m \wedge f(1_L)/0_L\Big) = \{0\}.$$

Then, as L is retractable, we have that $m \wedge f(1_L) = 0_L$. This implies that $f(1_L) = 0_L$ because m is essential in n, so that $f = 0$. Therefore, $X = \{0\}$, whence $Hom_{\mathcal{L_M}}\big(L, m/0_L\big)$ is essential in $Hom_{\mathcal{L_M}}\big(L, n/0_L\big)$.

(c) Note that $\bigvee_{i \in I} i(1_L) \leq \bigvee_{j \in J} j(1_L)$, so that

$$I \subseteq Hom_{\mathcal{L_M}}\big(L, \bigvee_{i \in I} i(1_L)/0_L\big) \subseteq Hom_{\mathcal{L_M}}\big(L, \bigvee_{j \in J} j(1_L)/0_L\big) = J.$$

Thus, as I is essential in J by hypothesis, we have that $Hom_{\mathcal{L_M}}(L, (\bigvee_{i \in I} i(1_L))/0_L)$ is essential in $Hom_{\mathcal{L_M}}(L, (\bigvee_{j \in J} j(1_L))/0_L)$. Therefore, by (a), it follows that $\bigvee_{i \in I} i(1_L)$ is essential in $\bigvee_{j \in J} j(1_L)$.

(d) We are supposing that L is semi-projective and that $\bigvee_{i \in I} i(1_L)$ is essential in $\bigvee_{j \in J} j(1_L)$. Hence, by (b), we obtain that $I = Hom_{\mathcal{L_M}}\big(L, \bigvee_{i \in I} i(1_L)/0_L\big)$ is essential in $Hom_{\mathcal{L_M}}\big(L, \bigvee_{j \in J} j(1_L)/0_L\big)$. Now, let $K \in {}^{J}/\{0\}$ such that $I \cap K = \{0\}$. Since $K \subseteq J \subseteq Hom_{\mathcal{L_M}}\big(L, \bigvee_{j \in J} j(1_L)/0_L\big)$ and I is essential in $Hom_{\mathcal{L_M}}\big(L, \bigvee_{j \in J} j(1_L)/0_L\big)$, K must be trivial. Therefore, I is essential in J.

(e) Let $x, y \in m/0_L$, with $0_L < x, y$. Since L is retractable by hypothesis, we have that

$$Hom_{\mathcal{L_M}}(L, x/0_L) \neq \{0\} \neq Hom_{\mathcal{L_M}}(L, y/0_L).$$

Further, by assumption, $Hom_{\mathcal{L_M}}\big(L, m/0_L\big)$ is a uniform right ideal of S, and so,

$$\{0\} \neq Hom_{\mathcal{L_M}}\big(L, x/0_L\big) \cap Hom_{\mathcal{L_M}}\big(L, y/0_L\big) = Hom_{\mathcal{L_M}}\big(L, \big(x \wedge y)/0_L\big)\big).$$

Therefore, $x \wedge y \neq 0_L$, thus showing that m is uniform in L.

For the converse, let I and J be nonzero right ideals contained in $Hom_{\mathcal{L_M}}\big(L, m/0_L\big)$. Then, we can choose two non-trivial linear morphisms $i, j : L \longrightarrow m/0_L$, where $i \in I$ and $j \in J$. Note that $i(1_L) \neq 0_L \neq j(1_L)$, and $i(1_L), j(1_L) \leq m$. Since m is uniform in L, it follows that $i(1_L) \wedge j(1_L) \neq 0_L$. Further, we have that

$$Hom_{\mathcal{L_M}}\big(L, i(1_L)/0_L\big) \cap Hom_{\mathcal{L_M}}\big(L, j(1_L)/0_L\big) = \\ Hom_{\mathcal{L_M}}\big(L, \big(i(1_L) \wedge j(1_L)\big)/0_L\big) \neq \{0\},$$

because L is retractable by hypothesis. It follows that $I \cap J \neq 0$, because

$$\{0\} \neq Hom_{\mathcal{L_M}}\big(L, i(1_L)/0_L\big) \cap Hom_{\mathcal{L_M}}\big(L, j(1_L)/0_L\big) \subseteq (i \circ S) \cap (j \circ S) \subseteq I \cap J,$$

where the first inclusion holds because L is semi-projective (see Theorem 3.11).

Therefore, $Hom_{\mathcal{L_M}}\big(L, m/0_L\big)$ is uniform in $\mathcal{R}(S)$.

(f) Let $\bigvee_{i\in I} i(1_L)$ be a uniform element in L. Then, by part (e), $Hom_{\mathcal{L_M}}\big(L, \bigvee_{i\in I} i(1_L)/0_L\big)$ is a uniform right ideal of S. Moreover, as $I \subseteq Hom_{\mathcal{L_M}}\big(L, \bigvee_{i\in I} i(1_L)/0_L\big)$, it follows that I is a uniform right ideal of S.

(g) Note first that (i) implies that $0_L < m$. Let $k \in m/0_L$ with $k \neq 0_L$. Then, since L is retractable by hypothesis, we have that

$$\{0\} \neq Hom_{\mathcal{L_M}}\big(L, k/0_L\big) \subseteq Hom_{\mathcal{L_M}}\big(L, m/0_L\big)$$

because $k \leq m$. Furthermore, the fact that $Hom_{\mathcal{L_M}}\big(L, m/0_L\big)$ is simple implies that

$$Hom_{\mathcal{L_M}}\big(L, k/0_L\big) = Hom_{\mathcal{L_M}}\big(L, m/0_L\big).$$

Now, as $m/0_L$ is generated by L, there exists a family of linear morphisms $\{f_t\}_{t\in T}$ such that $m = \bigvee_{t\in T} f_t(1_L)$. This way, $f_t \in Hom_{\mathcal{L_M}}\big(L, m/0_L\big) = Hom_{\mathcal{L_M}}\big(L, k/0_L\big)$, so that $f_t(1_L) \leq k$ for all $t \in T$. Hence,

$$\bigvee_{t\in T} f_t(1_L) \leq k \leq m = \bigvee_{t\in T} f_t(1_L),$$

which implies that $k = m$. Therefore, $m/0_L$ is simple.

For the converse, note first that, as L is retractable, $Hom_{\mathcal{L_M}}\big(L, m/0_L\big) \neq \{0\}$. Let $f \in Hom_{\mathcal{L_M}}\big(L, m/0_L\big)$ with $f \neq 0$. Then, by simpleness, $f(L) = m/0_L$. Hence, by Theorem 3.11, L being semi-projective implies that

$$f \circ S = Hom_{\mathcal{L_M}}\big(L, f(L)\big) = Hom_{\mathcal{L_M}}\big(L, m/0_L\big),$$

thus showing that $Hom_{\mathcal{L_M}}\big(L, m/0_L\big)$ is simple.

(h) Necessity follows directly from (g).

For sufficiency, note first that, by (g), $Hom_{\mathcal{L_M}}\big(L, \bigvee_{i\in I} i(1_L)/0_L\big)$ is simple. Now, since $\bigvee_{i\in I} i(1_L) \neq 0_L$, we have that

$$\{0\} \neq I \subseteq Hom_{\mathcal{L_M}}\big(L, \bigvee_{i\in I} i(1_L)/0_L\big),$$

whence $I = Hom_{\mathcal{L_M}}\big(L, \bigvee_{i\in I} i(1_L)/0_L\big)$. □

Definition 3.18. Let $L \in \mathcal{L_M}$. We say that L is *co-Hopfian* if every linear monomorphism[2] $L \xrightarrow{f} L$ is an epimorphism.

[2]By this we mean a monomorphism in the category $\mathcal{L_M}$.

Example 3.19. Consider $\mathcal{L}(\mathbb{Z})$, the lattice of all submodules of $\mathbb{Z}_{\mathbb{Z}}$. Observe that both $\mathbb{Z}$ and $2\mathbb{Z}$ are elements of $\mathcal{L}(\mathbb{Z})$. Since $2\mathbb{Z}$ is isomorphic to $\mathbb{Z}$, it follows that $\mathcal{L}(2\mathbb{Z})$ is isomorphic to $\mathcal{L}(\mathbb{Z})$. Thus, $2\mathbb{Z}/0$ is an initial interval of $\mathcal{L}(\mathbb{Z})$ isomorphic to $\mathcal{L}(\mathbb{Z})$. But also, $2\mathbb{Z}$ is a proper submodule of $\mathbb{Z}$. Therefore, the lattice $\mathcal{L}(\mathbb{Z})$ is not co-Hopfian, as evidenced by the linear monomorphism:

$$\mathcal{L}(\mathbb{Z}) \xrightarrow{\cong} 2\mathbb{Z}/0 \hookrightarrow \mathcal{L}(\mathbb{Z})$$

which is not an epimorphism.

Proposition 3.20. *Any finite lattice $L \in \mathcal{L}_{\mathcal{M}}$ is co-Hopfian.*

Proof. Let $L \in \mathcal{L}_{\mathcal{M}}$ be a finite lattice, and let $f : L \longrightarrow L$ be a linear monomorphism. We know that the kernel of f is trivial, so the induced lattice isomorphism has the form $\overline{f} : L \longrightarrow f(1_L)/0_L$. In this way, since L is a finite lattice, the initial interval $f(1_L)/0_L$ cannot be proper in L. Therefore, $f(1_L) = 1_L$, showing that f is surjective. □

Definition 3.21. Let $L \in \mathcal{L}_{\mathcal{M}}$, and let $S = End_{\mathcal{L}_{\mathcal{M}}}(L)$. A function $\varphi : S \longrightarrow S$ is called a *right S-endomorphism* (or, in this paper, simply an *S-endomorphism*) if for any $f, g \in S$, one has that $\varphi(f \circ g) = \varphi(f) \circ g$.

Remark 3.22. It follows from Definition 3.21 that the image $\varphi(S)$ of any S-endomorphism $\varphi : S \longrightarrow S$ is a right ideal of the monoid S.

We will call an S-endomorphism $\varphi : S \longrightarrow S$ an *S-monomorphism* if, for any two S-endomorphisms $\psi, \psi' : S \longrightarrow S$ such that $\varphi \circ \psi = \varphi \circ \psi'$, it follows that $\psi = \psi'$.

Proposition 3.23. *Let $L \in \mathcal{L}_{\mathcal{M}}$, and $S = End_{\mathcal{L}_{\mathcal{M}}}(L)$. If the mapping $\varphi : S \longrightarrow S$ is an S-monomorphism, then*

$$\big\{f \in S | \varphi(f) = 0\big\} = \{0\}.$$

Proof. Assume that there exists $0 \neq g \in S$ such that $\varphi(g) = 0$. We define the function $\psi_g : S \longrightarrow S$ such that $\psi_g(h) = g \circ h$. Note that ψ_g is an S-endomorphism. Then, for $h \in S$,

$$(\varphi \circ \psi_g)(h) = \varphi \circ (\psi_g(h)) = \varphi(g \circ h) = \varphi(g) \circ h = 0 \circ h = 0.$$

This way, $\varphi \circ \psi_g = 0 = \varphi \circ 0$, from which we obtain that $\psi_g = 0$ because φ is an S-monomorphism. However, one also has that $\psi_g \neq 0$ since $\psi_g(Id_L) = g \circ Id_L = g \neq 0$. This is a contradiction. □

Definition 3.24. Let $L \in \mathcal{L}_{\mathcal{M}}$ and $S = End_{\mathcal{L}_{\mathcal{M}}}(L)$. We call the monoid S *co-Hopfian* if every S-monomorphism $\varphi : S \longrightarrow S$ is surjective.

Definition 3.25. Let $L \in \mathcal{L}_{\mathcal{M}}$ and $S = End_{\mathcal{L}_{\mathcal{M}}}(L)$. We say that $f \in S$ is *right regular* if $f \circ g = 0$ with $g \in S$ implies that $g = 0$.

Lemma 3.26. *For a lattice $L \in \mathcal{L}_{\mathcal{M}}$, the monoid $S = End_{\mathcal{L}_{\mathcal{M}}}(L)$ is co-Hopfian if every right regular element in S is a unit.*

Proof. Let $\varphi : S \longrightarrow S$ be an S-monomorphism. We claim that, for the linear morphism $Id_L \in S$, the element $\varphi(Id_L) \in S$ is right regular. Indeed, if this were not the case, there would exist some $0 \neq g \in S$ such that

$$0 = \varphi(Id_L) \circ g = \varphi(Id_L \circ g) = \varphi(g).$$

However, this contradicts Proposition 3.23, as φ is an S-monomorphism.

Now, by hypothesis, we obtain that $\varphi(Id_L)$ is a unit. In particular, there exists $h \in S$ such that

$$\varphi(h) = \varphi(Id_L \circ h) = \varphi(Id_L) \circ h = Id_L.$$

Thus, for any $g \in S$, it happens that

$$\varphi(h \circ g) = \varphi(h) \circ g = Id_L \circ g = g.$$

Hence, $\varphi(S) = S$, that is, φ is surjective. □

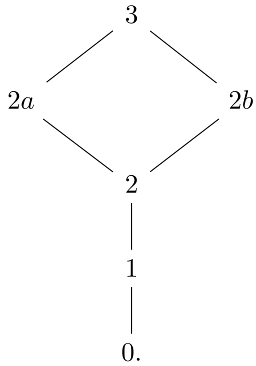
Definition 3.27. Let $L \in \mathcal{L}_{\mathcal{M}}$ and $S = End_{\mathcal{L}_{\mathcal{M}}}(L)$. We say that L is *quasi-injective* if for any linear monomorphism $m : N \longrightarrow L$ and any linear morphism $f : N \longrightarrow L$, there exists $f' \in S$ such that the following diagram is commutative:

$$\begin{array}{ccccc} 0 & \longrightarrow & N & \xrightarrow{m} & L \\ & & {\scriptstyle f}\downarrow & \swarrow {\scriptstyle f'} & \\ & & L. & & \end{array}$$

(In module-theoretic language, this notion can be rendered as "L is L-injective", as "L belongs to its own injectivity class", or as "L belongs to its own injectivity domain".)

For example, the simple lattice $\{0, 1\}$ is a quasi-injective lattice. In contrast, the next example provides a lattice that is not quasi-injective.

Example 3.28. Let P be the following complete modular lattice:



Consider the linear modular lattice $2/0$:

Then, the mapping $f : 2/0 \longrightarrow P$ such that $f(0) = 0$, $f(1) = 0$, and $f(2) = 1$ is a linear morphism (with kernel 1). Now, if P were a quasi-injective lattice, for the inclusion mapping $\iota : 2/0 \longrightarrow P$ and the linear morphism f, there would exist a linear morphism $f' : P \longrightarrow P$ such that $f' \circ \iota = f$. Then

$$f'(0) = f'(\iota(0)) = f(0) = 0 = f(1) = f'(\iota(1)) = f'(1).$$

Moreover, since $f'(2) = f'(\iota(2)) = f(2) = 1$, it follows that $k_{f'} = 1$. Thus, according to the definition of a linear morphism, we obtain a lattice isomorphism between the quotient lattice $\frac{3}{1}$ and an initial interval of P, which is evidently impossible. Consequently, P cannot be a quasi-injective lattice.

Definition 3.29. For $L \in \mathcal{L}_{\mathcal{M}}$ and $f \in S = End_{\mathcal{L}_{\mathcal{M}}}(L)$, the *right annihilator* of f is

$$Ann_{\mathfrak{r}}(f) = \{g \in S | f \circ g = 0\}.$$

Remark 3.30. $Ann_{\mathfrak{r}}(f)$ is a right ideal of the monoid S, for any $L \in \mathcal{L}_{\mathcal{M}}$.

Definition 3.31. Let $L \in \mathcal{L}_{\mathcal{M}}$, and let $S = End_{\mathcal{L}_{\mathcal{M}}}(L)$. The *right singular ideal* of S is

$$Z_{\mathfrak{r}}(S) = \{f \in S | Ann_{\mathfrak{r}}(f) \text{ is essential in } \mathcal{R}(S)\}.$$

Remark 3.32. $Z_{\mathfrak{r}}(S)$ is a right ideal of the monoid S. Indeed, let $f \in Z_{\mathfrak{r}}(S)$ and $g \in S$. Let $\{0\} \neq I \in \mathcal{R}(S)$, and choose some $0 \neq h \in I$. If $g \circ h = 0$, then $0 \neq h \in I \cap Ann_{\mathfrak{r}}(f \circ g)$. If $g \circ h \neq 0$, then, as $Ann_{\mathfrak{r}}(f)$ is essential in

$\mathcal{R}(S)$, one has that $(g \circ h) \circ S \cap Ann_{\mathfrak{r}}(f) \neq \{0\}$, that is, there is $k \in S$ such that $0 \neq g \circ h \circ k \in Ann_{\mathfrak{r}}(f)$. Thus, $f \circ g \circ h \circ k = 0$, so that $0 \neq h \circ k \in I \cap Ann_{\mathfrak{r}}(f \circ g)$. In either case, $I \cap Ann_{\mathfrak{r}}(f \circ g) \neq \{0\}$. Therefore, $f \circ g \in Z_{\mathfrak{r}}(S)$.

(In fact, $Z_{\mathfrak{r}}(S)$ is also closed under left composition with elements of S, which makes it a two-sided ideal of the monoid S. To verify this, observe that, for $f \in Z_{\mathfrak{r}}(S)$ and $g \in S$, $Ann_{\mathfrak{r}}(f) \subseteq Ann_{\mathfrak{r}}(g \circ f)$.)

Definition 3.33. Let $L \in \mathcal{L}_{\mathcal{M}}$, and let $\mathcal{A}_L$ be the set of all atoms of L. Then, the *socle* of L is

$$Soc(L) = \bigvee_{x \in \mathcal{A}_L} x/0_L.$$

For a lattice $L \in \mathcal{L}_{\mathcal{M}}$, a right ideal J of $S = End_{\mathcal{L}_{\mathcal{M}}}(L)$, and an initial interval K of L, we denote by $(J)(K)$ (or simply JK, when there is no risk of ambiguity) the initial interval of L determined by $\bigvee_{g \in J} g(1_K)$. That is,

$$(J)(K) = \bigvee_{g \in J} g(1_K)/0_L.$$

Lemma 3.34. *Let $L \in \mathcal{L}_{\mathcal{M}}$ be a retractable lattice, and let $S = End_{\mathcal{L}_{\mathcal{M}}}(L)$. Then $f \in S$ is a linear monomorphism if and only if $S \xrightarrow{f \circ \_} S$ is an S-monomorphism.*

Proof. ($\Longrightarrow$) For convenience, let us write $\varphi_f = f \circ \_$. Let $\psi, \psi' : S \longrightarrow S$ be two S-endomorphisms such that $\varphi_f \circ \psi = \varphi_f \circ \psi'$. Then, for all $g \in S$,

$$f \circ (\psi(g)) = (\varphi_f \circ \psi)(g) = (\varphi_f \circ \psi')(g) = f \circ (\psi'(g)).$$

Since f is a linear monomorphism, it follows that $\psi(g) = \psi'(g)$, $\forall g \in S$. Therefore, $\psi = \psi'$, showing that $f \circ \_$ is an S-monomorphism.
($\Longleftarrow$) If $0 < k_f$, since L is retractable, there is a nonzero linear morphism $g \in Hom_{\mathcal{L}_{\mathcal{M}}}(L, k_f/0)$, so that $f \circ g = 0$, which is a contradiction because we assumed that $f \circ \_$ was an S-monomorphism. □

Proposition 3.35. *Let $L \in \mathcal{L}_{\mathcal{M}}$ be a retractable lattice, and let $S = End_{\mathcal{L}_{\mathcal{M}}}(L)$. Then, the following statements hold:*

(a) *$f \in S$ is a linear monomorphism if and only if f is right regular in S.*
(b) *L is co-Hopfian if and only if S is co-Hopfian.*
(c) *If L is quasi-injective, then each right regular element in S has a left inverse in S.*
(d) *$Z_{\mathfrak{r}}(S) \subseteq \{f \in S | k_f$ is essential in $L\}$, and further, $(Z_{\mathfrak{r}}(S))(Soc(L)) = 0$.*

Proof. (a) Note first that

$$Ann_{\mathfrak{r}}(f) = \{g \in S | f \circ g = 0\} = Hom_{\mathcal{L}_{\mathcal{M}}}(L, k_f/0_L),$$

where k_f denotes the kernel of f. Therefore, f is a monomorphism if and only if $k_f = 0_L$, which, by retractability of L, holds if and only if $Ann_{\mathfrak{r}}(f) = 0$, which is equivalent to f being right regular.

(b) ($\Longrightarrow$) By part (a), any right regular element $f \in S$ is a linear monomorphism. Furthermore, since by hypothesis L is co-Hopfian, these linear monomorphisms are surjective. Thus, every right regular element of S is an isomorphism, that is, a unit in S. Therefore, by Lemma 3.26, S is co-Hopfian.

($\Longleftarrow$) Let $f : L \longrightarrow L$ be a linear monomorphism. By Lemma 3.34, the mapping $f \circ \_$ is an S-monomorphism. Furthermore, since S is co-Hopfian by hypothesis, $f \circ \_$ is surjective. Therefore, there exists $g \in S$ such that $f \circ g = Id_L$. Thus, f is surjective.

(c) Let $f \in S$ be right regular. By part (a), f is a linear monomorphism, so it induces a lattice isomorphism of the form $\overline{f} : L \longrightarrow {}^{f(1_L)}/_{0_L}$. Further, as L is quasi-injective, there exists $g \in S$ that makes the following diagram, where ι is the inclusion mapping, commutative:

$$\begin{array}{ccccc} 0 & \longrightarrow & {}^{f(1)}/_{0_L} & \overset{\iota}{\hookrightarrow} & L \\ & & {\scriptstyle (\overline{f})^{-1}}\downarrow & \swarrow {\scriptstyle g} & \\ & & L & & \end{array}$$

For $x \in L$, $f(x) \in {}^{f(1_L)}/_{0_L}$, so that

$$g\big(f(x)\big) = (\overline{f})^{-1}\big(f(x)\big) = x.$$

Therefore, g is a left inverse of f in S.

(d) As noted in part (a), $Ann_{\mathfrak{r}}(f) = Hom_{\mathcal{L}_{\mathcal{M}}}(L, k_f/0_L)$ for each $f \in S$. In particular, when $f \in Z_{\mathfrak{r}}(S)$, we have that $Hom_{\mathcal{L}_{\mathcal{M}}}(L, k_f/0_L)$ is an essential right ideal of S. Thus, by Theorem 3.17(a), we obtain that k_f is essential in L. Therefore,

$$Z_{\mathfrak{r}}(S) \subseteq \{f \in S | k_f \text{ is essential in } L\}.$$

Now, for the second statement, let $x \in L$ be an atom and let $f \in Z_{\mathfrak{r}}(S)$. Then, as k_f is essential in L, we obtain that $k_f \wedge x = x$, that is, $x \leq k_f$. It follows that $\bigvee_{x \in \mathcal{A}_L} x \leq k_f$, and hence, that $f\big(\bigvee_{x \in \mathcal{A}_L} x\big) = 0_L$. Therefore,

$$(Z_{\mathfrak{r}}(S))(Soc(L)) = 0. \qquad \square$$

Definition 3.36. Let $L \in \mathcal{L}_{\mathcal{M}}$, and let $S = End_{\mathcal{L}_{\mathcal{M}}}(L)$. We call the set

$$Soc_r(S) = \bigcup \{I \mid I \text{ is a simple right ideal of } S\}$$

the *right socle* of S.

Lemma 3.37. *A lattice $L \in \mathcal{L}_{\mathcal{M}}$ is semi-projective if and only if $I = Hom_{\mathcal{L}_{\mathcal{M}}}(L, IL)$ for any principal right ideal I of S.*

Proof. Let $g \in S$ and $I = g \circ S$. Since $g \in I$ and $f(1_L) \leq g(1_L)$ for any $f \in I$, it holds that $\bigvee_{f \in I} f(1_L) = g(1_L)$, that is, $IL = g(S)$.

The result now follows from Theorem 3.11. □

Proposition 3.38. *Let $L \in \mathcal{L}_{\mathcal{M}}$ be retractable and semi-projective, and let $S = End_{\mathcal{L}_{\mathcal{M}}}(L)$. Then,*

(a) $Z_{\mathfrak{r}}(S) = \{f \in S |\ k_f$ *is essential in* $L\}$*;*

(b) $\big(Soc_r(S)\big)(L) = Soc(L)$*;*

(c) $Soc_r(S) \subseteq Hom_{\mathcal{L}_{\mathcal{M}}}\big(L, Soc(L)\big)$.

Proof. (a) By Proposition 3.35(d), $Z_{\mathfrak{r}}(S) \subseteq \{f \in S |\ k_f$ is essential in $L\}$. For the reverse inclusion, let $f \in S$ be such that k_f is essential in L, that is, k_f is essential in 1_L. Theorem 3.17(b) implies that $Hom_{\mathcal{L}_{\mathcal{M}}}\big(L, k_f/0_L\big)$ is essential in $Hom_{\mathcal{L}_{\mathcal{M}}}\big(L, 1_L/0_L\big) = S$. Also, note that $Ann_{\mathfrak{r}}(f) = Hom_{\mathcal{L}_{\mathcal{M}}}\big(L, k_f/0_L\big)$. Therefore, $f \in Z_{\mathfrak{r}}(S)$.

(b) As any simple ideal I of S is principal, we can express $I = Hom_{\mathcal{L}_{\mathcal{M}}}(L, IL)$, because of Lemma 3.37. Bearing this in mind, according to Theorem 3.17(h), we have that $IL = a_I/0_L$, where a_I is an atom of L. This holds for any simple right ideal I of S, implying that $(Soc_r(S))(L) \subseteq Soc(L)$.

For the reverse inclusion, take any atom $a \in L$, and write $I = Hom_{\mathcal{L}_{\mathcal{M}}}(L, a/0_L)$. By Theorem 3.17(g), I is a simple right ideal of S. Given this and the fact that $IL = a/0_L$, it follows that $Soc(L) \subseteq (Soc_r(S))(L)$.

(c) Due to Lemma 3.37, any simple right ideal I of S can be expressed as $I = Hom_{\mathcal{L}_{\mathcal{M}}}(L, IL)$. Thus, by Theorem 3.17(g), the initial interval IL is simple, thus obtaining that $I \subseteq Hom_{\mathcal{L}_{\mathcal{M}}}(L, Soc(L))$. As the latter holds for any simple right ideal I of S, it follows that $Soc_r(S) \subseteq Hom_{\mathcal{L}_{\mathcal{M}}}(L, Soc(L))$. □

Definition 3.39. Let $L \in \mathcal{L}_{\mathcal{M}}$ and $S = End_{\mathcal{L}_{\mathcal{M}}}(L)$. We say that L is *weakly co-Hopfian* if for each linear monomorphism $f \in S$, one has that $f(1_L)$ is an essential element in L.

Regarding the monoid S, we say it is *weakly co-Hopfian* if for each S-monomorphism $\varphi : S \longrightarrow S$, one has that $\varphi(S)$ is an essential right ideal of S.

Example 3.40. Let $\mathcal{P}(\mathbb{N})$ denote the power set of the set of natural numbers $\mathbb{N}$. As any power set, $\mathcal{P}(\mathbb{N})$ is partially ordered by inclusion, and moreover, $\mathcal{P}(\mathbb{N})$ forms a lattice with the join and meet operations defined as the union and intersection of sets, respectively. Of course, $0_{\mathcal{P}(\mathbb{N})} = \emptyset$ and $1_{\mathcal{P}(\mathbb{N})} = \mathbb{N}$. Now, we define $f : \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$ through $f : X \mapsto 2X$, where $2X = \{2x \mid x \in X\}$. Note that f is a linear monomorphism in $S = End_{\mathcal{L_M}}(\mathcal{P}(\mathbb{N}))$ with $f(\mathbb{N}) = 2\mathbb{N}$, the set of all even natural numbers. However, the element $2\mathbb{N}$ is not essential in $\mathcal{P}(\mathbb{N})$: $2\mathbb{N} \cap \{3\} = \emptyset$. Consequently, $\mathcal{P}(\mathbb{N})$ is not weakly co-Hopfian.

Example 3.41. Consider the unit interval $[0, 1]$ with the order induced by the usual order of $\mathbb{R}$. The function $f : [0, 1] \to [0, 1]$ defined by the rule $f(x) = x/2$ is a linear monomorphism. However, f is not surjective, so that the lattice $[0, 1]$ is not co-Hopfian.

Nevertheless, for any linear monomorphism $g : [0, 1] \to [0, 1]$, the element $g(1_L)$ is greater than zero. This implies that $g(1_L)$ is essential in $[0, 1]$. Consequently, $[0, 1]$ is weakly co-Hopfian.

Recall that a complete modular lattice L is *upper semicontinuous* —or *upper continuous* for short— if for every $a \in L$ and any upper directed set $D \subseteq L$, one has that

$$a \wedge (\bigvee_{d \in D} d) = \bigvee_{d \in D} (a \wedge d).$$

Definition 3.42. We say that an upper continuous complete modular lattice L has *finite uniform dimension* if L has a finite maximal independent subset of uniform elements.

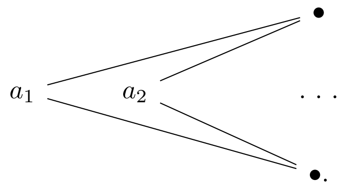
According to [4, Theorem 8.1], the uniform dimension of an upper continuous complete modular lattice of finite uniform dimension can be defined as the cardinality of a maximal independent subset of nonzero uniform elements.

Also, [4, Theorem 8.2] ensures that an upper continuous complete modular lattice L has finite uniform dimension if and only if L does not contain infinite independent subsets.

Definition 3.43. Let $L \in \mathcal{L_M}$, and let $S = End_{\mathcal{L_M}}(L)$. We say that S has *finite right uniform dimension* if $\mathcal{R}(S)$ has finite uniform dimension, that is, if there is a finite maximal independent subset of uniform right ideals of S.

In [5, Theorem 2.6(a)], the authors proved that a retractable semi-projective module M has finite uniform dimension if and only if its endomorphism ring S has finite right uniform dimension. Furthermore, these dimensions coincide. The next example shows that the corresponding statement for linear modular lattices does not hold.

Example 3.44. Let $L \in \mathcal{L}_\mathcal{M}$ be the lattice of length 2 with countably infinite many atoms:



We established in Example 3.8 that L is retractable and semi-projective. One verifies easily that L is upper continuous, and has a finite uniform dimension equal to two. However, its endomorphism monoid S has a countably infinite independent set of uniform right ideals, each of which has the form $Hom_{\mathcal{L}_\mathcal{M}}(L, a_k/0_L)$ with a_k an atom of L (these right ideals are uniform because of Theorem 3.17(e)).

Definition 3.45. We say that a lattice $L \in \mathcal{L}_\mathcal{M}$ is *weakly compressible* if for all $a \in L$ with $a \neq 0_L$, there exists a linear morphism $f : L \longrightarrow a/0_L$ such that $f^2 \neq 0$.

Example 3.46. The simple lattice is weakly compressible. In contrast, the lattice $L = \{0, 1, 2\}$, with the order induced by $\mathbb{N}$, is not weakly compressible. Indeed, the only nonzero linear morphism $f : \{0, 1, 2\} \longrightarrow \{0, 1\}$ satisfies $f^2 = 0$.

Definition 3.47. A lattice $L \in \mathcal{L}_\mathcal{M}$ is *cocyclic* if it has an essential atom. Likewise, we say that the monoid $S = End_{\mathcal{L}_\mathcal{M}}(L)$ is *cocyclic* if there exists a non-trivial endomorphism $f \in S$ that lies in every nonzero right ideal of S.

Definition 3.48. A lattice $L \in \mathcal{L}_\mathcal{M}$ is called *finitely cogenerated* if for any subset $\{x_a\}_{a\in A} \subseteq L$ such that $\bigwedge_{a\in A} x_a = 0_L$, there exists a finite subset F of A with $\bigwedge_{a\in F} x_a = 0_L$.

Similarly, we say that the monoid $S = End_{\mathcal{L}_\mathcal{M}}(L)$ is *finitely cogenerated* if for any collection of right ideals $\{I_a\}_{a\in A}$ of S such that $\bigcap_{a\in A} I_a = \{0\}$, there exists a finite subset F of A with $\bigcap_{a\in F} I_a = \{0\}$.

We say that S is *semiprime* if for every right ideal I and $0 \neq n \in \mathbb{N}$,

$$I \neq \{0\} \Rightarrow I^n \neq \{0\},$$

where $I^n = \{f_1 \circ \ldots \circ f_n \mid f_i \in I \text{ for each } 1 \leq i \leq n\}$.

Theorem 3.49. *Let $L \in \mathcal{L}_\mathcal{M}$ be a retractable and semi-projective lattice, and let $S = End_{\mathcal{L}_\mathcal{M}}(L)$. Then, the following statements are true:*

(a) *L is weakly co-Hopfian if and only if S is weakly co-Hopfian.*

(b) $Hom_{\mathcal{L_M}}(1_L/a, L) = \{0\}$ *for every essential element* $a \in L$ *if and only if* $Z_{\mathfrak{r}}(S) = \{0\}$.

(c) *If* L *is weakly compressible, then* S *is semiprime.*

(d) L *is cocyclic if and only if* S *is cocyclic.*

(e) *If* S *is finitely cogenerated, then* L *is finitely cogenerated.*

Proof. (a) ($\Longrightarrow$) Let $\varphi : S \longrightarrow S$ be an S-monomorphism, and let $f = \varphi(Id_L)$. Note that, for any $g \in S$,

$$\varphi(g) = \varphi(Id_L \circ g) = \varphi(Id_L) \circ g = f \circ g.$$

Thus, we can express φ as $\varphi = f \circ \_$. Then, L being retractable, Lemma 3.34 ensures that f is a monomorphism, and hence, $f(1_L)$ is essential in L because L is weakly co-Hopfian. Recall also that, by Theorem 3.11, $f \circ S = Hom_{\mathcal{L_M}}\big(L, {}^{f(1_L)}/{}_{0_L}\big)$. Considering the above, if we let $I = \varphi(S) = f \circ S$ and $J = S$, then, observing that $\bigvee_{i \in I} i(1_L) = f(1_L)$ and $\bigvee_{j \in J} j(1_L) = 1_L$, Theorem 3.17(d) provides that $\varphi(S)$ is essential in S. Therefore, S is weakly co-Hopfian.

($\Longleftarrow$) Let $f : L \longrightarrow L$ be a linear monomorphism. Since L is retractable, we can appeal to Lemma 3.34 to obtain the S-monomorphism $(f \circ \_) : S \longrightarrow S$. Thus, as S is weakly co-Hopfian, $f \circ S$ is an essential right ideal of S. Then, by Theorem 3.17(c), we have that $f(1_L) = \bigvee_{s \in S} (f \circ s)(1_L)$ is essential in $1_L = \bigvee_{s \in S} s(1_L)$. Therefore, L is weakly co-Hopfian.

(b) Note first that, since L is retractable and semi-projective,

$$Z_{\mathfrak{r}}(S) = \{f \in S | k_f \text{ is essential in } L\}$$

due to Proposition 3.38(a).

($\Longrightarrow$) Let $f \in S$ be such that its kernel k_f is essential in L. Write $\overline{f} : 1_L/k_f \longrightarrow f(1_L)/0_L$ for the induced lattice isomorphism and $\iota : f(1_L)/0_L \hookrightarrow L$ for the inclusion mapping. Then, the composite $\iota \circ \overline{f} : 1_L/k_f \longrightarrow L$ lies in $Hom_{\mathcal{L_M}}\big(1_L/k_f, L\big) = \{0\}$. Thus, $\iota \circ \overline{f}$ is the trivial morphism, so that $k_f = 1_L$. Therefore, $f = 0$.

($\Longleftarrow$) Let $a \in L$ be an essential element, and let $f \in Hom_{\mathcal{L_M}}\big(1_L/a, L\big)$. If $L \xrightarrow{a \vee \_} 1_L/a$ denotes the linear morphism such that $x \longmapsto x \vee a$, then the composite $f \circ (a \vee \_) \in S$. Furthermore, the kernel of $f \circ (a \vee \_)$ is greater than or equal to a, so it is essential in L. Hence, $f \circ (a \vee \_) \in Z_{\mathfrak{r}}(S) = \{0\}$, which implies that $f = 0$. Therefore, $Hom_{\mathcal{L_M}}\big(1_L/a, L\big) = \{0\}$.

(c) Let us assume that L is weakly compressible and that S is not semiprime. Then, S has a right ideal $I \neq \{0\}$ such that $I^k = \{0\}$ for some $k \geq 2$. Take the least such k, so that $I^{k-1} \neq \{0\}$. Now, let J be a non-trivial principal right ideal contained in I^{k-1}. Then, since L is semi-projective and weakly compressible, in view of Lemma 3.37, there exists a linear morphism $f \in Hom_{\mathcal{L_M}}(L, JL) = J$ such that $f^2 \neq 0$. However, $f^2 \in I^{2k-2} = \{0\}$ since, as $k \geq 2$, $2k-2 \geq k$. Therefore, S is semiprime.

(d) ($\Longrightarrow$) Let $a \in L$ be an essential atom. Since L is semi-projective, Theorem 3.17, parts (b) and (g), gives that $Hom_{\mathcal{L_M}}(L, a/0_L)$ is a simple and essential right ideal of S. It follows that any nonzero element of $Hom_{\mathcal{L_M}}(L, a/0_L)$ belongs to every non-trivial right ideal of S.

($\Longleftarrow$) There exists $0 \neq g \in S$ such that g belongs to every non-trivial right ideal of S. For each $0_L \neq x \in L$, L being retractable, $g \in Hom_{\mathcal{L_M}}(L, x/0_L)$, so that $0_L \neq g(1_L) \leq x$. This shows that $g(1_L)$ is an essential atom of L.

(e) Let $\{x_a\}_{a\in A} \subseteq L$ be such that $\bigwedge\limits_{a\in A} x_a = 0_L$. Then, the family $\big\{Hom_{\mathcal{L_M}}(L, x_a/0_L)\big\}_{a\in A}$ of right ideals of S satisfies that

$$\bigcap_{a\in A} Hom_{\mathcal{L_M}}\big(L, x_a/0_L\big) = Hom_{\mathcal{L_M}}\big(L, (\textstyle\bigwedge\limits_{a\in A} x_a)/0_L\big) = Hom_{\mathcal{L_M}}\big(L, 0\big) = \{0\}.$$

Thus, since S is finitely cogenerated, there exists a finite subset F of A such that

$$\bigcap_{a\in F} Hom_{\mathcal{L_M}}(L, x_a/0_L) = \{0\}.$$

Moreover, as

$$Hom_{\mathcal{L_M}}(L, (\textstyle\bigwedge\limits_{a\in F} x_a)/0_L) = \displaystyle\bigcap_{a\in F} Hom_{\mathcal{L_M}}(L, x_a/0_L) = \{0\},$$

and L is retractable, it follows that $\bigwedge\limits_{a\in F} x_a = 0_L$. Therefore, L is finitely cogenerated. □

Definition 3.50. We say a lattice $L \in \mathcal{L_M}$ is *directly finite* if it is not isomorphic to any initial interval $a/0_L$, where $a < 1_L$ and a has a complement in L.

Lemma 3.51. *Let $L \in \mathcal{L_M}$ and suppose that $f, g \in End_{\mathcal{L_M}}(L)$ are such that $f \circ g = Id_L$. Then, f is an epimorphism, g is a monomorphism, and $g(1_L)$ and k_f are complements of each other in L.*

Proof. Since $f(g(1_L)) = Id_L(1_L) = 1_L$, f is an epimorphism. Further, since

$$k_g = Id_L(k_g) = f(g(k_g)) = f(0_L) = 0_L,$$

g is a monomorphism.

Now, as $f\big(g(1_L) \vee k_f\big) = f(g(1_L)) = 1_L = f(1_L)$, injectivity of $1_L/k_f \xrightarrow{\overline{f}} L$ gives that $g(1_L) \vee k_f = 1_L$. Moreover, surjectivity of $L \xrightarrow{\overline{g}} {}^{g(1_L)}/_{0_L}$ gives $y \in L$ such that $g(y) = g(1_L) \wedge k_f$, so that

$$y = Id_L(y) = f(g(y)) = f(g(1) \wedge k_f) = 0_L.$$

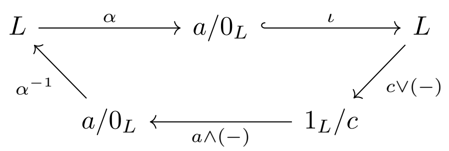
Therefore, $g(1_L) \wedge k_f = g(0_L) = 0_L$. □

Lemma 3.52. *A lattice $L \in \mathcal{L}_{\mathcal{M}}$ is directly finite if and only if, for any $f, g \in End_{\mathcal{L}_{\mathcal{M}}}(L)$,*

$$f \circ g = Id_L \Rightarrow g \circ f = Id_L.$$

Proof. ($\Longrightarrow$) If $f \circ g = Id_L$, then, by Lemma 3.51, g is a monomorphism and $g(1_L)$ has a complement in L. Now, as g induces a lattice isomorphism $\overline{g} : L \longrightarrow g(1_L)/0_L$, the hypothesis of L being directly finite implies that $g(1_L) = 1_L$. Therefore, g is a lattice isomorphism. It now follows easily that f and g are inverse isomorphisms.

($\Longleftarrow$) Suppose that L is not directly finite, that is, there exists an isomorphism $\alpha : L \to a/0_L$ where $a \neq 1_L$ has a complement $c \in L$. By modularity, $a \wedge (-) : 1_L/c \to a/0_L$ is an isomorphism with inverse $c \vee (-) : a/0_L \to 1_L/c$. Denote by $\iota : a/0_L \to L$ the inclusion mapping. Consider the following diagram:

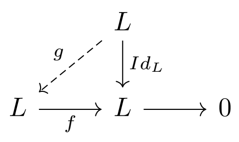


Set $f = \alpha^{-1} \circ (a \wedge (-)) \circ (c \vee (-)) : L \to L$ and $g = \iota \circ \alpha : L \longrightarrow L$. Then, clearly, $f \circ g = Id_L$, but $(g \circ f)(c) = (\iota \circ \alpha \circ \alpha^{-1})(0_L) = 0_L \neq c$, so $g \circ f \neq Id_L$. □

Definition 3.53. A lattice $L \in \mathcal{L}_{\mathcal{M}}$ is *Hopfian* if every linear epimorphism[3] $f : L \longrightarrow L$ is a linear monomorphism.

Proposition 3.54. *If $L \in \mathcal{L}_{\mathcal{M}}$ is semi-projective and co-Hopfian, then L is Hopfian.*

Proof. Let $f : L \longrightarrow L$ be a linear epimorphism. Since L is semi-projective, there exists a linear morphism $g : L \longrightarrow L$ that makes the diagram



commutative, that is, $f \circ g = Id_L$. By Lemma 3.51, g is a monomorphism. Since L is co-Hopfian, g is also an epimorphism and hence an isomorphism. It now follows easily that f and g are inverse isomorphisms. In particular, f is a monomorphism, which shows that L is Hopfian. □

[3]Let us call a linear morphism that is an epimorphism in the category $\mathcal{L}_{\mathcal{M}}$ a *linear epimorphism.*

Proposition 3.55. *If $L \in \mathcal{L}_{\mathcal{M}}$ is Hopfian, then L is directly finite.*

Proof. If L is not directly finite, there exists an isomorphism $\gamma : L \longrightarrow a/0_L$, where $a \neq 1_L$ has a complement $c \in L \setminus \{0\}$. By modularity, we also have an isomorphism $a \wedge (-) : 1_L/c \longrightarrow a/0_L$. Set as $f : L \to L$ the composite

$$L \overset{c\vee(-)}{\longrightarrow} 1_L/c \underset{\cong}{\overset{a\wedge(-)}{\longrightarrow}} a/0_L \underset{\cong}{\overset{\gamma^{-1}}{\longrightarrow}} L.$$

Observe that f is surjective but not injective, as $f(c) = 0_L = f(0_L)$ and $c \neq 0_L$. This implies that L is not Hopfian. □

According to [9, Proposition 3.5], in the context of a semi-projective module, the attributes of being co-Hopfian, Hopfian, and directly finite are equivalent. Nevertheless, this equivalence does not hold for linear modular lattices, as demonstrated by the following example.

Example 3.56. Consider the lattice $L = \{1/n\}_{n\in\mathbb{N}} \cup \{0\}$, with the order induced by $\mathbb{R}$. We claim that L is a semi-projective lattice. Let ${}^{1/n}/_0$ be a non-trivial initial interval of L, and let $L \overset{f}{\longrightarrow} {}^{1/n}/_0$ be a linear epimorphism. Observe that, due to the definition of L, each non-trivial initial interval is an infinite lattice, whereas each quotient interval, apart from L itself, is a finite lattice. With this in mind, since f induces a lattice isomorphism $\bar{f} : 1/k_f \longrightarrow {}^{1/n}/_0$, the only option for k_f is $k_f = 0$. Furthermore,

$$f(1/x) = {}^1/_{x\,+\,(n\,-\,1)},$$

for all $x \geq 1$.

The above argument applies to any non-trivial linear morphism $g : L \longrightarrow {}^{1/n}/_0$: g induces $\bar{g} : 1/k_g \to {}^{g(1)}/_0$, so that $k_g = 0$, and, putting $g(1) = 1/m$ (with $m \geq n$), it holds that

$$g(1/x) = {}^{1_L}/_{x\,+\,(m\,-\,1)},$$

for all $x \geq 1$. Therefore, the linear morphism $L \overset{h}{\longrightarrow} L$, given by $h(0) = 0$ and $h(1/x) = {}^1/_{x\,+\,(m\,-\,n)}$, satisfies that $f \circ h = g$. Therefore, L is semi-projective.

Let us now note that the lattice L is directly finite. This follows from the fact that L is totally ordered, and therefore only 0 and 1 are complemented in L. Moreover, L is not co-Hopfian since the linear morphism $k : L \longrightarrow L$ such that $k(0) = 0$ and $k(1/n) = 1/(n+1)$ for $n \geq 1$ is an injective mapping which is not surjective.

Lemma 3.57. *Let $L \overset{f}{\longrightarrow} L$ be a linear epimorphism, and let $a \in L$. Then, the mapping $1_L/a \overset{f|}{\longrightarrow} {}^{1_L}/_{f(a)}$ is a linear morphism with kernel $k_f \vee a$.*

Proof. For each $x \in 1_L/a$,

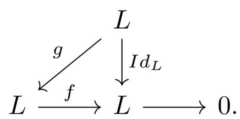
$$f|\big((k_f \vee a) \vee x\big) = f(k_f \vee a \vee x) = f(k_f \vee x) = f(x) = f|(x).$$

Furthermore, the mapping $\overline{f|} : {}^{1_L}/_{k_f \vee a} \longrightarrow {}^{1_L}/_{f(a)}$ is a lattice isomorphism, as it coincides with the restriction and corestriction of the lattice isomorphism $\bar{f}$ induced by f. □

Recall that an element $a \in L$ is called *superfluous* in L if $a \vee b \neq 1_L$ holds for every $b \in L\backslash\{1_L\}$.

Proposition 3.58. *Let $L \in \mathcal{L}_{\mathcal{M}}$ be semi-projective, and let $a \in L$ be superfluous and strongly invariant in L. Then, L is Hopfian if and only if $1_L/a$ is Hopfian.*

Proof. ($\Longleftarrow$) Let $f : L \longrightarrow L$ be a linear epimorphism. As L is semi-projective, there exists a linear morphism $g : L \longrightarrow L$ such that the following diagram is commutative:



Thus, $f \circ g = Id_L$, so that, by Lemma 3.51, g is a monomorphism and the elements $g(1_L)$ and k_f are complements of each other.

Now, by Lemma 3.57, the mapping $1_L/a \xrightarrow{f|} {}^{1_L}/_{f(a)}$ is a linear morphism with kernel $k_f \vee a$. Note that $f(a) \leq a$ because a is strongly invariant in L. Then, the composition $f' = (\_ \vee a) \circ f| : 1_L/a \longrightarrow 1_L/a$ produces a linear epimorphism. Moreover, as $1_L/a$ is Hopfian, f' is an isomorphism. Hence,

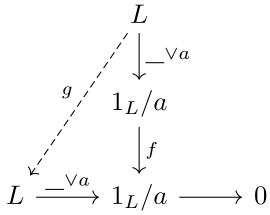
$$f'(k_f \vee a) = \big((\_ \vee a) \circ f|\big)(k_f \vee a) = f(a) \vee a = a = f'(a).$$

Therefore, $k_f \vee a = a$, that is, $k_f \leq a$. Since a is superfluous in L, so is k_f. Then, since $g(1_L) \vee k_f = 1_L$, one has that $g(1_L) = 1_L$. Thus,

$$k_f = 1_L \wedge k_f = g(1_L) \wedge k_f = 0_L.$$

Consequently, f is a monomorphism, showing that L is Hopfian.

($\Longrightarrow$) Let $1_L/a \xrightarrow{f} 1_L/a$ be a linear epimorphism. Since L is semi-projective, there exists a linear morphism $g : L \longrightarrow L$ that makes the diagram



commutative. Hence, $(\_ \vee a) \circ g$ is a linear epimorphism, and thus, $g(1_L) \bigvee a = 1_L$. Since a is superfluous in L, it follows that $g(1_L) = 1_L$, that is, g is an epimorphism. Then, L being Hopfian implies that g is an isomorphism. Now, due to the commutativity of the above diagram, we have that $g(k_f) \vee a = a$. Thus, $g(k_f) \leq a$, so that

$$k_f \leq g^{-1}(a) \leq a \leq k_f,$$

because of the strong invariance of a. Thus, $k_f = a$, indicating that f is a lattice monomorphism. Therefore, the lattice $1_L/a$ is Hopfian. □

We say that a lattice $L \in \mathcal{L}_{\mathcal{M}}$ is *coatomic* if for every $a \in L \backslash \{1_L\}$, the quotient interval $1_L/a$ contains maximal proper elements. These maximal proper elements are called coatoms, and the set of all coatoms of L is denoted by $\mathcal{C}_L$.

Corollary 3.59. *Let $L \in \mathcal{L}_{\mathcal{M}}$ be semi-projective and coatomic. Then, L is Hopfian if and only if $1_L / \bigwedge_{c \in \mathcal{C}_L} c$ is Hopfian.*

Proof. By Proposition 3.58, it suffices to show that the element $Jac(L) = \bigwedge_{c \in \mathcal{C}_L} c$ is superfluous and strongly invariant in L. Now, since L is coatomic, [3, Proposition 4.4](3) ensures that $Jac(L)$ is a superfluous element in L and that

$$Jac(L) = Rad(L) = \bigvee_{c \in S(L)} c,$$

where $S(L)$ denotes the set of all superfluous elements of L. (Bear in mind that, throughout [3], coatomicity is called "condition (KL)" — see the definition before [3, Remark 4.2].) Lastly, it follows from [2, Proposition 2.2] that $Rad(L)$ is strongly invariant in L. □

Definition 3.60. For a lattice $L \in \mathcal{L}_{\mathcal{M}}$, we introduce the following conditions:

- D1: For each $a \in L$, there exists a complement $c \in L$ such that $c \leq a$ and a is superfluous in $1_L/c$.
- D2: If $a \in L$ is such that $1_L/a \cong c/0_L$, with c a complement in L, then a is a complement.
- D3: Given two complements k and c in L with $k \vee c = 1_L$, the element $k \wedge c$ is a complement.

Lemma 3.61. *Let $L \in \mathcal{L}_{\mathcal{M}}$ be a D2 lattice, and let $k, c \in L$ be two complements. If $k/0_L \xrightarrow{f} c/0_L$ is a linear epimorphism, then the kernel k_f is a complement in L.*

Proof. As f is a linear epimorphism, $k/k_f \cong c/0_L$. Further, if k^* denotes a complement of k in L, it holds, by modularity, that

$$k/k_f = {}^{k}/_{k_f \vee (k^* \wedge k)} = {}^{k}/_{(k_f \vee k^*) \wedge k} \cong {}^{k \vee (k^* \vee k_f)}/_{k_f \vee k^*} = {}^{1_L}/_{k_f \vee k^*}.$$

Hence, as L satisfies condition D2, $k_f \vee k^*$ is a complement in L. Let $w \in L$ be a complement of $k_f \vee k^*$. We claim that $k^* \vee w$ is a complement of k_f in L. Indeed, on one end,

$$k_f \vee (k^* \vee w) = (k_f \vee k^*) \vee w = 1_L.$$

On the other end, by modularity,

$$k^* \vee (k_f \wedge (k^* \vee w)) = (k^* \vee k_f) \wedge (w \vee k^*) = \big((k^* \vee k_f) \wedge w\big) \vee k^* = k^*.$$

Thus, $k_f \wedge (k^* \vee w) \leq k^*$. Also, $k_f \wedge (k^* \vee w) \leq k_f \leq k$, so that

$$k_f \wedge (k^* \vee w) \leq k^* \wedge k = 0_L. \qquad \square$$

Proposition 3.62. *Let $L \in \mathcal{L}_{\mathcal{M}}$. If L is a D2 lattice, then L is a D3 lattice.*

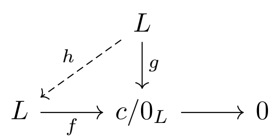
Proof. Let k and c be two complements in L with $k \vee c = 1_L$. If k^* is a complement of k in L, then, by modularity,

$$k^*/0_L \cong 1_L/k = {}^{c \vee k}/_{k} \cong {}^{c}/_{k \wedge c}.$$

Then, there is a lattice isomorphism ${}^{c}/_{k \wedge c} \xrightarrow{h} k^*/0_L$. Thus, writing $c/0_L \xrightarrow{\rho} c/(k \wedge c)$ for the canonical epimorphism, the composition $f = h \circ \rho : c/0_L \longrightarrow k^*/0_L$ yields a linear epimorphism. Hence, by Lemma 3.61, we have that $k \wedge c = k_f$ is a complement in L. $\square$

Lemma 3.63. *Any semi-projective lattice $L \in \mathcal{L}_{\mathcal{M}}$ is a D2 lattice.*

Proof. Let $a, c \in L$ such that $1_L/a \overset{\beta}{\cong} c/0_L$, with c a complement in L. Then, there exists a linear epimorphism $f = \beta \circ (\_\_ \vee a) : L \longrightarrow c/0_L$. Let c' be a complement of c in L, so that, by modularity, $1_L/c' \overset{\gamma}{\cong} c/0_L$. Then, there is a linear epimorphism $g = \gamma \circ (\_\_ \vee c') : L \longrightarrow c/0_L$. As L is semi-projective, there exists a linear morphism $h : L \longrightarrow L$ such that the diagram



is commutative. We claim that $h(c)$ is a complement of a in L. On the one hand, observe that

$$f(h(c) \vee a) = f(h(c)) = g(c) = g(c \vee c') = g(1_L) = c.$$

Since $\overline{f} : 1_L/a \longrightarrow c/0_L$ is a lattice isomorphism, it follows that $h(c) \bigvee a = 1_L$. On the other hand, as the restriction $g| : c/0_L \longrightarrow c/0_L$ is a lattice isomorphism, so is $(f \circ h)| : c/0_L \longrightarrow c/0_L$. Thus, the restriction of h to $c/0_L$ is an injective linear morphism, so it induces a lattice isomorphism from $c/0_L$ to ${}^{h(c)}/{}_{0_L}$. Let then $y \in c/0_L$ such that $h(y) = h(c) \wedge a$. Then,

$$0_L = f(h(c) \wedge a) = f(h(y)) = g(y).$$

Therefore, $y = 0_L$, and thence, $h(c) \wedge a = 0_L$. □

Definition 3.64. Let $L \in \mathcal{L}_{\mathcal{M}}$. We say that L is *lifting* if it satisfies condition D1. Moreover, we say that L is *discrete* if it is lifting and satisfies condition D2 and that it is *quasi-discrete* if it is lifting and satisfies condition D3.

Theorem 3.65. *Let $L \in \mathcal{L}_{\mathcal{M}}$ be semi-projective. Then, the following statements are equivalent:*

(1) *L is discrete.*
(2) *L is quasi-discrete.*
(3) *L is lifting.*

Proof. (1) $\Longrightarrow$ (2) By Proposition 3.62.

(2) $\Longrightarrow$ (3) By definition.

(3) $\Longrightarrow$ (1) By Lemma 3.63, seeing as L is semi-projective. □

Definition 3.66. A lattice $L \in \mathcal{L}_{\mathcal{M}}$ is *hollow* if each proper element of L is superfluous in L.

Definition 3.67. We say that a lattice $L \in \mathcal{L}_{\mathcal{M}}$ is *indecomposable* if the only complements in L are 0_L and 1_L.

Proposition 3.68. *Let $L \in \mathcal{L}_{\mathcal{M}}$ be indecomposable and semi-projective. Then, L is discrete if and only if L is hollow.*

Proof. ($\Longrightarrow$) As L is discrete, it is lifting. Then, for proper $x \in L$, there exists a complement $c \leq x$ such that x is superfluous in $1_L/c$. Since L is indecomposable, necessarily $c = 0_L$, so that x is superfluous in L. Hence, L is hollow.

($\Longleftarrow$) Note first that, trivially, 1_L is superfluous in $1_L/1_L$. For proper $x \in L$, x is superfluous in $L = 1_L/0_L$, because L is hollow. Since 0_L and 1_L are always complements in L, it follows that L is lifting. Now, as L is semi-projective, Lemma 3.63 provides that L satisfies condition D2. Therefore, L is discrete. □

Definition 3.69. We say that a lattice $L \in \mathcal{L}_{\mathcal{M}}$ is *pseudo semi-projective* if for any two linear morphisms $f, g \in End_{\mathcal{L}_{\mathcal{M}}}(L)$ with $f(1_L) = g(1_L)$, there exists $h \in End_{\mathcal{L}_{\mathcal{M}}}(L)$ such that $f \circ h = g$.

Remark 3.70. Any semi-projective lattice is pseudo semi-projective.

Next, we show a lattice that is not pseudo semi-projective, and a lattice that is pseudo semi-projective but not semi-projective.

Example 3.71. Consider the lattice $L = \{1\} \cup \{1 - {}^{1}\!/\!{}_{n}\}_{n \in \mathbb{N}}$ with the order induced by $\mathbb{R}$. Observe that the only infinite initial interval of L is L. Further, the quotient interval $1_L/k_f$ is infinite for any $0 \neq f \in End_{\mathcal{L}_{\mathcal{M}}}(L)$. Thus, the induced lattice isomorphism $\overline{f} : 1/k_f \longrightarrow {}^{f(1)}\!/\!{}_{0}$ shows that any nonzero endomorphism must be a linear epimorphism.

Now, let us define the linear epimorphism $g : L \longrightarrow L$ such that $g(1) = 1$, $g(0) = 0$, $g(1/2) = 0$ and $g(1 - {}^{1}\!/\!{}_{n}) = 1 - {}^{1}\!/\!{}_{n-1}$ for all $n \geq 3$. Then, for the diagram

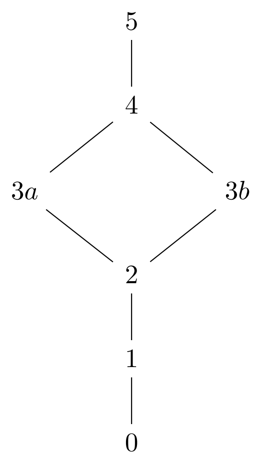
$$\begin{array}{ccccc} & & L & & \\ & & \big\downarrow{\scriptstyle Id_L} & & \\ L & \xrightarrow[g]{} & L & \longrightarrow & 0, \end{array}$$

there is no $h \in End_{\mathcal{L}_{\mathcal{M}}}(L)$ such that $g \circ h = Id_L$. Indeed, if such an epimorphism existed, there would be $x, y \in L$ such that $h(x) = 0$ and $h(y) = 1/2$. Then

$$x = Id_L(x) = (g \circ h)(x) = 0 = (g \circ h)(y) = Id_L(y) = y,$$

so that $h(x) = h(y)$, a contradiction. Hence, L is not a pseudo semi-projective lattice.

Example 3.72. Let us denote by L the following lattice:



We claim that L is not semi-projective. Indeed, consider the linear morphisms $f : L \longrightarrow 2/0$ and $g : L \longrightarrow 2/0$, with induced lattice isomorphisms $\overline{f} : 5/3a \longrightarrow 2/0$ and $\overline{g} : 5/4 \longrightarrow 1/0$, respectively. Then, if $h \in End_{\mathcal{L_M}}(L)$ were such that $f \circ h = g$, we would have that $f(h(5)) = g(5) = 1$, so that $h(5) = 4$ or $h(5) = 3b$. However, both options are not possible since L has no quotient intervals isomorphic to $4/0$ nor to $3b/0$.

Let us now show that L is pseudo semi-projective. We have already observed that only the initial intervals $0/0$, $1/0$, $2/0$ and L itself are isomorphic to a quotient interval of L. Thus, the kernel of any linear endomorphism of L lies in the set $\{5, 4, 3a, 3b, 0\}$. Let $f, g \in End_{\mathcal{L_M}}(L)$ be such that $f(5) = g(5)$. As

$$5/k_f \cong {}^{f(5)}/_0 = {}^{g(5)}/_0 \cong 5/k_g,$$

either $k_f = k_g$ or k_f and k_g are different and non-comparable.

Suppose first that $k_f \neq k_g$. Then, either $k_f = 3a$ and $k_g = 3b$, or $k_f = 3b$ and $k_g = 3a$. As these two cases are symmetric, we will assume that $k_f = 3a$ and $k_g = 3b$. Thus, since $5/3a \overset{\bar{f}}{\cong} {}^{f(5)}/_0$ and $5/3b \overset{\bar{g}}{\cong} {}^{g(5)}/_0$, it follows that $f(5) = 2 = g(5)$ and $f(4) = 1 = g(4)$. Also, $f(3a/0) = f(k_f/0_L) = \{0\} = g(3b/0)$ and $f(3b) = f(k_f \vee 3b) = f(3a \vee 3b) = f(4) = 1 = g(3a)$. Set $h : L \to L$ such that $h(3a) = 3b$, $h(3b) = 3a$ and $h(x) = x$ for every other $x \in L$. Then, clearly, h is a lattice isomorphism and $f \circ h = g$.

Similar arguments show that $f = g$ whenever $k_f = k_g = 3a$, $k_f = k_g = 3b$, $k_f = k_g = 4$ or $k_f = k_g = 5$. Therefore, in these four cases $f \circ Id_L = g$.

Lastly, if $k_f = k_g = 0$, then f and g are linear monomorphisms, and thus, L being finite, lattice isomorphisms. Hence, $f \circ (f^{-1} \circ g) = g$.

Therefore, L is a pseudo semi-projective lattice.

The reasoning behind Proposition 3.6 may also be used to obtain

Lemma 3.73. *A lattice $L \in \mathcal{L_M}$ is pseudo semi-projective if and only if for any $f, g \in End_{\mathcal{L_M}}(L)$ with $f(1_L) = g(1_L)$, one has that $f \circ End_{\mathcal{L_M}}(L) = g \circ End_{\mathcal{L_M}}(L)$.*

Remark 3.74. In the proof of Lemma 3.63, it holds that $f(1_L) = g(1_L)$. Thus, we can strengthen that lemma by saying that every pseudo semi-projective lattice is a D2 lattice.

Remark 3.75. For pseudo semi-projective $L \in \mathcal{L_M}$, if $f, g \in End_{\mathcal{L_M}}(L)$ are such that $f(1_L) = g(1_L)$ and k_f is superfluous in L, then the linear morphism h that makes the diagram

$$\begin{array}{ccccc} & & L & & \\ & \overset{h}{\swarrow} & \downarrow g & & \\ L & \xrightarrow[f]{} & f(1_L)/0_L & \longrightarrow & 0 \end{array}$$

commutative is an epimorphism. Indeed,

$$f(h(1_L) \vee k_f) = f(h(1_L)) = g(1_L) = f(1_L).$$

Since $\bar{f} : 1_L/k_f \longrightarrow f(1_L)/0_L$ is a lattice isomorphism, $h(1_L) \vee k_f = 1_L$. Then, $h(1_L) = 1_L$, so that h is a linear epimorphism.

Proposition 3.76. *If $L \in \mathcal{L}_{\mathcal{M}}$ is pseudo semi-projective and hollow, then L is Hopfian.*

Proof. Note that the zero lattice is Hopfian. Assume then that L is non-trivial. Let $f \in End_{\mathcal{L}_{\mathcal{M}}}(L)$ be an epimorphism, and let us consider the following commutative diagram:

$$\begin{array}{ccccc} & & L & & \\ & \overset{h}{\swarrow} & \downarrow Id_L & & \\ L & \xrightarrow[f]{} & L & \longrightarrow & 0 \end{array}$$

Since L is hollow, the element k_f is superfluous in L, so, by Remark 3.75, h is an epimorphism. Let $y \in L$ be such that $h(y) = k_f$. Then,

$$y = Id_L(y) = f(h(y)) = f(k_f) = 0_L.$$

So that $k_f = h(0_L) = 0_L$. Therefore, L is Hopfian. □

Proposition 3.55 directly provides

Corollary 3.77. *If $L \in \mathcal{L}_{\mathcal{M}}$ is pseudo semi-projective and hollow, then L is directly finite.*

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References

[1] T. Albu and M. Iosif, *The category of linear modular lattices*, Bull. Math. Soc. Sci. Math. Roumanie (N. S.), 56(104) (2013), 33-46.
[2] T. Albu and M. Iosif, *On socle and radical of modular lattices*, Ann. Univ. Buchar. Math. Ser., 5(63) (2014), 187-194.

[3] T. Albu and M. Iosif, *Lattice preradicals with applications to Grothendieck categories and torsion theories*, J. Algebra, 444 (2015), 339-366.

[4] G. Călugăreanu, Lattice Concepts of Module Theory, Kluwer Texts in the Mathematical Sciences, 22, Kluwer Academic Publishers, Dordrecht, 2000.

[5] A. Haghany and M. R. Vedadi, *Study of semi-projective retractable modules*, Algebra Colloq., 14(3) (2007), 489-496.

[6] S. Mac Lane, Categories for the Working Mathematician, Second edition, Graduate Texts in Mathematics, 5, Springer-Verlag, New York, 1998.

[7] S. Pardo-Guerra, H. A. Rincón-Mejía and M. G. Zorrilla-Noriega, *Some isomorphic big lattices and some properties of lattice preradicals*, J. Algebra Appl., 19(7) (2020), 2050140 (29 pp).

[8] S. Pardo-Guerra, H. A. Rincón-Mejía and M. G. Zorrilla-Noriega, *Big lattices of hereditary and natural classes of linear modular lattices*, Algebra Universalis, 82(4) (2021), 52 (15 pp).

[9] M. K. Patel, *Properties of semi-projective modules and their endomorphism rings*, In: Algebra and its Applications, Springer Proc. Math. Stat., Springer, Singapore, 174 (2016), 321-328.

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