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COFREE COM-PRELIE ALGEBRAS

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Abstract. A Com-PreLie bialgebra is a commutative bialgebra with an extra preLie product satisfying some compatibilities with the product and the coproduct. We here give examples of cofree Com-PreLie bialgebras, including all the ones such that the preLie product is homogeneous of degree $\geqslant -1$. We also give a graphical description of free unitary Com-PreLie algebras, explicit their canonical bialgebra structure and exhibit with the help of a rigidity theorem certain cofree quotients, including the Connes-Kreimer Hopf algebra of rooted trees. We finally prove that the dual of these bialgebras are also enveloping algebras of preLie algebras, combinatorially described.

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1. Introduction

Com-PreLie bialgebras, introduced in [5,6], are commutative bialgebras with an extra preLie product, compatible with the product and coproduct, see Definition 2.1 below. They appeared in Control Theory, as the Lie algebra of the group of Fliess operators [7] naturally owns a Com-PreLie bialgebra structure, and its underlying bialgebra is a shuffle Hopf algebra. Free (non unitary) Com-PreLie bialgebras were also described, in terms of partitioned rooted trees.

We here give examples of connected cofree Com-PreLie bialgebras. As cocommutative cofree bialgebras are, up to isomorphism, shuffle algebras $Sh(V) = (T(V), \sqcup\!\sqcup, \Delta)$, where V is the space of primitive elements, we firstly characterize Com-PreLie bialgebras structures on $Sh(V)$ in term of operators $\varpi : T(V) \otimes T(V) \longrightarrow V$, satisfying two identities, see Proposition 3.4. In particular, if we assume that the obtained preLie bracket is homogeneous of degree 0 for the graduation of $Sh(V)$ by the length, then ϖ is reduced to a linear map $f : V \longrightarrow V$, and

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the obtained preLie product is given by (Proposition 3.6):

$$\forall x_1,\ldots,x_m,y_1,\ldots,y_n \in V,$$
$$x_1\ldots x_m \bullet y_1\ldots y_n=\sum_{i=0}^n x_1\ldots x_{i-1}f(x_i)(x_{i+1}\ldots x_m \sqcup\!\sqcup\, y_1\ldots y_n).$$

In particular, if $V=\mathrm{Vect}(x_0,x_1)$ and f is defined by $f(x_0)=0$ and $f(x_1)=x_0$, we obtain the Com-PreLie bialgebra of Fliess operators in dimension 1. If we assume that the obtained preLie bracket is homogeneous of degree -1, then ϖ is given by two bilinear products $*$ and $\{-,-\}$ on V such that $*$ is preLie, $\{-,-\}$ is antisymmetric and for all $x,y,z\in V$,

$$x*\{y,z\}=\{x*y,z\},$$
$$\{x,y\}*z=\{x*y,z\}+\{x,y*z\}+\{\{x,y\},z\}.$$

This includes preLie products on V when $\{-,-\}=0$ and nilpotent Lie algebras of nilpotency order 2 when $*=0$, see Proposition 3.9.

We then extend the construction of free Com-PreLie algebras of [5] in terms of partitioned trees (see Definition 4.1) to free unitary Com-PreLie algebras $UCP(\mathcal{D})$, with the help of a complementary decoration by integers. We obtain free Com-PreLie algebras $CP(\mathcal{D})$ as the augmentation ideal of a quotient of $UCP(\mathcal{D})$, the right action of the unit $\emptyset$ on the generators of $UCP(\mathcal{D})$ being arbitrarily chosen (Proposition 4.8). Recall that partitioned trees are rooted forests with an extra structure of a partition of its vertices into blocks; forgetting the blocks, we obtain the Connes-Kreimer Hopf algebra $\mathcal{H}_{CK}$ of rooted trees [3], which is given in this way a natural structure of Com-PreLie bialgebra (Proposition 4.10). Using Livernet's rigidity theorem for preLie algebras, we prove that the augmentation ideals of $UCP(\mathcal{D})$ and $CP(\mathcal{D})$ are free as preLie algebras. Theorem 5.11 is a rigidity theorem which gives a simple criterion for a connected (as a coalgebra) Com-PreLie bialgebra to be cofree, in terms of the right action of the unit on its primitive elements. Applied to $CP(\mathcal{D})$ and $\mathcal{H}_{CK}$, it proves that they are isomorphic to shuffle bialgebras, which was already known for $\mathcal{H}_{CK}$. We also consider the dual Hopf algebras of $UCP(\mathcal{D})$ and $CP(\mathcal{D})$: as these Hopf algebras are right-sided combinatorial in the sense of [11], there dual are enveloping algebras of other preLie algebras, which we explicitly describe in Theorem 5.14, and then compare to the original Com-PreLie algebras.

This text is organized as follows: the first section contains reminders and lemmas on Com-PreLie algebras, including the extension of the Guin-Oudom extension of the preLie product in the Com-PreLie case. The second section deals with the characterization of preLie products on shuffle algebras. The next section contains the description of free unitary Com-PreLie algebras and two families of quotients, whereas the fifth and last one contains results on the bialgebraic structures of these objects: existence of the coproduct, the rigidity theorem 5.11 and its applications, the dual preLie algebras, and an application to a family of subalgebras, named Connes-Moscovici subalgebras.

Notations 1.1.
(1) Let $\mathbb{K}$ be a commutative field of characteristic zero. All the objects (vector spaces, algebras, coalgebras, preLie algebras, ...) in this text will be taken over $\mathbb{K}$.
(2) For all $n \in \mathbb{N}$, we denote by $[n]$ the set $\{1, \ldots, n\}$. In particular, $[0] = \emptyset$.

2. Reminders on Com-PreLie algebras

Let V be a vector space.

- We denote by $T(V)$ the tensor algebra of V. Its unit is the empty word, which we denote by $\emptyset$. The element $v_1 \otimes \ldots \otimes v_n \in V^{\otimes n}$, with $v_1, \ldots, v_n \in V$, will be shortly denoted by $v_1 \ldots v_n$. The deconcatenation coproduct of $T(V)$ is defined by
$$\forall v_1, \ldots, v_n \in V, \qquad \Delta(v_1 \ldots v_n) = \sum_{i=0}^{n} v_1 \ldots v_i \otimes v_{i+1} \ldots v_n.$$
The shuffle product of $T(V)$ is denoted by $\sqcup\!\sqcup$. Recall that it can be inductively defined by
$$\forall x, y \in V,\ \forall u, v \in T(V), \quad \emptyset \sqcup\!\sqcup v = 0, \quad xu \sqcup\!\sqcup yv = x(u \sqcup\!\sqcup yv) + y(xu \sqcup\!\sqcup v).$$
For example, if $v_1, v_2, v_3, v_4 \in V$,
$$v_1 \sqcup\!\sqcup v_2v_3v_4 = v_1v_2v_3v_4 + v_2v_1v_3v_4 + v_2v_3v_1v_4 + v_2v_3v_4v_1,$$
$$v_1v_2 \sqcup\!\sqcup v_3v_4 = v_1v_2v_3v_4 + v_1v_3v_2v_4 + v_1v_3v_4v_2 + v_3v_1v_2v_4 + v_3v_1v_4v_2 + v_3v_4v_1v_2,$$
$$v_1v_2v_3 \sqcup\!\sqcup v_4 = v_1v_2v_3v_4 + v_1v_2v_4v_3 + v_1v_2v_4v_3 + v_1v_4v_2v_3 + v_4v_1v_2v_3.$$
$Sh(V) = (T(V), \sqcup\!\sqcup, \Delta)$ is a Hopf algebra, known as the shuffle algebra of V.
- $S(V)$ is the symmetric algebra of V. It is a Hopf algebra, with the coproduct defined by
$$\forall v \in V, \qquad \Delta(v) = v \otimes \emptyset + \emptyset \otimes v.$$
- $coS(V)$ is the subalgebra of $(T(V), \sqcup\!\sqcup)$ generated by V. It is the greatest cocommutative Hopf subalgebra of $(T(V), \sqcup\!\sqcup, \Delta)$, and is isomorphic to $S(V)$ via the algebra morphism
$$\theta : \begin{cases} (S(V), m, \Delta) & \longrightarrow & (coS(V), \sqcup\!\sqcup, \Delta) \\ v_1 \ldots v_k & \longrightarrow & v_1 \sqcup\!\sqcup \ldots \sqcup\!\sqcup v_k. \end{cases}$$

2.1. Definitions.

Definition 2.1. (1) A *Com-PreLie algebra* [5,6] is a family $A = (A, \cdot, \bullet)$, where A is a vector space, $\cdot$ and $\bullet$ are bilinear products on A, such that

$$\begin{array}{lrl} \forall a, b \in A, & a \cdot b = b \cdot a, & \\ \forall a, b, c \in A, & (a \cdot b) \cdot c = a \cdot (b \cdot c), & \\ \forall a, b, c \in A, & (a \bullet b) \bullet c - a \bullet (b \bullet c) = (a \bullet c) \bullet b - a \bullet (c \bullet b) & \text{(preLie identity)}, \\ \forall a, b, c \in A, & (a \cdot b) \bullet c = (a \bullet c) \cdot b + a \cdot (b \bullet c) & \text{(Leibniz identity)}. \end{array}$$

In particular, $(A, \cdot)$ is an associative, commutative algebra and $(A, \bullet)$ is a right preLie algebra. We shall say that A is unitary if the algebra $(A, \cdot)$ is unitary.

(2) A *Com-PreLie bialgebra* is a family $(A, \cdot, \bullet, \Delta)$, such that
  (a) $(A, \cdot, \bullet)$ is a unitary Com-PreLie algebra.
  (b) $(A, \cdot, \Delta)$ is a bialgebra.
  (c) For all $a, b \in A$, $\Delta(a \bullet b) = a^{(1)} \otimes a^{(2)} \bullet b + a^{(1)} \bullet b^{(1)} \otimes a^{(2)} \cdot b^{(2)}$, with Sweedler's notation $\Delta(x) = x^{(1)} \otimes x^{(2)}$.

Remark 2.2. If $(A, \cdot, \bullet, \Delta)$ is a Com-PreLie bialgebra, then for any $\lambda \in \mathbb{K}$, also is $(A, \cdot, \lambda\bullet, \Delta)$.

Lemma 2.3. (1) *Let $(A, \cdot, \bullet)$ be a unitary Com-PreLie algebra. Its unit is denoted by $\emptyset$. For all $a \in A$, $\emptyset \bullet a = 0$.*
(2) *Let A be a Com-PreLie bialgebra, with counit ε. For all $a, b \in A$, $\varepsilon(a \bullet b) = 0$.*

Proof. (1) Indeed, $\emptyset \bullet a = (\emptyset \cdot \emptyset) \bullet a = (\emptyset \bullet a) \cdot \emptyset + \emptyset \cdot (\emptyset \bullet a) = 2(\emptyset \bullet a)$, so $\emptyset \bullet a = 0$.
(2) For all $a, b \in A$,

$$\begin{aligned} \varepsilon(a \bullet b) &= (\varepsilon \otimes \varepsilon) \circ \Delta(a \bullet b) \\ &= \varepsilon(a^{(1)})\varepsilon(a^{(2)} \bullet b) + \varepsilon(a^{(1)} \bullet b^{(1)})\varepsilon(a^{(2)} \cdot b^{(2)}) \\ &= \varepsilon(a^{(1)})\varepsilon(a^{(2)} \bullet b) + \varepsilon(a^{(1)} \bullet b^{(1)})\varepsilon(a^{(2)})\varepsilon(b^{(2)}) \\ &= \varepsilon(a \bullet b) + \varepsilon(a \bullet b), \end{aligned}$$

so $\varepsilon(a \bullet b) = 0$. □

Remark 2.4. Let us give a few reminders on the (dual) Hochschild cohomology for coalgebras, also called Cartier-Quillen cohomology, see [3]. Let (C, Δ) be a coalgebra, and (M, ρ_L, ρ_R) be a bicomodule over C, with left coaction ρ_L and right coaction ρ_R. An n-cochain is a map $L : M \longrightarrow C^{\otimes n}$. The coboundary d is given on any n-cochain n by

$$d(L) = (\mathrm{Id} \otimes L) \circ \rho_L + \sum_{i=1}^{n} (\mathrm{Id}^{\otimes(i-1)} \otimes \Delta \otimes \mathrm{Id}^{\otimes(n-i)}) \circ L + (-1)^{n-1}(L \otimes \mathrm{Id}) \circ \rho_R.$$

In particular, if (B, m, Δ) is a bialgebra, we can consider the bicomodule (M, ρ_L, ρ_R) defined by $M = B$, $\rho_R = \Delta$ and

$$\forall x \in B, \qquad \rho_L(x) = 1 \otimes x.$$

A 1-cocycle is then a map $L : B \longrightarrow B$ such that for any $x \in B$,

$$\Delta \circ L(x) = 1 \otimes L(x) + (L \otimes \mathrm{Id}) \circ \Delta.$$

Observe that in any Com-PreLie bialgebra A, if a is primitive, for any $b \in A$,

$$\Delta(a \bullet b) = \emptyset \otimes a \bullet b + a \bullet b^{(1)} \otimes b^{(2)}. \tag{1}$$

Therefore, the map $b \mapsto a \bullet b$ is a 1-cocycle for this cohomology.

2.2. Linear endomorphism on primitive elements.

Notations 2.5. If A is a bialgebra, we denote by $\mathrm{Prim}(A)$ the space of its primitive elements.

Proposition 2.6. *Let A be a Com-PreLie bialgebra. Its unit is denoted by $\emptyset$.*

(1) *If $x \in \mathrm{Prim}(A)$, then $x \bullet \emptyset \in \mathrm{Prim}(A)$. We denote by f_A the map*

$$f_A : \begin{cases} \mathrm{Prim}(A) & \longrightarrow & \mathrm{Prim}(A) \\ a & \longrightarrow & a \bullet \emptyset. \end{cases}$$

(2) $\mathrm{Prim}(A)$ *is a preLie subalgebra of* $(A, \bullet)$ *if and only if* $f_A = 0$.

Proof. (1) Indeed, if a is primitive, then

$$\Delta(a \bullet \emptyset) = a \otimes \emptyset \bullet \emptyset + \emptyset \otimes a \bullet \emptyset + a \bullet \emptyset \otimes \emptyset \cdot \emptyset + \emptyset \bullet \emptyset \otimes a \cdot \emptyset = 0 + \emptyset \otimes \emptyset \bullet a + a \bullet \emptyset \otimes \emptyset + 0,$$

so $a \bullet \emptyset$ is primitive.

(2) Let $a, b \in \mathrm{Prim}(A)$. Then

$$\begin{aligned} \Delta(a \bullet b) &= a \otimes \emptyset \bullet b + \emptyset \otimes a \bullet b + \emptyset \bullet \emptyset \otimes a \cdot b + a \bullet \emptyset \otimes b + \emptyset \bullet b \otimes a + a \bullet b \otimes \emptyset \\ &= \emptyset \otimes a \bullet b + a \bullet b \otimes \emptyset + f_A(a) \otimes b. \end{aligned}$$

Hence, $\mathrm{Prim}(A)$ is a preLie subalgebra if and only if for any $a, b \in A$, $f_A(a) \otimes b = 0$, that is to say if and only if $f_A = 0$. □

2.3. Extension of the pre-Lie product. Let A be a Com-PreLie algebra. It is a Lie algebra, with the bracket defined by

$$\forall x, y \in A, \qquad [x, y] = x \bullet y - y \bullet x.$$

We shall use the Oudom-Guin construction of its enveloping algebra [12,13]. In order to avoid confusions, we shall denote by $\times$ the usual product of $S(A)$ and by 1 its unit. We extend the preLie product $\bullet$ into a product from $S(A) \otimes S(A)$ into $S(A)$ by

- If $a_1, \ldots, a_k \in A$, $(a_1 \times \ldots \times a_k) \bullet 1 = a_1 \times \ldots \times a_k$.
- If $a, a_1, \ldots, a_k \in A$,

$$a \bullet (a_1 \times \ldots \times a_k) = (a \bullet (a_1 \times \ldots \times a_{k-1})) \bullet a_k - \sum_{i=1}^{k-1} a \bullet (a_1 \times \ldots \times (a_i \bullet a_k) \times \ldots \times a_{k-1}).$$

- If $x, y, z \in S(A)$, $(x \times y) \bullet z = (x \bullet z^{(1)}) \times (y \bullet z^{(2)})$, where $\Delta(z) = z^{(1)} \otimes z^{(2)}$ is the usual coproduct of $S(A)$.

Notations 2.7. If $c_1, \ldots, c_n \in A$ and $I = \{i_1, \ldots, i_k\} \subseteq [n]$, we put

$$\prod_{i \in I}^{\times} c_i = c_{i_1} \times \ldots \times c_{i_k}.$$

Proposition 2.8. (1) *Let A be a Com-PreLie algebra. If $a, b, c_1, \ldots, c_n \in A$,*

$$(a \cdot b) \bullet (c_1 \times \ldots \times c_k) = \sum_{I \subseteq [n]} \left(a \bullet \prod_{i \in I}^{\times} c_i \right) \cdot \left(b \bullet \prod_{i \notin I}^{\times} c_i \right).$$

(2) *Let A be a Com-PreLie bialgebra. If $a, b_1, \ldots, b_n \in A$,*

$$\Delta(a \bullet (b_1 \times \ldots \times b_n)) = \sum_{I \subseteq [n]} a^{(1)} \bullet \left(\prod_{i \in I}^{\times} b_i^{(1)} \right) \otimes \left(\prod_{i \in I} b_i^{(2)} \right) a^{(2)} \bullet \left(\prod_{i \notin I}^{\times} b_i \right).$$

Proof. These are proved by direct, but quite long, inductions on n. $\square$

Lemma 2.9. *Let A be a Com-PreLie bialgebra. For all $a \in \mathrm{Prim}(A)$, $k \geqslant 0$, $b_1, \ldots, b_l \in A$, $a \bullet \emptyset^{\times k} \times b_1 \times \ldots \times b_l = f_A^k(a) \bullet b_1 \times \ldots \times b_l$.*

Proof. This is obvious if $k=0$. Let us prove it for $k=1$ by induction on l. It is obvious if $l=0$. Let us assume the result at rank $l-1$. Then

$$\begin{aligned}
a\bullet\emptyset\times b_1\times\ldots\times b_l &= (a\bullet\emptyset\times b_1\times\ldots\times b_{l-1})\bullet b_l + a\bullet(\emptyset\bullet b_l)\times b_1\times\ldots\times b_{l-1}\\
&\quad+\sum_{i=1}^{l-1} a\bullet\emptyset\times b_1\times\ldots\times(b_i\bullet b_l)\times\ldots\times b_{l-1}\\
&= (f_A(a)\bullet b_1\times\ldots\times b_{l-1})\bullet b_l + 0\\
&\quad+\sum_{i=1}^{l-1} f_A(a)\bullet b_1\times\ldots\times(b_i\bullet b_l)\times\ldots\times b_{l-1}\\
&= f_A(a)\bullet b_1\times\ldots\times b_l.
\end{aligned}$$

The result is proved for $k\geqslant 2$ by an induction on k. $\square$

3. Examples on shuffle algebras

3.1. Preliminary lemmas. We shall denote by $\pi : T(V) \longrightarrow V$ the canonical projection.

Lemma 3.1. *Let $\varpi : T(V)\otimes T(V) \longrightarrow V$ be a linear map.*

(1) *There exists a unique map $\bullet : T(V)\otimes T(V) \longrightarrow T(V)$ such that*
 (a) $\pi\circ\bullet=\varpi$.
 (b) *For all $u,v\in T(V)$,*

$$\Delta(u\bullet v)=u^{(1)}\otimes u^{(2)}\bullet v+u^{(1)}\bullet v^{(1)}\otimes u^{(2)}\sqcup\!\sqcup\, v^{(2)}. \tag{2}$$

 This product $\bullet$ is given by

$$\forall u,v\in T(V),\qquad u\bullet v=u^{(1)}\varpi(u^{(2)}\otimes v^{(1)})(u^{(3)}\sqcup\!\sqcup\, v^{(2)}). \tag{3}$$

(2) *The following conditions are equivalent:*
 (a) *For all $u,v,w\in T(V)$,*

$$(u\sqcup\!\sqcup\, v)\bullet w=(u\bullet w)\sqcup\!\sqcup\, v+u\sqcup\!\sqcup\,(v\bullet w).$$

 (b) *For all $u,v,w\in T(V)$,*

$$\varpi((u\sqcup\!\sqcup\, v)\otimes w)=\varepsilon(u)\varpi(v\otimes w)+\varepsilon(v)\varpi(u\otimes w). \tag{4}$$

(3) *Let $N\in\mathbb{Z}$. The following conditions are equivalent:*
 (a) *$\bullet$ is homogeneous of degree N, that is to say*

$$\forall k,l\geqslant 0,\qquad V^{\otimes k}\bullet V^{\otimes l}\subseteq V^{\otimes(k+l+N)}.$$

 (b) *For all $k,l\geqslant 0$, such that $k+l+N\neq 1$, $\varpi(V^{\otimes k}\otimes V^{\otimes l})=(0)$.*

 We use the convention $V^{\otimes p}=(0)$ if $p<0$.

Proof. (1) *Existence.* Let $\bullet$ be the product on $T(V)$ defined by

$$\forall u,v\in T(V),\qquad u\bullet v=u^{(1)}\varpi(u^{(2)}\otimes v^{(1)})(u^{(3)}\sqcup\!\sqcup\, v^{(2)}).$$

As ϖ takes its values in V, for all $u,v\in T(V)$,

$$\begin{aligned}\pi(u\bullet v)&=\varepsilon(u^{(1)})\varpi(u^{(2)}\otimes v^{(1)})\varepsilon(u^{(3)}\sqcup\!\sqcup\, v^{(2)})\\&=\varepsilon(u^{(1)})\varpi(u^{(2)}\otimes v^{(1)})\varepsilon(u^{(3)})\varepsilon(v^{(2)})\\&=\varpi(u\otimes v).\end{aligned}$$

We denote by m the concatenation product of $T(V)$. As $(T(V),m,\Delta)$ is an infinitesimal bialgebra (see [9,10]), for all $u,v\in T(V)$,

$$\begin{aligned}\Delta(u\bullet v)&=u^{(1)}\otimes u^{(2)}\varpi(u^{(3)}\otimes v^{(1)})(u^{(4)}\sqcup\!\sqcup\, v^{(2)})+u^{(1)}\varpi(u^{(2)}\otimes v^{(1)})\otimes u^{(3)}\sqcup\!\sqcup\, v^{(2)}\\&\quad+u^{(1)}\otimes\varpi(u^{(2)}\otimes v^{(1)})(u^{(3)}\sqcup\!\sqcup\, v^{(2)})\\&\quad+u^{(1)}\varpi(u^{(2)}\otimes v^{(1)})(u^{(3)}\sqcup\!\sqcup\, v^{(2)})\otimes u^{(4)}\sqcup\!\sqcup\, v^{(3)}\\&\quad-u^{(1)}\varpi(u^{(2)}\otimes v^{(1)})\otimes u^{(3)}\sqcup\!\sqcup\, v^{(2)}-u^{(1)}\otimes\varpi(u^{(2)}\otimes v^{(1)})(u^{(3)}\otimes v^{(2)})\\&=u^{(1)}\otimes u^{(2)}\varpi(u^{(3)}\otimes v^{(1)})(u^{(4)}\sqcup\!\sqcup\, v^{(2)})\\&\quad+u^{(1)}\varpi(u^{(2)}\otimes v^{(1)})(u^{(3)}\sqcup\!\sqcup\, v^{(2)})\otimes u^{(4)}\sqcup\!\sqcup\, v^{(3)}\\&=u^{(1)}\otimes u^{(2)}\bullet v+u^{(1)}\bullet v^{(1)}\otimes u^{(2)}\sqcup\!\sqcup\, v^{(2)}.\end{aligned}$$

Unicity. Let $\diamond$ be another product satisfying the required properties. Let us denote that $u\diamond v=u\bullet v$ for any words u,v of respective lengths k and l. If $k=0$, then we can assume that $u=\emptyset$. We proceed by induction on l. If $l=0$, then we can assume that $v=\emptyset$. By (2), $\emptyset\bullet\emptyset$ and $\emptyset\diamond\emptyset$ are primitive elements of $T(V)$, so belong to V. Hence,

$$\emptyset\bullet\emptyset=\pi(\emptyset\bullet\emptyset)=\varpi(\emptyset\otimes\emptyset)=\pi(\emptyset\diamond\emptyset)=\emptyset\diamond\emptyset.$$

If $l\geqslant 1$, then, by (2),

$$\begin{aligned}\Delta(\emptyset\bullet v)&=\emptyset\otimes\emptyset\bullet v+\emptyset\bullet v\otimes\emptyset+\emptyset\bullet\emptyset\otimes v+\emptyset\bullet v^{(1)}\otimes v^{(2)},\\\tilde\Delta(\emptyset\bullet v)&=\emptyset\bullet\emptyset\otimes v+\emptyset\bullet v^{(1)}\otimes v^{(2)}.\end{aligned}$$

The same computation for $\diamond$ and the induction hypothesis on l, applied to $(\emptyset,v^{(1)})$, imply that $\tilde\Delta(\emptyset\bullet v-\emptyset\diamond v)=0$, so $\emptyset\bullet v-\emptyset\diamond v\in V$. Finally,

$$\emptyset\bullet v-\emptyset\diamond v=\pi(\emptyset\bullet v-\emptyset\diamond v)=\varpi(\emptyset\otimes v-\emptyset\otimes v)=0.$$

If $k\geqslant 1$, we proceed by induction on l. If $l=0$, we can assume that $v=\emptyset$; (2) implies that $\tilde\Delta(u\bullet\emptyset-u\diamond\emptyset)=0$, so $u\bullet\emptyset-u\diamond\emptyset=0$ and, applying π, finally

$u \bullet \emptyset = u \diamond \emptyset$. If $l \geqslant 1$, by (2), the induction hypothesis on k applied to $(u^{(1)}, v)$ and the induction hypothesis on l applied to $(u, \emptyset)$ and $(u, v^{(1)})$ gives

$$\begin{aligned}\tilde{\Delta}(u \bullet v) &= u^{(1)} \otimes u^{(2)} \bullet v + u \bullet \emptyset \otimes v + u \bullet v^{(1)} \otimes v^{(2)} \\ &= u^{(1)} \otimes u^{(2)} \diamond v + u \diamond \emptyset \otimes v + u \diamond v^{(1)} \otimes v^{(2)} = \tilde{\Delta}(u \diamond v).\end{aligned}$$

As before, $u \bullet v = u \diamond v$.

(2) $\Longrightarrow$ As ϖ takes its values in V, we have

$$\begin{aligned}\varpi(u \sqcup\!\sqcup v) \otimes w) &= \varpi((u \bullet w) \sqcup\!\sqcup v + u \sqcup\!\sqcup (v \bullet w)) \\ &= \varepsilon(v)\varpi(u \otimes w) + \varepsilon(u)\varpi(v \otimes w).\end{aligned}$$

$\Longleftarrow$ For all $u, v, w \in T(V)$,

$$\begin{aligned}(u \sqcup\!\sqcup v) \bullet w &= (u^{(1)} \sqcup\!\sqcup v^{(1)})\varpi((u^{(2)} \sqcup\!\sqcup v^{(2)}) \otimes w^{(1)})(u^{(3)} \sqcup\!\sqcup v^{(3)} \sqcup\!\sqcup w^{(2)}) \\ &= \varepsilon(u^{(2)})(u^{(1)} \sqcup\!\sqcup v^{(1)})\varpi(v^{(2)} \otimes w^{(1)})(u^{(3)} \sqcup\!\sqcup v^{(3)} \sqcup\!\sqcup w^{(2)}) \\ &+ \varepsilon(v^{(2)})(u^{(1)} \sqcup\!\sqcup v^{(1)})\varpi(u^{(2)} \otimes w^{(1)})(u^{(3)} \sqcup\!\sqcup v^{(3)} \sqcup\!\sqcup w^{(2)}) \\ &= (u^{(1)} \sqcup\!\sqcup v^{(1)})\varpi(v^{(2)} \otimes w^{(1)})(u^{(2)} \sqcup\!\sqcup v^{(3)} \sqcup\!\sqcup w^{(2)}) \\ &+ (u^{(1)} \sqcup\!\sqcup v^{(1)})\varpi(u^{(2)} \otimes w^{(1)})(u^{(3)} \sqcup\!\sqcup v^{(2)} \sqcup\!\sqcup w^{(2)}) \\ &= u \sqcup\!\sqcup \left(v^{(1)}\varpi(v^{(2)} \otimes w^{(1)})(v^{(3)} \sqcup\!\sqcup w^{(2)})\right) \\ &+ v \sqcup\!\sqcup \left(u^{(1)}\varpi(u^{(2)} \otimes w^{(1)})(u^{(3)} \sqcup\!\sqcup w^{(2)})\right) \\ &= u \sqcup\!\sqcup (v \bullet w) + (u \bullet w) \sqcup\!\sqcup v.\end{aligned}$$

So the compatibility between $\sqcup\!\sqcup$ and $\bullet$ is satisfied.

(3) $(a) \Longrightarrow (b)$ immediately implied by $\varpi = \pi \circ \bullet$. $(b) \Longrightarrow (a)$ comes from (3). □

Remark 3.2. If (4) is satisfied, for $u = v = \emptyset$, we obtain

$$\forall w \in T(V), \qquad \varpi(\emptyset \otimes w) = 0.$$

Lemma 3.3. *Let $\varpi : T(V) \otimes T(V) \longrightarrow V$, satisfying (4), and let $\bullet$ be the product associated to ϖ in Lemma 3.1. Then $(T(V), \sqcup\!\sqcup, \bullet, \Delta)$ is a Com-PreLie bialgebra if and only if*

$$\forall u, v, w \in T(V),$$
$$\varpi(u \bullet v \otimes w) - \varpi(u \otimes v \bullet w) = \varpi(u \bullet w \otimes v) - \varpi(u \otimes w \bullet v). \tag{5}$$

Proof. $\Longrightarrow$ This is immediately obtained by applying π to the preLie identity, as $\varpi = \pi \circ \bullet$.

$\Longleftarrow$ By Lemma 3.1, it remains to prove that $\bullet$ is preLie. For any $u, v, w \in T(V)$, we put $PL(u, v, w) = (u \bullet v) \bullet w - u \bullet (v \bullet w) - (u \bullet w) \bullet v + u \bullet (w \bullet v)$. By hypothesis,

$\pi \circ PL(u,v,w) = 0$ for any $u,v,w \in T(V)$. Let us prove that $PL(u,v,w) = 0$ for any $u,v,w \in T(V)$. A direct computation using (2) shows that

$$\Delta(PL(u,v,w)) = u^{(1)} \otimes PL(u^{(2)},v,w) \otimes u^{(1)} \\ + PL(u^{(1)},v^{(1)},w^{(1)}) \otimes u^{(2)} \sqcup v^{(2)} \sqcup w^{(2)}. \quad (6)$$

Let $v \in T(V)$. Then

$$\emptyset \bullet v = (\emptyset \sqcup \emptyset) \bullet v = (\emptyset \bullet v) \sqcup \emptyset + \emptyset \sqcup (\emptyset \bullet v) = 2\emptyset \bullet v,$$

so $\emptyset \bullet v = 0$ for any $v \in T(V)$. Hence, for any $v,w \in T(V)$, $PL(\emptyset,v,w) = 0$: by trilinearity of PL, we can assume that $\varepsilon(u) = 0$. In this case, (6) becomes

$$\Delta(PL(u,v,w)) = \emptyset \otimes PL(u,v,w) + PL(u,v^{(1)},w^{(1)}) \otimes v^{(2)} \sqcup w^{(2)} \\ + PL(u^{(1)},v^{(1)},w^{(1)}) \otimes u^{(2)} \sqcup v^{(2)} \sqcup w^{(2)}.$$

We assume that u,v,w are words of respective lengths k, l and n, with $k \geqslant 1$. Let us first prove that $PL(u,v,w) = 0$ if $l = 0$, or equivalently if $v = \emptyset$, by induction on n. If $n = 0$, then we can take $w = \emptyset$ and, obviously, $PL(u,\emptyset,\emptyset) = 0$. If $n \geqslant 1$, (6) becomes

$$\Delta(PL(u,\emptyset,w)) = \emptyset \otimes PL(u,v,w) + PL(u,\emptyset,w^{(1)}) \otimes w^{(2)} \\ = \emptyset \otimes PL(u,v,w) + PL(u,\emptyset,w) \otimes \emptyset + PL(u,\emptyset,w^{(1)}) \otimes w^{(2)}.$$

By the induction hypothesis on n, $PL(u,\emptyset,w^{(1)}) = 0$, so $PL(u,\emptyset,w)$ is primitive, so belongs to V. As $\pi \circ PL = 0$, $PL(u,\emptyset,w) = 0$.

Therefore, we can now assume that $l \geqslant 1$. By symmetry in v and w, we can also assume that $n \geqslant 1$. Let us now prove that $PL(u,v,w) = 0$ by induction on k. If $k = 0$, there is nothing more to prove. If $k \geqslant 1$, we proceed by induction on $l+n$. If $l+n \leqslant 1$, there is nothing more to prove. Otherwise, using both induction hypotheses, (6) becomes

$$\Delta(PL(u,v,w)) = PL(u,v,w) \otimes \emptyset + \emptyset \otimes PL(u,v,w).$$

So $PL(u,v,w) \in V$. As $\pi \circ PL = 0$, $PL(u,v,w) = 0$. □

Consequently:

Proposition 3.4. *Let $\varpi : T(V) \otimes T(V) \longrightarrow V$ be a linear map such that (4) and (5) are satisfied. The product $\bullet$ defined by (2) makes $(T(V), \sqcup, \bullet, \Delta)$ a Com-PreLie bialgebra. We obtain in this way all the preLie products $\bullet$ such that $(T(V), \sqcup, \bullet, \Delta)$ a Com-PreLie bialgebra. Moreover, for any $N \in \mathbb{Z}$, $\bullet$ is homogeneous of degree N if and only if*

$$\forall k,l \in \mathbb{N}, \qquad k + l + N \neq 1 \Longrightarrow \varpi(V^{\otimes k} \otimes V^{\otimes l}) = (0). \quad (7)$$

Remark 3.5. Let $\varpi : T(V) \otimes T(V) \longrightarrow V$, satisfying (7) for a given $N \in \mathbb{Z}$. Then

(1) (4) is satisfied if and only if for all $k, l, n \in \mathbb{N}$ such that $k + l + n = 1 - N$,

$$\forall u \in V^{\otimes k},\ \forall v \in V^{\otimes l},\ \forall w \in V^{\otimes n}, \quad \varpi((u \sqcup\!\sqcup v) \otimes w) = \varepsilon(u)\varpi(v \otimes w) + \varepsilon(v)\varpi(u \otimes w).$$

(2) (5) is satisfied if and only if for all $k, l, n \in \mathbb{N}$ such that $k + l + n = 1 - 2N$,

$$\forall \in V^{\otimes k},\ \forall v \in V^{\otimes l},\ \forall w \in V^{\otimes n}, \qquad \varpi(u \bullet v \otimes w) - \varpi(u \otimes v \bullet w) = \varpi(u \bullet w \otimes v) - \varpi(u \otimes w \bullet v).$$

Note that (4) is always satisfied if $u = \emptyset$ or $v = \emptyset$, that is to say if $k = 0$ or $l = 0$.

In the next paragraphs, we shall look at $N \geqslant 0$ and $N = -1$.

3.2. PreLie products of positive degree.

Proposition 3.6. *Let f be a linear endomorphism of V. We define a product $\bullet$ on $T(V)$ by*

$$\forall x_1, \ldots, x_m, y_1, \ldots, y_n \in V,$$
$$x_1 \ldots x_m \bullet y_1 \ldots y_n = \sum_{i=0}^{n} x_1 \ldots x_{i-1} f(x_i)(x_{i+1} \ldots x_m \sqcup\!\sqcup y_1 \ldots y_n). \tag{8}$$

Then $(T(V), \sqcup\!\sqcup, \bullet, \Delta)$ is a Com-PreLie bialgebra denoted by $T(V, f)$. Conversely, if $\bullet$ is a product on $T(V)$, homogeneous of degree $N \geqslant 0$, there exists a unique $f : V \longrightarrow V$ such that $(T(V), \sqcup\!\sqcup, \bullet, \Delta) = T(V, f)$.

Proof. We look for all possible ϖ, homogeneous of a certain degree $N \geqslant 0$, such that (4) and (5) are satisfied. Let us consider such a ϖ. For any $k, l \in \mathbb{N}$, we denote by $\varpi_{k,l}$ the restriction of ϖ to $V^{\otimes k} \otimes V^{\otimes l}$. By (7), $\varpi_{k,l} = 0$ if $k + l \neq 1$. As (4) implies that $\varpi_{0,1} = 0$, the only possibly nonzero $\varpi_{k,l}$ is $\varpi_{1,0} : V \longrightarrow V$, which we denote by f. Then (2) gives (8).

Let us consider any linear endomorphism f of V and consider ϖ such that the only nonzero component of ϖ is $\varpi_{1,0} = f$. Let us prove (4) for $u \in V^{\otimes k}$, $v \in V^{\otimes l}$, $w \in V^{\otimes n}$, with $k + l + n = 1 - N$. For all the possibilities for (k, l, n), $0 \in \{k, l, n\}$, and the result is then obvious.

Let us prove (4) for $u \in V^{\otimes k}$, $v \in V^{\otimes l}$, $w \in V^{\otimes n}$, with $k + l + n = 1 - 2N$. We obtain two possibilities:

- $(k, l, n) = (0, 1, 0)$ or $(0, 0, 1)$. We can assume that $u = \emptyset$. As $\emptyset \bullet x = 0$ for any $x \in T(V)$, the result is obvious.
- $(k, l, n) = (1, 0, 0)$. We can assume that $v = w = \emptyset$, and the result is then obvious.

This concludes the proof. □

Remark 3.7. (1) If $N \geqslant 1$, necessarily $f = 0$, so $\bullet = 0$.
(2) With the notation of Proposition 2.6, $f_{T(V,f)} = f$.

We obtain in this way the family of Com-PreLie bialgebras of [5], coming from a problem of composition of Fliess operators in Control Theory. So we have from [5]:

Corollary 3.8. *Let $k, l \geqslant 0$. We denote by $\mathcal{S}h(k,l)$ the set of (k,l)-shuffles, that is to say permutations $\sigma \in \mathfrak{S}_{k+l}$ such that*

$$\sigma(1) < \ldots < \sigma(k), \qquad \sigma(k+1) < \ldots < \sigma(k+l).$$

If $\sigma \in \mathcal{S}h(k,l)$, we put

$$m_k(\sigma) = \max\{i \in [k] \mid \sigma(1) = 1, \ldots, \sigma(i) = i\},$$

with the convention $m_k(\sigma) = 0$ if $\sigma(1) \neq 1$. Then, in $T(V,f)$, if $v_1, \ldots, v_{k+l} \in V$,

$$\begin{aligned} &v_1 \ldots v_k \bullet v_{k+1} \ldots v_{k+l} \\ &= \sum_{\sigma \in \mathcal{S}h(k,l)} \sum_{i=1}^{m_k(\sigma)} (\mathrm{Id}^{\otimes(i-1)} \otimes f \otimes \mathrm{Id}^{\otimes(k+l-i)})(v_{\sigma^{-1}(1)} \ldots v_{\sigma^{-1}(k+l)}). \end{aligned} \tag{9}$$

3.3. PreLie products of degree -1.

Proposition 3.9. *Let $*$ and $\{-,-\}$ be two bilinear products on V such that*

$$\forall x, y, z \in V, \quad (x * y) * z - x * (y * z) = (x * z) * y - x * (z * y), \tag{10}$$
$$\{x, y\} = -\{y, x\},$$
$$x * \{y, z\} = \{x * y, z\},$$
$$\{x, y\} * z = \{x * z, y\} + \{x, y * z\} + \{\{x, y\}, z\}.$$

We define a product $\bullet$ on $T(V)$ in the following way: for all $x_1, \ldots, x_m, y_1, \ldots, y_n \in V$,

$$\begin{aligned} x_1 \ldots x_m \bullet y_1 \ldots y_n &= \sum_{i=1}^{n} x_1 \ldots x_{i-1}(x_i * y_1)(x_{i+1} \ldots x_m \sqcup\!\sqcup y_2 \ldots y_n) \\ &+ \sum_{i=1}^{k-1} x_1 \ldots x_{i-1}\{x_i, x_{i+1}\}(x_{i+2} \ldots x_m \sqcup\!\sqcup y_1 \ldots y_n). \end{aligned} \tag{11}$$

Then $(T(V), \sqcup\!\sqcup, \bullet, \Delta)$ is a Com-PreLie bialgebra, and we obtain in this way all the possible preLie products $\bullet$, homogeneous of degree -1, such that $(T(V), \sqcup\!\sqcup, \bullet, \Delta)$ is a Com-PreLie bialgebra.

Proof. Let us consider a linear map $\varpi : T(V) \otimes T(V) \longrightarrow V$, satisfying (7) for $N = -1$. Denoting by $\varpi_{k,l} = \varpi_{|V^{\otimes k} \otimes V^{\otimes l}}$ for any k, l, the only possibly nonzero $\varpi_{k,l}$ are for $(k, l) = (2, 0)$, $(1, 1)$ and $(0, 2)$. For all $x, y \in V$, we put

$$x * y = \varpi_{1,1}(x \otimes y), \qquad \{x, y\} = \varpi_{2,0}(xy \otimes \emptyset).$$

(4) is equivalent to

$$\forall w \in V^{\otimes 2}, \qquad \varpi_{0,2}(\emptyset \otimes w) = 0,$$
$$\forall x, y \in V, \qquad \varpi_{2,0}((xy + yx) \otimes \emptyset) = 0.$$

Hence, we now assume that $\varpi_{0,2} = 0$, and we obtain that (4) is equivalent to (10)-2. The nullity of $\varpi_{0,2}$ and (2) give (11).

Let us now consider (5), with $u \in V^{\otimes k}$, $v \in V^{\otimes l}$, $w \in V^{\otimes n}$, $k+l+n = 1-2N = 3$. By symmetry between v and w, and by nullity of $\varpi_{0,l}$ for all l, we have to consider two cases:

- $k = l = n = 1$. We put $u = x$, $v = y$, $w = z$, with $x, y, z \in V$. Then (5) is equivalent to

  $$(x * y) * z - x * (y * z) = (x * z) * y - x * (z * y),$$

  that is to say to (10)-1.
- $k = 1$, $l = 2$, $z = 0$. We put $u = x$, $v = yz$, $w = \emptyset$, with $x, y, z \in V$. Then (5) is equivalent to

  $$\{x * y, z\} - x * \{y, z\} = 0,$$

  that is to say to (10)-3.
- $k = 2$, $l = 1$, $z = 0$. We put $u = xy$, $v = z$, $w = \emptyset$, with $x, y, z \in V$. Then (5) is equivalent to

  $$\{x * z, y\} + \{x, y * z\} + \{\{x, y\}, z\} = \{x, y\} * z,$$

  that is to say to (10)-4.

We conclude with Proposition 3.4. □

Remark 3.10. (1) In particular, $*$ is a preLie product on V, and for all $x, y \in V$, $x \bullet y = x * y$.
(2) If $x_1, \ldots, x_m \in V$,

$$x_1 \ldots x_m \bullet \emptyset = \sum_{i=1}^{m-1} x_1 \ldots x_{i-1}\{x_i, x_{i+1}\}x_{i+2} \ldots x_m.$$

Example 3.11. (1) If $*$ is a preLie product on V, we can take $\{-, -\} = 0$, and (10) is satisfied. Using the classification of preLie algebras of dimension

2 over $\mathbb{C}$ of [1], it is not difficult to show that if the dimension of V is 1 or 2, then necessarily $\{-,-\}$ is zero.

(2) If $* = 0$, then (10) becomes

$$\forall x, y \in V, \qquad \{x, y\} = -\{y, x\},$$
$$\forall x, y, z \in V, \qquad \{\{x, y\}, z\} = 0,$$

that is to say $(V, \{-,-\})$ is a nilpotent Lie algebra, which nilpotency order is (2).

(3) Here is a family of examples where both $*$ and $\{-,-\}$ are nonzero. Let V be 3-dimensional space, with basis (x, y, z), and let a, b, c be scalars. We consider the products given by the following arrays:

$\bullet$	x	y	z
x	x	y	z
y	0	0	0
z	0	0	0

$\{-,-\}$	x	y	z
x	0	$ay + bz$	$cy + (1-a)z$
y	$-ay - bz$	0	0
z	$(a-1)z - cy$	0	0

Then $(V, \bullet, \{-,-\})$ satisfies (10) if and only if $a^2 - a + bc = 0$, or equivalently,

$$(2a-1)^2 + (b+c)^2 - (b-c)^2 = 1.$$

This equation defines a hyperboloid of one sheet.

4. Free Com-PreLie algebras and quotients

4.1. Description of free Com-PreLie algebras. We described in [5] free Com-PreLie algebras in terms of decorated rooted partitioned trees. We now work with free unitary Com-PreLie algebras.

Definition 4.1. (1) A *partitioned forest* is a pair (F, I) such that

(a) F is a rooted forest (the edges of F being oriented from the roots to the leaves). The set of its vertices is denoted by $V(F)$.

(b) I is a partition of the vertices of F with the following condition: if x, y are two vertices of F which are in the same part of I, then either they are both roots, or they have the same direct ascendant.

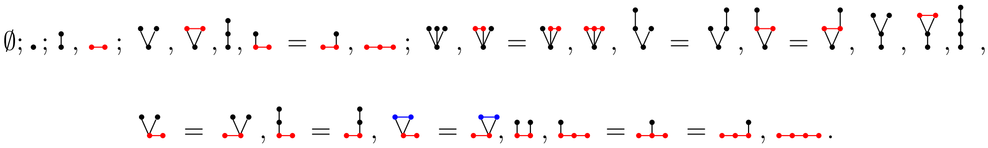
The parts of the partition are called *blocks.*

(2) We shall say that a partitioned forest F is a *partitioned tree* if all the roots are in the same block. Note that in this case, one of the blocks of F is the set of roots of F. By convention, the empty forest $\emptyset$ is considered as a partitioned tree.

(3) Let $\mathcal{D}$ be a set. A *partitioned tree decorated by $\mathcal{D}$* is a triple (T, I, d), where (T, I) is a partitioned tree and d is a map from the set of vertices of T into $\mathcal{D}$. For any vertex x of T, $d(x)$ is called the *decoration* of x.

(4) The set of isoclasses of partitioned trees, included the empty tree, will be denoted by $\mathcal{PT}$. For any set $\mathcal{D}$, the set of isoclasses of partitioned trees decorated by $\mathcal{D}$ will be denoted by $\mathcal{PT}(\mathcal{D})$; the set of isoclasses of partitioned trees decorated by $\mathbb{N}\times\mathcal{D}$ will be denoted by $\mathcal{UPT}(\mathcal{D}) = \mathcal{PT}(\mathbb{N}\times\mathcal{D})$.

Example 4.2. We represent partitioned trees by the underlying rooted forest, the blocks of cardinality $\geqslant 2$ being represented by horizontal edges of different colors. Here are the partitioned trees with $\leqslant 4$ vertices:



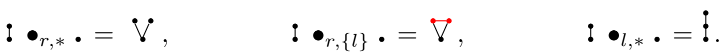
Let us fix a set $\mathcal{D}$.

Definition 4.3. Let $T = (T, I, d)$ and $T' = (T', J, d') \in \mathcal{UPT}(\mathcal{D})$.

(1) The partitioned tree $T \cdot T'$ is defined as follows.
   (a) As a rooted forest, $T \cdot T'$ is TT'.
   (b) We put $I = \{I_1, \ldots, I_k\}$ and $J = \{J_1, \ldots, J_l\}$ and we assume that the block of roots of T is I_1 and the block of roots of T' is J_1. The partition of the vertices of $T \cdot T'$ is $\{I_1 \sqcup J_1, I_2, \ldots, I_k, J_2, \ldots, J_l\}$.

   $(\mathcal{UPT}(\mathcal{D}), \cdot)$ is a commutative monoid, of unit $\emptyset$.
(2) Let s be a vertex of T'.
   (a) We denote by $\mathcal{B}l(s)$ the set of blocks of T, children of s.
   (b) Let $b \in \mathcal{B}l(s) \sqcup \{*\}$. We denote by $T \bullet_{s,b} T'$ the partitioned tree obtained in this way:
      - Graft T' on s, that is to say add edges from s to any root of T'.
      - If $b \in \mathcal{B}l(s)$, join the block b and the block of roots of T'.
   (c) Let $k \in \mathbb{Z}$. The decoration of s is denoted by (i, d). The element $T[k]_s \in \mathcal{UPT}(\mathcal{D}) \sqcup \{0\}$ is defined by the following:
      - If $i + k \geqslant 0$, replace the decoration of s by $(i + k, d)$.
      - If $i + k < 0$, $T[k]_s = 0$.

Example 4.4. Let $T =$, $T' =$. We denote by r the root of T and by l the leaf of T. Then



Lemma 4.5. *Let $A_+ = (A_+, \cdot, \bullet)$ be a Com-PreLie algebra, and $f : A_+ \longrightarrow A_+$ be a linear map such that*

$$\forall x, y \in A_+, \qquad f(x \cdot y) = f(x) \cdot y + x \cdot f(y),$$
$$f(x \bullet y) = f(x) \bullet y + x \bullet f(y).$$

We put $A = A_+ \oplus \mathrm{Vect}(\emptyset)$. Then A is given a unitary Com-PreLie algebra structure, extending the one of A_+, by

$$\emptyset \cdot \emptyset = \emptyset, \qquad \emptyset \bullet \emptyset = 0,$$
$$\forall x \in A_+, \qquad x \cdot \emptyset = x, \qquad \emptyset \cdot x = x,$$
$$x \bullet \emptyset = f(x), \qquad \emptyset \bullet x = 0.$$

Proof. Obviously, $(A, \cdot)$ is a commutative, unitary associative algebra. Let us prove the PreLie identity for $x, y, z \in A_+ \sqcup \{\emptyset\}$.

- If $x = \emptyset$, then $x \bullet (y \bullet z) = (x \bullet y) \bullet z = x \bullet (z \bullet y) = (x \bullet z) \bullet y = 0$. We now assume that $x \in A_+$.
- If $y = z = \emptyset$, then obviously the PreLie identity is satisfied.
- If $y = \emptyset$ and $z \in A_+$, then

$$x \bullet (y \bullet z) = 0, \qquad (x \bullet y) \bullet z = f(x) \bullet z,$$
$$x \bullet (z \bullet y) = x \bullet f(z), \qquad (x \bullet z) \bullet y = f(x \bullet z).$$

  As f is a derivation for $\bullet$, the PreLie identity is satisfied. By symmetry, it is also true if $y \in A_+$ and $z = \emptyset$.

Let us now prove the Leibniz identity for $x, y, z \in A_+ \sqcup \{\emptyset\}$. It is obviously satisfied if $x = \emptyset$ or $y = \emptyset$; we assume that $x, y \in A_+$. If $z = \emptyset$, then

$$(x \cdot y) \bullet z = f(x \cdot y), \qquad (x \bullet z) \cdot y = f(x) \cdot y, \qquad x \cdot (y \bullet z) = x \cdot f(y).$$

As f is a derivation for $\cdot$, the Leibniz identity is satisfied. □

Proposition 4.6. *Let $UCP(\mathcal{D})$ be the vector space generated by $\mathcal{UPT}(\mathcal{D})$. We extend $\cdot$ by bilinearity and the PreLie product $\bullet$ is defined by*

$$\forall T, T' \in \mathcal{UPT}(\mathcal{D}), \qquad T \bullet T' = \begin{cases} \displaystyle\sum_{s \in V(t)} T \bullet_{s,*} T' \text{ if } t \neq \emptyset, \\ \displaystyle\sum_{s \in V(t)} T[+1]_s \text{ if } t = \emptyset. \end{cases}$$

Then $UCP(\mathcal{D})$ is the free unitary Com-PreLie algebra generated by the elements $\bullet_{(0,d)}$, $d \in D$.

Proof. We denote by $UCP_+(\mathcal{D})$ the subspace of $UCP(\mathcal{D})$ generated by nonempty trees. By [5, Proposition 18], this is the free Com-PreLie algebra generated by the elements $\bullet_{(k,\,d)}$, $k \in \mathbb{N}$, $d \in \mathcal{D}$. We define a map $f : UCP_+(\mathcal{D}) \longrightarrow UCP_+(\mathcal{D})$ by

$$\forall T \in \mathcal{UPT}(\mathcal{D}) \setminus \{\emptyset\}, \qquad f(T) = \sum_{s \in V(t)} T[+1]_s.$$

This is a derivation for both $\cdot$ and $\bullet$; by Lemma 4.5, $UCP(\mathcal{D})$ is a unitary Com-PreLie algebra.

Observe that for all $d \in \mathcal{D}$, $k \in \mathbb{N}$,

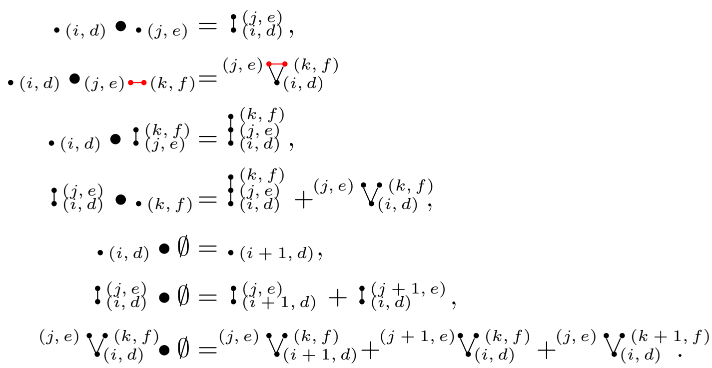
$$\bullet_{(0,\,d)} \bullet \emptyset^{\times k} = \bullet_{(k,\,d)}.$$

Let A be a unitary Com-PreLie algebra and, for all $d \in \mathcal{D}$, let $a_d \in A$. By [5, Proposition 18], we define a unique Com-PreLie algebra morphism by

$$\theta : \begin{cases} UCP_+(\mathcal{D}) & \longrightarrow & A \\ \bullet_{(k,\,d)} & \longrightarrow & a_d \times 1_A^{\times k}. \end{cases}$$

We extend it to $UCP(\mathcal{D})$ by sending $\emptyset$ to 1_A, and we obtain in this way a unitary Com-PreLie algebra from $UCP(\mathcal{D})$ to A, sending $\bullet_{(0,\,d)}$ to a_d for any $d \in \mathcal{D}$. This morphism is clearly unique. $\square$

Example 4.7. Let $i, j, k \in \mathbb{N}$ and $d, e, f \in \mathcal{D}$.



4.2. Quotients of $UCP(\mathcal{D})$.

Proposition 4.8. *We put $V_0 = \mathrm{Vect}(\bullet_{(0,\,d)}, d \in \mathcal{D})$, identified with $\mathrm{Vect}(\bullet_d, d \in \mathcal{D})$. Let $f : V_0 \longrightarrow V_0$ be any linear map. We consider the Com-PreLie ideal I_f of $UCP(\mathcal{D})$ generated by the elements $\bullet_{(1,\,d)} - f(\bullet_{(0,\,d)})$, $d \in \mathcal{D}$.*

1. *We denote by $\mathcal{UPT}'(\mathcal{D})$ the set of trees $T \in \mathcal{UPT}(\mathcal{D})$ such that for any vertex s of T, the decoration of s is of the form $(0, d)$, with $d \in \mathcal{D}$. It is trivially identified with $\mathcal{PT}(\mathcal{D})$. Then the family $(T + I_f)_{T \in \mathcal{UPT}'(\mathcal{D})}$ is a basis of $UCP(\mathcal{D})/I_f$.*
2. *In $UCP(\mathcal{D})/I_f$, for any $d \in \mathcal{D}$, $\bullet_{(0,\,d)} \bullet \emptyset = f(\bullet_{(0,\,d)})$.*

Proof. *First step.* We fix $d \in \mathcal{D}$. Let us first prove that for all $k \geqslant 0$,

$$\bullet_{(k,\, d)} + I_f = f^k(\bullet_{(0,\, d)}) + I_f.$$

It is obvious if $k = 0, 1$. Let us assume the result at rank $k-1$. We put $f(\bullet_{(0,\, d)}) = \sum a_e \bullet_{(0,\, e)}$. Then

$$\begin{aligned}\bullet_{(k,\, d)} + I_f &= \bullet_{(1,\, d)} \bullet \emptyset^{\times(k-1)} + I_f\\ &= \sum a_e \bullet_{(0,\, e)} \bullet \emptyset^{\times(k-1)} + I_f\\ &= \sum a_e f^{k-1}(\bullet_{(0,\, e)}) + I_f\\ &= f^k(\bullet_{(0,\, d)}) + I_f,\end{aligned}$$

so the result holds for all k.

Second step. Let $T \in \mathcal{UPT}(\mathcal{D})$; let us prove that there exists $x \in \mathrm{Vect}(\mathcal{UPT}'(\mathcal{D}))$, such that $T + I_f = x + I_f$. We proceed by induction on $|T|$. If $|T| = 0$, then $t = \emptyset$ and we can take $x = T$. If $|T| = 1$, then $T = \bullet_{(k,\, d)}$ and we can take, by the first step, $x = f^k(\bullet_{(0,\, d)})$. Let us assume the result at all ranks $< |T|$. If T has several roots, we can write $T = T_1 \cdot T_2$, with $|T_1|, |T_2| < |T|$. Hence, there exists $x_i \in \mathrm{Vect}(\mathcal{UPT}'(\mathcal{D}))$, such that $T_i + I_f = x_i + I_f$ for all $i \in [2]$, and we take $x = x_1 \cdot x_2$. Otherwise, we can write

$$T = \bullet_{(k,\, d)} \bullet T_1 \times \ldots \times T_k,$$

where $T_1, \ldots, T_k \in \mathcal{UPT}(\mathcal{D})$. By the induction hypothesis, there exists $x_i \in \mathrm{Vect}(\mathcal{UPT}'(\mathcal{D}))$ such that $T_i + I_f = x_i + I_f$ for all $i \in [k]$. We then take $x = f^k(\bullet_{(0,\, d)}) \bullet x_1 \times \ldots \times x_k$.

Third step. We give $CP_+(\mathcal{D}) = \mathrm{Vect}(\mathcal{PT}(\mathcal{D}) \setminus \{\emptyset\})$ a Com-PreLie structure by

$$\forall T, T' \in \mathcal{PT}(\mathcal{D}) \setminus \{\emptyset\}, \qquad T \bullet T' = \sum_{s \in V(t)} T \bullet_{s,*} T'.$$

We consider the map

$$F : \begin{cases} CP_+(\mathcal{D}) & \longrightarrow & CP_+(\mathcal{D})\\ T & \longrightarrow & \displaystyle\sum_{s \in V(T)} f_s(T),\end{cases}$$

where, $f_s(T)$ is the linear span of decorated partitioned trees obtained by replacing the decoration d_s of s by $f(d_s)$, the trees being considered as linear in any of their decorations. This is a derivation for both $\cdot$ and $\bullet$, so by Lemma 4.5, $CP(\mathcal{D})$ inherits a unitary Com-PreLie structure such that for any $d \in \mathcal{D}$,

$$\bullet_d \bullet \emptyset = f(\bullet_d).$$

By the universal property of $UCP(\mathcal{D})$, there exists a unique unitary Com-PreLie algebra morphism $\phi : UCP(\mathcal{D}) \longrightarrow CP(\mathcal{D})$, such that $\phi(\bullet_{(0,\, d)}) = \bullet_d$ for any $d \in \mathcal{D}$.

Then $\phi(\bullet_{(1,d)}) = f(\bullet_d)) = \phi(f(\bullet_{(0,d)})$ for any $d \in D$, so ϕ induces a morphism $\overline{\phi} : UCP(\mathcal{D})/I_f \longrightarrow CP(\mathcal{D})$. It is not difficult to prove that for any $T \in \mathcal{UPT}'(\mathcal{D})$, $\phi(T) = T$. As the family $\mathcal{PT}(\mathcal{D})$ is a basis of $CP(\mathcal{D})$, the family $(T + I_f)_{T \in \mathcal{UPT}'(\mathcal{D})}$ is linearly independent in $UCP(\mathcal{D})/I_f$. By the second step, it is a basis. □

Example 4.9. We choose $f = \mathrm{Id}_{V_0}$. The product in $UCP(\mathcal{D})/I_{\mathrm{Id}_{V_0}}$ is the one of Definition 4.3. If $T, T' \in \mathcal{PT}(\mathcal{D})$ and $T' \neq \emptyset$, then $T \bullet T'$ is the sum of all graftings of T' over T. Moreover,

$$T \bullet \emptyset = |T|T.$$

Hence, we now consider $CP(\mathcal{D})$, augmented by an unit $\emptyset$, as a unitary Com-PreLie algebra.

Proposition 4.10. *Let J be the Com-PreLie ideal of $CP(\mathcal{D})$ generated by the elements*

$$\bullet_d \bullet (F_1 \times F_2) - \bullet_d \bullet (F_1 \cdot F_2),$$

with $d \in \mathcal{D}$ and $F_1, F_2 \in \mathcal{PT}(\mathcal{D})$.

(1) *Let T and T' be two elements of $\mathcal{PT}(\mathcal{D})$ which are equal as decorated rooted forests. Then $T + J = T' + J$. Consequently, if F is a decorated rooted forest, the element $T' + I$ does not depend of the choice of $T' \in \mathcal{UPT}(\mathcal{D})$ such that $T' = F$ as a decorated rooted forest. This element is identified with F.*

(2) *The set of decorated rooted forests is a basis of $UCP(\mathcal{D})/J$.*

$CP(\mathcal{D})/J$ is then, as an algebra, identified with the Connes-Kreimer algebra $H_{CK}^{\mathcal{D}}$ of decorated rooted trees [3]*, which is in this way a unitary Com-PreLie algebra.*

Proof. (1) *First step.* Let us show that for any $x_1, \ldots, x_n \in UCP(\mathcal{D})$, $\bullet_d \bullet (x_1 \times \ldots \times x_n) + J = \bullet_d \bullet (x_1 \cdot \ldots \cdot x_n) + J$ by induction on n. It is obvious if $n = 1$, and it comes from the definition of J if $n = 2$. Let us assume the result at rank $n - 1$.

$$\begin{aligned}
&\bullet_d \bullet (x_1 \times \ldots \times x_n) + J\\
&= (\bullet_d \bullet (x_1 \times \ldots \times x_{n-1})) \bullet x_n - \sum_{i=1}^{n-1} \bullet_d \bullet (x_1 \times \ldots \times (x_i \bullet x_n) \times \ldots \times x_{n-1}) + J\\
&= (\bullet_d \bullet (x_1 \cdot \ldots \cdot x_{n-1})) \bullet x_n - \sum_{i=1}^{n-1} \bullet_d \bullet (x_1 \cdot \ldots \cdot (x_i \bullet x_n) \cdot \ldots \cdot x_{n-1}) + J\\
&= (\bullet_d \bullet (x_1 \cdot \ldots \cdot x_{n-1})) \bullet x_n - \bullet_d \bullet ((x_1 \cdot \ldots \cdot x_{n-1}) \bullet x_n) + J\\
&= \bullet_d \bullet ((x_1 \cdot \ldots \cdot x_{n-1}) \times x_n) + J\\
&= \bullet_d \bullet (x_1 \cdot \ldots x_{n-1} \cdot x_n) + J.
\end{aligned}$$

So the result holds for all n.

Second step. Let $F, G \in \mathcal{PT}(\mathcal{D})$, such that the underlying rooted decorated forests are equal. Let us prove that $F + J = G + J$ by induction on $n = |F| = |G|$. If $n = 0$, then $F = G = 1$ and it is obvious. If $n = 1$, then $F = G = \bullet_d$ and it is obvious. Let us assume the result at all ranks $< n$.

First case. If F has $k \geqslant 2$ roots, we can write $F = T_1 \cdot \ldots \cdot T_k$ and $G = T'_1 \cdot \ldots \cdot T'_k$, such that, for all $i \in [k]$, T_i and T'_i have the same underlying decorated rooted forest; By the induction hypothesis, $T_i + J = T'_i + J$ for all i, so $F + J = G + J$.

Second case. Let us assume that F has only one root. We can write $F = \bullet_d \bullet (F_1 \times \ldots \times F_k)$ and $G = \bullet_d \bullet (G_1 \times \ldots \times G_l)$. Then $F_1 \cdot \ldots \cdot F_k$ and $G_1 \cdot \ldots \cdot G_l$ have the same underlying decorated forest; by the induction hypothesis, $F_1 \cdot \ldots \cdot F_k + J = G_1 \cdot \ldots \cdot G_l + J$, so $\bullet_d \bullet (F_1 \cdot \ldots \cdot F_k) + J = \bullet_d \bullet (G_1 \cdot \ldots \cdot G_l) + J$. By the first step,

$$F + J = \bullet_d \bullet (F_1 \cdot \ldots \cdot F_k) + J = \bullet_d \bullet (G_1 \cdot \ldots \cdot G_l) + J = G + J.$$

(2) The set $\mathcal{RF}(\mathcal{D})$ of rooted forests linearly spans $CP(\mathcal{D})/J$ by the first point. Let J' be the subspace of $CP(\mathcal{D})$ generated by the differences of elements of $\mathcal{PT}(\mathcal{D})$ with the same underlying decorated forest. It is clearly a Com-PreLie ideal, and $\mathcal{RF}(\mathcal{D})$ is a basis of $CP(\mathcal{D})/J'$. Moreover, for all $F_1, F_2 \in \mathcal{PT}(\mathcal{D})$, $\bullet_d \bullet (F_1 \times F_2) + J' = \bullet_s \bullet (F_1 \cdot F_2) + J'$, as the underlying forests of $\bullet_d \bullet (F_1 \times F_2)$ and $\bullet_s \bullet (F_1 \cdot F_2)$ are equal. Consequently, there exists a Com-PreLie morphism from $CP(\mathcal{D})/J$ to $CP(\mathcal{D})/J'$, sending any element of $\mathcal{RF}(\mathcal{D})$ over itself. As the elements of $RF(\mathcal{D})$ are linearly independent in $CP(\mathcal{D})/J'$, they also are in $CP(\mathcal{D})/J$. □

4.3. PreLie structure of $UCP(\mathcal{D})$ and $CP(\mathcal{D})$. Let us now consider $UCP(\mathcal{D})$ and $CP(\mathcal{D})$ as preLie algebras. Their augmentation ideals are respectively denoted by $UCP_+(\mathcal{D})$ and $CP_+(\mathcal{D})$. Note that, as a preLie algebra,

$$UCP_+(\mathcal{D}) = CP_+(\mathbb{N} \times \mathcal{D}).$$

Let $\mathcal{D}$ be any set, and let $T \in \mathcal{PT}(\mathcal{D})$. Then T can be written as

$$T = (\bullet_{d_1} \bullet (T_{1,1} \times \ldots \times T_{i,s_1})) \cdot \ldots \cdot (\bullet_{d_k} \bullet (T_{k,1} \times \ldots \times T_{k,s_k})),$$

where $d_1, \ldots, d_k \in \mathcal{D}$ and the $T_{i,j}$'s are nonempty elements of $\mathcal{PT}(\mathcal{D})$. We shortly denote this as

$$T = B_{d_1,\ldots,d_k}(T_{1,1} \ldots T_{1,s_1}; \ldots; T_{k,1} \ldots T_{k,s_k}).$$

The set of partitioned subtrees $T_{i,j}$ of T is denoted by $\mathcal{S}t(T)$.

Proposition 4.11. *Let $\mathcal{D}$ be any set. One defines a coproduct δ on $CP_+(\mathcal{D})$ by*

$$\forall T \in \mathcal{PT}(\mathcal{D}), \qquad \delta(T) = \sum_{T' \in \mathcal{S}t(T)} T \setminus T' \otimes T.$$

Then, as a preLie algebra, $CP_+(\mathcal{D})$ is freely generated by $\mathrm{Ker}(\delta)$.

Proof. In other words, for any $T \in \mathcal{PT}(\mathcal{D})$, writing

$$T = B_{d_1,\ldots,d_k}(T_{1,1}\ldots T_{1,s_1};\ldots;T_{k,1}\ldots T_{k,s_k}).$$

we can rewrite

$$\delta(T) = \sum_{i=1}^{s}\sum_{j=1}^{s_i} B_{d_1,\ldots,d_k}(T_{1,1}\ldots T_{1,s_1};\ldots;T_{i,1}\ldots \widehat{T_{i,j}}\ldots T_{i,s_i};\ldots;T_{k,1}\ldots T_{k,s_k})\otimes T_{i,j}.$$

This immediately implies that δ is permutative [8]:

$$(\delta\otimes \mathrm{Id})\circ\delta = (23).(\delta\otimes\mathrm{Id})\circ\delta.$$

Moreover, for any $x, y \in \mathcal{PT}_+(\mathcal{D})$, using Sweedler's notation $\delta(x) = x^{(1)}\otimes x^{(2)}$, we obtain

$$\delta(x\cdot y) = x^{(1)}\cdot y\otimes x^{(2)} + x\cdot y^{(1)}\otimes y^{(2)}.$$

For any partitioned tree $T \in \mathcal{PT}(\mathcal{D})$, we denote by $r(T)$ the number of roots of T and we put $d(T) = r(T)T$. The map d is linearly extended as an endomorphism of $\mathcal{PT}(\mathcal{D})$. As the product $\cdot$ is homogeneous for the number of roots, d is a derivation of the algebra $(CP(\mathcal{D}),\cdot)$. Let us prove that for any $x, y \in CP_+(\mathcal{D})$,

$$\delta(x\bullet y) = d(x)\otimes y + x^{(1)}\bullet y\otimes x^{(2)} + x^{(1)}\otimes x^{(2)}\bullet y.$$

We denote by A the set of elements of $x \in CP_+(\mathcal{D})$, such that for any $y \in CP_+(\mathcal{D})$, the preceding equality holds. If $x_1, x_2 \in A$, then for any $y \in CP_+(\mathcal{D})$,

$$\begin{aligned}
\delta((x_1\cdot x_2)\bullet y) &= \delta((x_1\bullet y)\cdot x_2) + \delta(x_1\cdot(x_2\bullet y))\\
&= (x_1\bullet y)^{(1)}\cdot x_2\otimes (x_1\bullet y)^{(2)} + (x_1\bullet y)\cdot x_2^{(1)}\otimes x_2^{(2)}\\
&+ x_1^{(1)}\cdot(x_2\bullet y)\otimes x_1^{(2)} + x_1\cdot(x_2\bullet y)^{(1)}\otimes(x_2\bullet y)^{(2)}\\
&= d(x_1)\cdot x_2\otimes y + (x_1^{(1)}\bullet y)\cdot x_2\otimes x_1^{(1)} + x_1^{(1)}\cdot x_2\otimes x_1^{(2)}\bullet y\\
&+ (x_1\bullet y)\cdot x_2^{(1)}\otimes x_2^{(2)} + x_1^{(1)}\cdot(x_2\bullet y)\otimes x_1^{(2)}\\
&+ x_1\cdot d(x_2)\otimes y + x_1\cdot(x_2^{(1)}\bullet y)\otimes x_2^{(2)} + x_1\cdot x_2^{(1)}\otimes x_2^{(2)}\bullet y\\
&= d(x_1\cdot x_2)\otimes y + (x_1^{(1)}\cdot x_2)\bullet y\otimes x_1^{(2)} + (x_1\cdot x_2^{(1)})\bullet y\otimes x_2^{(2)}\\
&+ (x_1\cdot x_2)^{(1)}\otimes(x_1\cdot x_2)^{(2)}\bullet y\\
&= d(x_1\cdot x_2)\otimes y + (x_1\cdot x_2)^{(1)}\bullet y\otimes(x_1\cdot x_2)^{(2)}\\
&+ (x_1\cdot x_2)^{(1)}\otimes(x_1\cdot x_2)^{(2)}\bullet y.
\end{aligned}$$

So $x_1\cdot x_2 \in A$.

Let $d \in \mathcal{D}$. Note that $\delta(\bullet_d) = 0$. Moreover, for any $y \in CP_+(\mathcal{D})$,

$$\delta(\bullet_d\bullet y) = \delta(B_d(y)) = \bullet_d\otimes y,$$

so $\bullet_d \in A$. Let $T_1, \ldots, T_k \in \mathcal{PT}(\mathcal{D})$, nonempty. We consider $x = B_d(T_1 \ldots T_k)$. For any $y \in CP_+(\mathcal{D})$,

$$\begin{aligned}\delta(x \bullet y) &= \delta(B_d(T_1 \ldots T_k y)) + \sum_{j=1}^{k} \delta(B_d(T_1 \ldots (T_j \bullet y) \ldots T_k) \\ &= B_d(T_1 \ldots T_k) \otimes y + \sum_{i=1}^{k} B_d(T_1 \ldots \widehat{T_i} \ldots T_k y) \otimes T_i \\ &+ \sum_{i=1}^{k} \sum_{j \neq i} B_d(T_1 \ldots \widehat{T_i} \ldots (T_j \bullet y) \ldots T_k) \otimes T_i + \sum_{i=1}^{k} B_d(T_1 \ldots \widehat{T_i} \ldots T_k) \otimes T_i \bullet y \\ &= d(x) \otimes y + \sum_{i=1}^{k} B_d(T_1 \ldots \widehat{T_i} \ldots T_k) \bullet y \otimes T_i + \sum_{i=1}^{k} B_d(T_1 \ldots \widehat{T_i} \ldots T_k) \otimes T_i \bullet y \\ &= d(x) \otimes y + x^{(1)} \bullet y \otimes x^{(2)} + x^{(1)} \otimes x^{(2)} \bullet y.\end{aligned}$$

Hence, $x \in A$. As A is stable under $\cdot$ and contains any partitioned tree with one root, $A = CP_+(\mathcal{D})$.

For any nonempty partitioned tree $T \in \mathcal{PT}(\mathcal{D})$, we put $\delta'(T) = \frac{1}{r(T)} \delta(T)$. Then

$$(\delta' \otimes \mathrm{Id}) \circ \delta'(T) = \frac{1}{r(T)^2} (\delta \otimes \mathrm{Id}) \circ \delta(T),$$

so δ' is also permutative; moreover, for any $x, y \in CP_+(\mathcal{D})$,

$$\delta'(x \bullet y) = x \otimes y + x^{(1)} \bullet y \otimes x^{(2)} + x^{(1)} \otimes x^{(2)} \bullet y.$$

By Livernet's rigidity theorem [8], the preLie algebra $CP_+(\mathcal{D})$ is freely generated by $\mathrm{Ker}(\delta')$. For any integer n, we denote by $CP_n(\mathcal{D})$ the subspace of $CP(\mathcal{D})$ generated by trees T such that $r(T) = n$. Then, for all n, $\delta(CP_n(\mathcal{D})) \subseteq CP_n(\mathcal{D}) \otimes CP_+(\mathcal{D})$, and $\delta_{|CP_n(\mathcal{D})} = n\delta'_{|CP_n(\mathcal{D})}$. This implies that $\mathrm{Ker}(\delta) = \mathrm{Ker}(\delta')$. □

Lemma 4.12. *In $CP_+(\mathcal{D})$ or $UCP_+(\mathcal{D})$, $\mathrm{Ker}(\delta) \bullet \emptyset \subseteq \mathrm{Ker}(\delta)$.*

Proof. Let us work in $UCP_+(\mathcal{D})$. Let us prove that for any $x \in UCP_+(\mathcal{D})$,

$$\delta(x \bullet \emptyset) = x^{(1)} \bullet \emptyset \otimes x^{(2)} + x^{(1)} \otimes x^{(2)} \bullet \emptyset.$$

We denote by A the subspace of elements $x \in UCP_+(\mathcal{D})$ such that this holds. If $x_1, x_2 \in A$, then

$$\begin{aligned}
\delta((x_1 \cdot x_2) \bullet \emptyset) &= \delta((x_1 \bullet \emptyset) \cdot x_2) + \delta(x_1 \cdot (x_2 \bullet \emptyset)) \\
&= (x_1^{(1)} \bullet \emptyset) \cdot x_2 \otimes x^{(1)} + x_1^{(1)} \cdot x_2 \otimes x_1^{(2)} \bullet \emptyset + (x_1 \bullet \emptyset) \cdot x_2^{(1)} \otimes x_2^{(2)} \\
&+ x_1 \cdot (x_2^{(1)} \bullet \emptyset) \otimes x_2^{(2)} + x_1 \cdot x_2^{(1)} \otimes x_2^{(2)} \bullet \emptyset + x_1^{(1)} \cdot (x_2 \bullet \emptyset) \otimes x_1^{(2)} \\
&= (x_1^{(1)} \cdot x_2) \bullet \emptyset \otimes x_1^{(2)} + x_1^{(1)} \cdot x_2 \otimes x_1^{(2)} \bullet \emptyset \\
&+ (x_1 \cdot x_2^{(1)}) \bullet \emptyset \otimes x_2^{(1)} + x_1 \cdot x_2^{(1)} \otimes x_2^{(2)} \bullet \emptyset \\
&= (x_1 \cdot x_2)^{(1)} \bullet \emptyset \otimes (x_1 \cdot x_2)^{(2)} + (x_1 \cdot x_2)^{(1)} \otimes (x_1 \cdot x_2)^{(2)} \bullet \emptyset,
\end{aligned}$$

so $x_1 \cdot x_2 \in A$. If $d \in \mathcal{D}$ and $T_1, \ldots, T_k \in \mathcal{UPT}(\mathcal{D})$, nonempty, if $x = B_d(T_1 \ldots T_k)$,

$$\begin{aligned}
\delta(x \bullet \emptyset) &= \delta(B_{d+1}(T_1 \ldots T_k)) + \sum_{i=1}^{k} \delta(B_d(T_1 \ldots (T_i \bullet \emptyset) \ldots T_k) \\
&= \sum_{i=1}^{k} B_{d+1}(T_1 \ldots \widehat{T_i} \ldots T_k) \otimes T_i + \sum_{j=1}^{k} \sum_{i \neq j} B_d(T_1 \ldots (T_j \bullet \emptyset) \ldots \widehat{T_i} \ldots T_k) \otimes T_i \\
&+ \sum_{i=1}^{k} B_d(T_1 \ldots \widehat{T_i} \ldots T_k) \otimes T_i \bullet \emptyset \\
&= \sum_{i=1}^{k} B_d(T_1 \ldots \widehat{T_i} \ldots T_k) \bullet \emptyset \otimes T_i + \sum_{i=1}^{k} B_d(T_1 \ldots \widehat{T_i} \ldots T_k) \otimes T_i \bullet \emptyset \\
&= x^{(1)} \bullet \emptyset \otimes x^{(2)} + x^{(1)} \otimes x^{(2)} \bullet \emptyset,
\end{aligned}$$

so $x \in A$. Hence, $A = UCP_+(\mathcal{D})$. Consequently, if $x \in \mathrm{Ker}(\delta)$, then $x \bullet \emptyset \in \mathrm{Ker}(\delta)$. The proof is immediate for $CP_+(\mathcal{D})$, as for any tree $T \in \mathcal{PT}(\mathcal{D})$, $T \bullet \emptyset = |T|T$. □

Notations 4.13. We denote by ϕ the endomorphism of $\mathrm{Ker}(\delta)$ defined by $\phi(x) = x \bullet \emptyset$.

Corollary 4.14. *The preLie algebra $UCP(\mathcal{D})$, respectively $CP(\mathcal{D})$, is generated by* $\mathrm{Ker}(\delta) \oplus (\emptyset)$*, with the relations*

$$\begin{aligned}
& & \emptyset \bullet \emptyset = 0, & & \\
&\forall x \in \mathrm{Ker}(\delta), & \emptyset \bullet x = 0, & & x \bullet \emptyset = \phi(x).
\end{aligned}$$

Remark 4.15. We give $CP(\mathcal{D})$ a graduation by putting the elements of $\mathcal{D}$ homogeneous of degree 1, and we put $|D| = d$. For any $n \geqslant 1$, we denote by $t_n(d)$ the number of partitioned trees decorated by $\mathcal{D}$ with n vertices and by $f_n(d)$ the number of partitioned forests decorated by $\mathcal{D}$ with n vertices. We consider the formal series

$$F(d,X)=\sum_{n=0}^\infty f_n(d)X^n, \qquad T(d,X)=\sum_{n=0}^\infty t_n(d)X^n.$$

As any partitioned forest is a monomial of partitioned trees, we obtain

$$F(d,X)=\prod_{n=1}^\infty \frac{1}{(1-X^n)^{t_n(d)}}.$$

As any partitioned tree can be seen as a monomial of pairs (e,F), where $e\in\mathcal{D}$ and F a partitioned forest, we obtain that

$$T(d,X)=\prod_{n=1}^\infty \frac{1}{(1-X^n)^{df_{n-1}(d)}}.$$

These two formulas allow to compute $t_n(d)$ by induction on n, see Table 4.15 (see also [5]). For $d=1$, this gives Entry A035052 of the OEIS [14]; for $d=2$, Entry A226269. Moreover, the sequence of the coefficients of $\binom{d}{n}$ in $t_n(d)$ is Entry A052888.

We denote by $k_n(d)$ the dimension of $\mathrm{Ker}(\tilde{\Delta})_n$ in $CP(\mathcal{D})$. As the preLie algebra $CP(\mathcal{D})$ is freely generated by $\mathrm{Ker}(\tilde{\Delta})$, we obtain that

$$T(d)=\left(\sum_{n=1}^\infty k_n(d)X^n\right)\prod_{n=1}^\infty \frac{1}{(1-X^n)^{t_n(d)}}.$$

This allows to compute the first values of $k_n(d)$, see Table 4.15.

5. Bialgebra structures on free Com-PreLie algebras

5.1. Tensor product of Com-PreLie algebras.

Lemma 5.1. *Let A_1,A_2 be two Com-PreLie algebras and let $\varepsilon:A_1\longrightarrow\mathbb{K}$ such that*

$$\forall a,b\in A_1, \qquad \varepsilon(a\bullet b)=\varepsilon(b\bullet a).$$

Then $A_1\otimes A_2$ is a Com-PreLie algebra, with the products defined by

$$\begin{aligned}(a_1\otimes a_2)(b_1\otimes b_2)&=a_1b_1\otimes a_2b_2,\\ (a_1\otimes a_2)\bullet_\varepsilon(b_1\otimes b_2)&=a_1\bullet b_1\otimes a_2b_2+\varepsilon(b_1)a_1\otimes a_2\bullet b_2.\end{aligned}$$

Proof. $A_1\otimes A_2$ is obviously an associative and commutative algebra, with unit $1\otimes 1$. We take $\alpha=a_1\otimes a_2,\beta=b_1\otimes b_2,\gamma=c_1\otimes c_2\in A_1\otimes A_2$. Let us prove the PreLie identity.

$$\begin{aligned}
t_1(d) &= d \\
&= \binom{d}{1}, \\
t_2(d) &= \frac{(3d+1)d}{2} \\
&= 2\binom{d}{1} + 3\binom{d}{2}, \\
t_3(d) &= \frac{(19d^2+9d+2)d}{6} \\
&= 5\binom{d}{1} + 22\binom{d}{2} + 19\binom{d}{3}, \\
t_4(d) &= \frac{(63d^3+34d^2+13d+2)d}{8} \\
&= 14\binom{d}{1} + 139\binom{d}{2} + 309\binom{d}{3} + 189\binom{d}{4}, \\
t_5(d) &= \frac{(644d^4+400d^3+175d^2+35d+6)d}{30} \\
&= 42\binom{d}{1} + 868\binom{d}{2} + 3735\binom{d}{3} + 5472\binom{d}{4} + 2576\binom{d}{5}, \\
t_6(d) &= \frac{(44683d^5+31695d^4+14635d^3+4185d^2+1162d+120)d}{720} \\
&= 134\binom{d}{1} + 5491\binom{d}{2} + 40882\binom{d}{3} + 107866\binom{d}{4} + 116990\binom{d}{5} + 44683\binom{d}{6}, \\
t_7(d) &= \frac{(941977d^6+754131d^5+375235d^4+125265d^3+35308d^2+5124d+720)d}{5040} \\
&= 444\binom{d}{1} + 35452\binom{d}{2} + 430446\binom{d}{3} + 1821848\binom{d}{4} + 3418190\binom{d}{5} + 2933664\binom{d}{6} \\
&+ 941977\binom{d}{7}.
\end{aligned}$$

TABLE 1. First values of $t_n(d)$

$$\begin{aligned}
(\alpha \bullet_\varepsilon \beta) \bullet_\varepsilon \gamma - \alpha \bullet_\varepsilon (\beta \bullet_\varepsilon \gamma) &= (a_1 \bullet b_1) \bullet c_1 \otimes a_2 b_2 c_2 + \varepsilon(c_1) a_1 \bullet b_1 \otimes (a_2 b_2) \bullet c_2 \\
&+ \varepsilon(b_1) a_1 \bullet c_1 \otimes (a_2 \bullet b_2) c_2 + \varepsilon(b_1)\varepsilon(c_1) a_1 \otimes (a_2 b \bullet_2) \bullet c_2 \\
&- a_1 \bullet (b_1 \bullet c_1) \otimes a_2 b_2 c_2 - \varepsilon(c_1) a_1 \bullet b_1 \otimes a_2 (b_2 \bullet c_2) \\
&- \varepsilon(c_1)\varepsilon(b_1) a_1 \otimes a_2 \bullet (b_2 \bullet c_2) - \varepsilon(b_1 \bullet c_1) a_1 \otimes a_2 \bullet (b_2 c_2) \\
&= ((a_1 \bullet b_1) \bullet c_1 - a_1 \bullet (b_1 \bullet c_1)) \otimes a_2 b_2 c_2 \\
&+ \varepsilon(b_1)\varepsilon(c_1) a_1 \otimes ((a_2 \bullet b_2) \bullet c_2 - a_2 \bullet (b_2 \bullet c_2)) \\
&+ \varepsilon(c_1) a_1 \bullet b_1 \otimes (a_2 \bullet c_2) b_2 + \varepsilon(b_1) a_1 \bullet c_1 \otimes (a_2 \bullet b_2) c_2 \\
&- \varepsilon(b_1 \bullet c_1) a_1 \otimes a_2 \bullet (b_2 c_2).
\end{aligned}$$

$$\begin{aligned}
k_1(d) &= d \\
&= \binom{d}{1}, \\
k_2(d) &= \frac{(d+1)d}{2} \\
&= \binom{d}{1} + \binom{d}{2}, \\
k_3(d) &= \frac{(2d^2+1)d}{3} \\
&= \binom{d}{1} + 4\binom{d}{2} + 4\binom{d}{3}, \\
k_4(d) &= \frac{(11d^3+2d^2+d+2)d}{8} \\
&= 2\binom{d}{1} + 21\binom{d}{2} + 51\binom{d}{3} + 33\binom{d}{4}, \\
k_5(d) &= \frac{(203d^4+60d^3-5d^2-30d+12)d}{60} \\
&= 4\binom{d}{1} + 114\binom{d}{2} + 543\binom{d}{3} + 836\binom{d}{4} + 406\binom{d}{5}, \\
k_6(d) &= \frac{(220d^5+89d^4+16d^3+3d^2+4d+4)d}{24} \\
&= 14\binom{d}{1} + 690\binom{d}{2} + 5531\binom{d}{3} + 15206\binom{d}{4} + 16945\binom{d}{5} + 6600\binom{d}{6}, \\
k_7(d) &= \frac{(66518d^6+33831d^5+9170d^4-735d^3-1708d^2-1596d+360)d}{2520} \\
&= 42\binom{d}{1} + 4258\binom{d}{2} + 55452\binom{d}{3} + 243536\binom{d}{4} + 468055\binom{d}{5} + 408774\binom{d}{6} \\
&+ 133036\binom{d}{7}.
\end{aligned}$$

TABLE 2. First values of $k_n(d)$

As A_1 and A_2 are PreLie, the first and second lines of the last equality are symmetric in β and γ; the third line is obviously symmetric in β and γ; as m is commutative and by the hypothesis on ε, the last line also is. So $\bullet_\varepsilon$ is PreLie.

$$\begin{aligned}
(\alpha\beta) \bullet_\varepsilon \gamma &= (a_1b_1) \bullet c_1 \otimes a_2b_2c_2 + \varepsilon(c_1)a_1b_1 \otimes (a_2b_2) \bullet c_2 \\
&= ((a_1 \bullet c_1)b_1 + a_1(b_1 \bullet c_1)) \otimes a_2b_2c_2 \\
&+ \varepsilon(c_1)a_1b_1 \otimes ((a_2 \bullet c_2)b_2 + a_2(b_2 \bullet c_2))
\end{aligned}$$

$$= (a_1 \bullet c_1 \otimes a_2c_2 + \varepsilon(c_1)a_1 \otimes a_2 \bullet c_2)(b_1 \otimes b_2)$$
$$+ (a_1 \otimes a_2)(b_1 \bullet c_1 \otimes b_2c_2 + \varepsilon(c_1)b_1 \otimes b_2 \bullet c_2)$$
$$= (\alpha \bullet_\varepsilon \gamma)\beta + \alpha(\beta \bullet_\varepsilon \gamma).$$

So $A_1 \otimes A_2$ is Com-PreLie. □

Remark 5.2. Consequently, if $(A, m, \bullet, \Delta)$ is a Com-PreLie bialgebra, with counit ε, then Δ is a morphism of Com-PreLie algebras from $(A, m, \bullet)$ to $(A \otimes A, m, \bullet_\varepsilon)$. Indeed, for all $a, b \in A$, $\varepsilon(a \bullet b) = \varepsilon(b \bullet a) = 0$ and

$$\begin{aligned}\Delta(a) \bullet_\varepsilon \Delta(b) &= a^{(1)} \bullet b^{(1)} \otimes a^{(2)}b^{(2)} + \varepsilon(b^{(1)})a^{(1)} \otimes a^{(2)} \bullet b^{(2)}\\ &= a^{(1)} \bullet b^{(1)} \otimes a^{(2)}b^{(2)} + a^{(1)} \otimes a^{(2)} \bullet b\\ &= \Delta(a \bullet b).\end{aligned}$$

Lemma 5.3. (1) *Let A, B, C be three Com-PreLie algebras, $\varepsilon_A : A \longrightarrow \mathbb{K}$ and $\varepsilon_B : B \longrightarrow \mathbb{K}$ with the condition of Lemma 5.1. Then $\varepsilon_A \otimes \varepsilon_B : A \otimes B \longrightarrow \mathbb{K}$ also satisfies the condition of Lemma 5.1. Moreover, the Com-PreLie algebras $(A \otimes B) \otimes C$ and $A \otimes (B \otimes C)$ are equal.*

(2) *Let A, B be two Com-PreLie algebras, and $\varepsilon : A \longrightarrow \mathbb{K}$ such that*

$$\forall a, b \in A, \qquad \varepsilon(ab) = \varepsilon(a)\varepsilon(b), \qquad \varepsilon(a \bullet b) = 0.$$

Then $\varepsilon \otimes \mathrm{Id} : A \otimes B \longrightarrow B$ is a morphism of Com-PreLie algebras.

(3) *Let A, A', B, B' be Com-PreLie algebras, $\varepsilon : A \longrightarrow \mathbb{K}$ and $\varepsilon' : A' \longrightarrow \mathbb{K}$ satisfying the condition of Lemma 5.1. Let $f : A \longrightarrow A'$, $g : B \longrightarrow B'$ be Com-PreLie algebra morphisms such that $\varepsilon' \circ f = \varepsilon$. Then $f \otimes g : A \otimes B \longrightarrow A' \otimes B'$ is a Com-PreLie algebra morphism.*

Proof. (1) Indeed, if $a_1, a_2 \in A$ and $b_1, b_2 \in B$,

$$\begin{aligned}\varepsilon_A \otimes \varepsilon_B((a_1 \otimes b_1) \bullet (a_2 \otimes b_2)) &= \varepsilon_A(a_1 \bullet a_2)\varepsilon_B(b_1b_2) + \varepsilon_A(a_1)\varepsilon_A(a_2)\varepsilon_B(b_1 \bullet b_2)\\ &= \varepsilon_A(a_2 \bullet a_1)\varepsilon_B(b_2b_1) + \varepsilon_A(a_2)\varepsilon_A(a_1)\varepsilon_B(b_2 \bullet b_1)\\ &= \varepsilon_A \otimes \varepsilon_B((a_2 \otimes b_2) \bullet (a_1 \otimes b_1)).\end{aligned}$$

Let $a_1, a_2 \in A$, $b_1, b_2 \in B$, $c_1, c_2 \in C$. In $(A \otimes B) \otimes C$,

$$(a_1 \otimes b_1 \otimes c_1) \bullet (a_2 \otimes b_2 \otimes c_2)$$
$$= ((a_1 \otimes b_1) \bullet (a_2 \otimes b_2)) \otimes c_1c_2 + \varepsilon_A \otimes \varepsilon_B(a_2 \otimes b_2)a_1 \otimes b_1 \otimes c_1 \bullet c_2$$
$$= a_1 \bullet a_2 \otimes b_1b_2 \otimes c_1c_2 + \varepsilon_A(a_2)a_1 \otimes b_1 \bullet b_2 \otimes c_1c_2 + \varepsilon_A(a_2)\varepsilon_B(b_2)a_1 \otimes b_1 \otimes c_1 \bullet c_2.$$

In $A\otimes(B\otimes C)$,

$$\begin{aligned}&(a_1\otimes b_1\otimes c_1)\bullet(a_2\otimes b_2\otimes c_2)\\&=a_1\bullet a_2\otimes b_1b_2\otimes c_1c_2+\varepsilon_A(a_2)a_1\otimes((b_1\otimes c_1)\bullet(b_2\otimes c_2))\\&=a_1\bullet a_2\otimes b_1b_2\otimes c_1c_2+\varepsilon_A(a_2)a_1\otimes b_1\bullet b_2\otimes c_1c_2+\varepsilon_A(a_2)\varepsilon_B(b_2)a_1\otimes b_1\otimes c_1\bullet c_2.\end{aligned}$$

So $(A\otimes B)\otimes C=A\otimes(B\otimes C)$.

(2) Let $a_1,a_2\in A$, $b_1,b_2\in B$.

$$\begin{aligned}&\varepsilon\otimes\mathrm{Id}((a_1\otimes b_1)(a_2\otimes b_2))\\&=\varepsilon(a_1a_2)b_1b_2\\&=\varepsilon(a_1)\varepsilon(a_2)b_1b_2\\&=\varepsilon\otimes\mathrm{Id}((a_1\otimes b_1)\varepsilon\otimes\mathrm{Id}(a_2\otimes b_2),\end{aligned}\qquad\begin{aligned}&\varepsilon\otimes\mathrm{Id}((a_1\otimes b_1)\bullet(a_2\otimes b_2))\\&=\varepsilon(a_1\bullet a_2)b_1b_2+\varepsilon(a_1)\varepsilon(a_2)b_1\bullet b_2\\&=\varepsilon(a_1)\varepsilon(a_2)b_1\bullet b_2\\&=\varepsilon\otimes\mathrm{Id}((a_1\otimes b_1)\bullet\varepsilon\otimes\mathrm{Id}(a_2\otimes b_2).\end{aligned}$$

So $\varepsilon\otimes\mathrm{Id}$ is a morphism.

(3) $f\otimes g$ is obviously an algebra morphism. If $a_1,a_2\in A$, $b_1,b_2\in B$,

$$\begin{aligned}&(f\otimes g)((a_1\otimes b_1)\bullet(a_2\otimes b_2))\\&=(f\otimes g)(a_1\bullet a_2\otimes b_1b_2+\varepsilon(a_2)a_1\otimes b_1\bullet b_2)\\&=f(a_1)\bullet f(a_2)\otimes g(b_1)g(b_2)+\varepsilon(f(a_2))f(a_1)\otimes g(b_1)\bullet g(b_2)\\&=(f(a_1)\otimes g(b_1))\bullet(f(a_2)\otimes g(b_2)).\end{aligned}$$

So $f\otimes g$ is a Com-PreLie algebra morphism. □

Lemma 5.4. *Let A be a unital associative commutative bialgebra, and V a subspace of A which generates A. Let $\bullet$ be a product on A such that*

$$\forall a,b,c\in A,\qquad (ab)\bullet c=(a\bullet c)b+a(b\bullet c).$$

Then A is a Com-PreLie bialgebra if and only if for all $x\in V$, and for all $b,c\in A$,

$$\begin{aligned}(x\bullet b)\bullet c-x\bullet(b\bullet c)&=(x\bullet c)\bullet b-x\bullet(c\bullet b),\\ \Delta(x\bullet b)&=x^{(1)}\otimes x^{(2)}\bullet b+x^{(1)}\bullet b^{(1)}\otimes x^{(2)}b^{(2)}.\end{aligned}$$

Proof. $\Longrightarrow$ Obvious, by definition of a Com-PreLie algebra.
$\Longleftarrow$ We consider

$$B=\{a\in A\mid\forall b,c\in A,(a\bullet b)\bullet c-a\bullet(b\bullet c)=(a\bullet c)\bullet b-a\bullet(c\bullet b)\}.$$

We denote by 1_A the unit of A. Copying the proof of Lemma 2.3-1, we obtain that $1_A.b=0$ for all $b\in A$. This easily implies that $1_A\in B$. By hypothesis, $V\subseteq B$.

Let $a_1, a_2 \in B$. For all $b, c \in A$,

$$\begin{aligned}
&((a_1a_2)\bullet b)\bullet c-(a_1a_2)\bullet(b\bullet c)\\
&=((a_1\bullet b)\bullet c)a_2+(a_1\bullet b)(a_2\bullet c)+(a_1\bullet c)(a_2\bullet b)+a_1((a_2\bullet b)\bullet c)\\
&-(a_1\bullet(b\bullet c))a_2-a_1(a_2\bullet(b\bullet c))\\
&=((a_1\bullet b)\bullet c-a_1\bullet(b\bullet c))a_2+a_1((a_2\bullet b)\bullet c-a_2\bullet(b\bullet c))\\
&+(a_1\bullet b)(a_2\bullet c)+(a_1\bullet c)(a_2\bullet b).
\end{aligned}$$

As $a_1, a_2 \in B$, this is symmetric in b, c, so $a_1a_2 \in B$. Hence, B is a unitary subalgebra of A which contains V, so is equal to A: A is a Com-PreLie algebra. Let us now consider

$$C=\{a\in A\mid \forall b\in A, \Delta(a\bullet b)=a^{(1)}\otimes a^{(2)}\bullet b+a^{(1)}\bullet b^{(1)}\otimes a^{(2)}b^{(2)}\}.$$

By hypothesis, $V \subseteq C$. Let $b \in B$.

$$1_A\otimes 1_A\bullet b+1_A\bullet b^{(1)}\otimes 1b^{(2)}=0=\Delta(1_A\bullet b),$$

so $1_A \in C$. Let $a_1, a_2 \in C$. For all $b \in A$,

$$\begin{aligned}
\Delta((a_1a_2)\bullet b)&=\Delta((a_1\bullet b)a_2+a_1(a_2\bullet b))\\
&=a_1^{(1)}a_2^{(1)}\otimes(a_1^{(2)}\bullet b)a_2^{(2)}+(a_1^{(1)}\bullet b^{(1)})a_2^{(1)}\otimes a_1^{(2)}b^{(2)}a_2^{(2)}\\
&a_1^{(1)}a_2^{(1)}\otimes a_1^{(2)}(a_2^{(2)}\bullet b)+a_1^{(1)}(a_2^{(1)}\bullet b^{(1)})\otimes a_1^{(2)}a_2^{(2)}b^{(2)}\\
&=a_1^{(1)}a_2^{(1)}\otimes(a_1^{(2)}a_2^{(2)})\bullet b+(a_1^{(1)}a_2^{(1)})\bullet b^{(1)}\otimes a_1^{(2)}a_2^{(2)}b^{(2)}\\
&=(a_1a_2)^{(1)}\otimes(a_1a_2)^{(2)}\bullet b+(a_1a_2)^{(1)}\bullet b^{(1)}\otimes(a_1a_2)^{(2)}b^{(2)}.
\end{aligned}$$

Hence, $a_1a_2 \in C$, and C is a unitary subalgebra of A. As it contains V, $C = A$ and A is a Com-PreLie bialgebra. □

5.2. Coproduct on $UCP(\mathcal{D})$.

Definition 5.5. (1) Let T be a partitioned tree and $I \subseteq V(T)$. We shall say that I is an ideal of T if for any vertex $v \in I$ and any vertex $w \in V(T)$ such that there exists an edge from v to w, then $w \in I$. The set of ideals of T is denoted by $\mathcal{I}d(T)$.

(2) Let T be partitioned forest decorated by $\mathbb{N} \times I$, and $I \in \mathcal{I}d(T)$.
   - By restriction, I is a partitioned decorated forest. The product $\cdot$ of the trees of I is denoted by $P^I(F)$.
   - By restriction, $T \setminus I$ is a partitioned decorated tree. For any vertex $v \in T \setminus I$, if we denote by (i, d) the decoration of v in T, we replace it by $(i + \iota_I(v), d)$, where $\iota_I(v)$ is the number of blocks C of T, included

in I, such that there exists an edge from v to any vertex of C. The partitioned decorated tree obtained in this way is denoted by $R^I(F)$.

Theorem 5.6. *We define a coproduct on* $UCP(\mathcal{D})$ *by*

$$\forall T \in \mathcal{PT}(\mathbb{N}\times\mathcal{D}), \qquad \Delta(T) = \sum_{I \in \mathcal{I}d(T)} R^I(T)\otimes P^I(T).$$

Then $UCP(\mathcal{D})$ *is a Com-PreLie bialgebra. Moreover,* $CP(\mathcal{D})$ *and* $\mathcal{H}_{CK}^{\mathcal{D}}$ *are Com-PreLie bialgebra quotients of* $UCP(\mathcal{D})$*, and* $\mathcal{H}_{CK}^{\mathcal{D}}$ *is the Connes-Kreimer Hopf algebra of decorated rooted trees* [3,4].

Proof. We consider

$$\varepsilon : \begin{cases} UCP(\mathcal{D}) & \longrightarrow & \mathbb{K} \\ F & \longrightarrow & \delta_{F,1}. \end{cases}$$

By Lemma 5.3-1, $UCP(\mathcal{D}) \otimes_\varepsilon UCP(\mathcal{D})$ is a Com-PreLie algebra. It is unitary, the unit being $\emptyset\otimes\emptyset$. Hence, there exists a unique Com-PreLie algebra morphism $\Delta' : UCP(\mathcal{D}) \longrightarrow UCP(\mathcal{D})\otimes_\varepsilon UCP(\mathcal{D})$, sending $\bullet_{(0,\,d)}$ over $\bullet_{(0,\,d)}\otimes\emptyset+\emptyset\otimes\bullet_{(0,\,d)}$ for all $d\in\mathcal{D}$. By Lemma 5.3-2, $(UCP(\mathcal{D})\otimes_\varepsilon UCP(\mathcal{D}))\otimes_{\varepsilon\otimes\varepsilon} UPC(\mathcal{D})$ and $UCP(\mathcal{D})\otimes_\varepsilon (UCP(\mathcal{D})\otimes_\varepsilon UCP(\mathcal{D}))$ are equal, and as both $(\mathrm{Id}\otimes\Delta')\circ\Delta'$ and $(\Delta'\otimes\mathrm{Id})\circ\Delta'$ are Com-PreLie algebra morphisms sending $\bullet_{(0,\,d)}$ over $\bullet_{(0,\,d)}\otimes\emptyset\otimes\emptyset+\emptyset\otimes\bullet_{(0,\,d)}\otimes\emptyset+\emptyset\otimes\emptyset\otimes\bullet_{(0,\,d)}$ for all $d\in\mathcal{D}$, they are equal: Δ' is coassociative. Moreover, $(\mathrm{Id}\otimes\varepsilon)\circ\Delta'$ and $(\varepsilon\otimes\mathrm{Id})\circ\Delta'$ are Com-PreLie endomorphisms of $UCP(\mathcal{D})$ sending $\bullet_{(0,\,d)}$ over itself for all $d\in\mathcal{D}$, so they are both equal to Id: ε is the counit of Δ'. Hence, with this coproduct Δ', $UCP(\mathcal{D})$ is a Com-PreLie bialgebra.

Let us now prove that $\Delta(\mathcal{T}) = \Delta'(\mathcal{T})$ for all $T\in\mathcal{PT}(\mathbb{N}\times\mathcal{D})$. We proceed by induction on the number of vertices n of T. If $n=0$ or $n=1$, it is obvious. Let us assume the result at all ranks $< n$. If T has strictly more than one root, we can write $T = T'\cdot T''$, where T' and T'' has strictly less that n vertices. It is easy to see that the ideals of T are the parts of $T'\sqcup T''$ of the form $I'\sqcup I''$, such that $I'\in\mathcal{I}d(\mathcal{T}')$ and $I''\in\mathcal{I}d(\mathcal{T}'')$. Moreover, for such an ideal of T,

$$R^{I'\sqcup I''}(T'\cdot T'') = R^{I'}(T')\cdot R^{I''}(T''), \quad P^{I'\sqcup I''}(T'\cdot T'') = P^{I'}(T')\cdot P^{I''}(T'').$$

Hence,

$$\begin{aligned}\Delta(T) &= \sum_{I'\in\mathcal{I}d(\mathcal{T}'),\, I''\in\mathcal{I}d(\mathcal{T}'')} R^{I'}(T')\cdot R^{I''}(T'')\otimes R^{I'}(T')R^{I''}(T'')\\ &= \Delta(T)\cdot\Delta(T'')\\ &= \Delta'(T')\cdot\Delta'(T'')\\ &= \Delta'(T\cdot T'')\\ &= \Delta(T).\end{aligned}$$

If T has only one root, we can write $T = \bullet_{(i,\,d)} \bullet (T_1 \times \ldots \times T_k)$, where $T_1, \ldots, T_k \in \mathcal{PT}(\mathbb{N} \times \mathcal{D})$. The induction hypothesis holds for $T_1, \ldots, T_N$. The ideals of T are

- T itself: for this ideal I, $P^I(T) = T$ and $R^I(T) = \emptyset$.
- Ideals $I_1 \sqcup \ldots \sqcup I_k$, where I_j is an ideal of T_j for all j. For such an ideal I, $P^I(T) = P^{I_1}(T_1) \cdot \ldots \cdot P^{I_k}(T_k)$. Let $J = \{i_1, \ldots, i_p\}$ be the set of indices i such that $I_i = T_i$, that is to say the number of blocks C of I such that is an edge from the root of T to any vertex of C. Then

$$\begin{aligned} R^I(T) &= \bullet_{(i+p,\,d)} \bullet \prod_{j \notin J}^{\times} R^{I_j}(T_j) \\ &= f^l_{UCP(\mathcal{D})}(\bullet_{(i,\,d)}) \bullet \prod_{j \notin J}^{\times} R^{I_j}(T_j) \\ &= \bullet_{(i,\,d)} \bullet \emptyset^{\times p} \times t \prod_{j \notin J}^{\times} R^{I_j}(T_j) \\ &= \bullet_{(i,\,d)} \bullet R^{I_1}(T_1) \times \ldots \times R^{I_k}(T_k). \end{aligned}$$

We used Lemma 2.9 for the third equality.

By Proposition 2.8, with $a = \bullet_{(i,\,d)}$ and $b_1 \times \ldots \times b_n = T_1 \times \ldots \times T_k$,

$$\begin{aligned} \Delta'(T) = &\sum_{I \subseteq [k]} \bullet_{(i,\,d)} \bullet \left(\prod_{i \in I}^{\times} T_i^{(1)}\right) \otimes \left(\prod_{i \in I} T_i^{(2)}\right) \emptyset \bullet \left(\prod_{i \notin I}^{\times} T_i\right) \\ &+ \sum_{I \subseteq [k]} \emptyset \bullet \left(\prod_{i \in I}^{\times} T_i^{(1)}\right) \otimes \left(\prod_{i \in I} T_i^{(2)}\right) \bullet_{(i,\,d)} \bullet \left(\prod_{i \notin I}^{\times} T_i\right) \\ &= \bullet_{(i,\,d)} \bullet T_1^{(1)} \times \ldots \times T_k^{(1)} \otimes T_1^{(2)} \cdot \ldots \cdot T_k^{(2)} + 0 \\ &+ \emptyset \otimes \bullet_{(i,\,d)} \bullet T_1 \times \ldots \times T_k \\ &= \sum_{I_j \in \mathcal{I}d(T_j)} \bullet_{(i,\,d)} \bullet R^{I_1}(T_1) \times \ldots \times R^{I_k}(T_k) \otimes P^{I_1}(T_1) \cdot \ldots \cdot P^{I_k}(T_k) + \emptyset \otimes T \end{aligned}$$

$$\begin{aligned} &= \sum_{I \in \mathcal{I}d(T),\, I \neq T} R^I(T) \otimes P^I(T) + \emptyset \otimes T \\ &= \sum_{I \in \mathcal{I}d(T)} R^I(T) \otimes P^I(T) \\ &= \Delta(T). \end{aligned}$$

Hence, $\Delta' = \Delta$.

For all $d \in \mathcal{D}$, $\bullet_{(0,\,d)} - \bullet_{(1,\,d)}$ is primitive, so $\Delta(\bullet_{(0,\,d)} - \bullet_{(1,\,d)}) \in I \otimes UCP(\mathcal{D}) + UCP(\mathcal{D}) \otimes I$. Consequently, I is a coideal, and the quotient $UCP(\mathcal{D})/I = CP(\mathcal{D})$ is a Com-PreLie bialgebra.

Let $x, y \in CP(\mathcal{D})$. By Proposition 2.8, as $\bullet_d$ is primitive,

$$\Delta(\bullet_d \bullet (x \times y)) = \bullet_d \bullet (x^{(1)} \times y^{(1)}) \otimes x^{(2)} \cdot y^{(2)} + 1 \otimes \bullet_d \bullet (x \times y),$$

whereas, by the 1-cocycle property,

$$\Delta(\bullet_d \bullet (x \cdot y)) = \bullet_d \bullet (x^{(1)} \cdot y^{(1)}) \otimes x^{(2)} \cdot y^{(2)} + \otimes \bullet_d \bullet (x \cdot y).$$

Hence,

$$\begin{aligned}\Delta(\bullet_d \bullet (x \times y) - \bullet_d \bullet (x \cdot y)) &= \underbrace{(\bullet_d \bullet (x^{(1)} \times y^{(1)}) - \bullet_d \bullet (x^{(1)} \cdot y^{(1)}))}_{\in J} \otimes x^{(2)} \cdot y^{(2)} \\ &\quad + 1 \otimes \underbrace{(\bullet_d \bullet (x \times y) - \bullet_d \bullet (x \cdot y))}_{\in J} \\ &\in J \otimes CP(\mathcal{D}) + CP(\mathcal{D}) \otimes J,\end{aligned}$$

so J is a coideal and $CP(\mathcal{D})/J = \mathcal{H}_{CK}^{\mathcal{D}}$ is a Com-PreLie bialgebra.

Let us consider

$$B_d : \begin{cases} \mathcal{H}_{CK}^{\mathcal{D}} & \longrightarrow & \mathcal{H}_{CK}^{\mathcal{D}} \\ T_1 \ldots T_k & \longrightarrow & \bullet_d \bullet T_1 \times \ldots \times T_k, \end{cases}$$

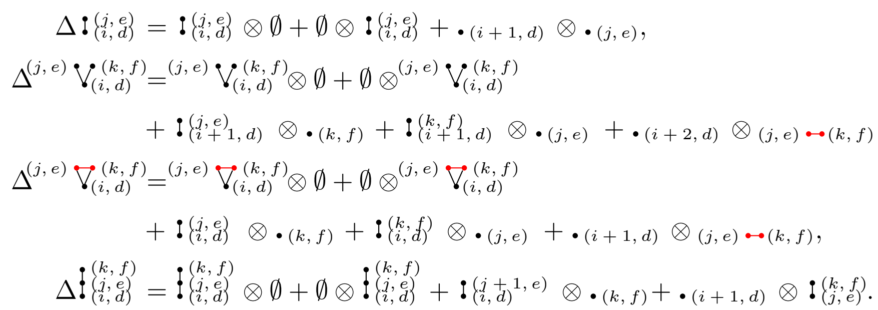
where $T_1, \ldots, T_k$ are rooted trees decorated by $\mathcal{D}$. In other terms, $B_d(T_1 \ldots T_k)$ is the tree obtained by grafting the forest $T_1 \ldots T_k$ on a common root decorated by d. By Proposition 2.8 and Lemma 2.9, for all forest $F = T_1 \ldots T_k \in \mathcal{H}_{CK}^{\mathcal{D}}$,

$$\begin{aligned}\Delta \circ B_d(F) &= \bullet_d \bullet T_1^{(1)} \times \ldots \times T_k^{(1)} \otimes T_1^{(2)} \ldots T_k^{(2)} + 0 + \emptyset \otimes \bullet_d \bullet T_1 \times \ldots \times T_k \\ &= B_d(F^{(1)}) \otimes F^{(2)} + \emptyset \otimes B_d(F).\end{aligned}$$

We recognize the 1-cocycle property which characterizes the Connes-Kreimer coproduct of rooted trees, so $\mathcal{H}_{CK}^{\mathcal{D}}$ is indeed the Connes-Kreimer Hopf algebra. $\square$

Example 5.7. Let $i, j, k \in \mathbb{N}$ and $d, e, f \in \mathcal{D}$. In $UCP(\mathcal{D})$,

$$\Delta \bullet_{(i,d)} = \bullet_{(i,d)} \otimes \emptyset + \emptyset \otimes \bullet_{(i,d)},$$



In $CP(\mathcal{D})$,

$$\begin{aligned}
\Delta \bullet_d &= \bullet_d \otimes \emptyset + \emptyset \otimes \bullet_d,\\
\Delta \mathbf{I}^{e}_{d} &= \mathbf{I}^{e}_{d} \otimes \emptyset + \emptyset \otimes \mathbf{I}^{e}_{d} + \bullet_d \otimes \bullet_e,\\
\Delta {}^{e}\mathbf{V}^{f}_{d} &= {}^{e}\mathbf{V}^{f}_{d} \otimes \emptyset + \emptyset \otimes {}^{e}\mathbf{V}^{f}_{d} + \mathbf{I}^{e}_{d} \otimes \bullet_f + \mathbf{I}^{f}_{d} \otimes \bullet_e + \bullet_d \otimes {}_e\bullet\!\!-\!\!\bullet_f,\\
\Delta {}^{e}\overline{\mathbf{V}}^{f}_{d} &= {}^{e}\overline{\mathbf{V}}^{f}_{d} \otimes \emptyset + \emptyset \otimes {}^{e}\overline{\mathbf{V}}^{f}_{d} + \mathbf{I}^{e}_{d} \otimes \bullet_f + \mathbf{I}^{f}_{d} \otimes \bullet_e + \bullet_d \otimes {}_e\bullet\!\!-\!\!\bullet_f,\\
\Delta \mathbf{I}^{f}_{e\,d} &= \mathbf{I}^{f}_{e\,d} \otimes \emptyset + \emptyset \otimes \mathbf{I}^{f}_{e\,d} + \mathbf{I}^{e}_{d} \otimes \bullet_f + \bullet_d \otimes \mathbf{I}^{f}_{e}.
\end{aligned}$$

In $\mathcal{H}^{\mathcal{D}}_{CK}$,

$$\begin{aligned}
\Delta \bullet_d &= \bullet_d \otimes \emptyset + \emptyset \otimes \bullet_d,\\
\Delta \mathbf{I}^{e}_{d} &= \mathbf{I}^{e}_{d} \otimes \emptyset + \emptyset \otimes \mathbf{I}^{e}_{d} + \bullet_d \otimes \bullet_e,\\
\Delta {}^{e}\mathbf{V}^{f}_{d} &= {}^{e}\mathbf{V}^{f}_{d} \otimes \emptyset + \emptyset \otimes {}^{e}\mathbf{V}^{f}_{d} + \mathbf{I}^{e}_{d} \otimes \bullet_f + \mathbf{I}^{f}_{d} \otimes \bullet_e + \bullet_d \otimes \bullet_e \bullet_f,\\
\Delta \mathbf{I}^{f}_{e\,d} &= \mathbf{I}^{f}_{e\,d} \otimes \emptyset + \emptyset \otimes \mathbf{I}^{f}_{e\,d} + \mathbf{I}^{e}_{d} \otimes \bullet_f + \bullet_d \otimes \mathbf{I}^{f}_{e}.
\end{aligned}$$

5.3. An application: Connes-Moscovici subalgebras. Let us fix a set $\mathcal{D}$ of decorations. For any $d \in \mathcal{D}$, we define an operator $N_d : \mathcal{H}^{\mathcal{D}}_{CK} \longrightarrow \mathcal{H}^{\mathcal{D}}_{CK}$ by

$$\forall x \in \mathcal{H}^{\mathcal{D}}_{CK}, \qquad N_d(x) = x \bullet \bullet_d.$$

In other words, if F is a rooted forest, $N_d(F)$ is the sum of all forests obtained by grafting a leaf decorated by d on a vertex of F: when $\mathcal{D}$ is reduced to a singleton, this is the growth operator N of [3].

For all $k \geqslant 1$, $i_1, \ldots, i_k \in \mathcal{D}$, we put

$$X_{i_1,\ldots,i_k} = N_{i_k} \circ \ldots \circ N_{i_2}(\bullet_{i_1}).$$

When $|\mathcal{D}| = 1$, these are the generators of the Connes-Moscovici subalgebra of [3].

Proposition 5.8. *Let $\mathcal{H}^{\mathcal{D}}_{CM}$ be the subalgebra of $\mathcal{H}^{\mathcal{D}}_{CK}$ generated by all the elements $X_{i_1,\ldots,i_k}$. Then $\mathcal{H}^{\mathcal{D}}_{CM}$ is a Hopf subalgebra.*

Proof. Note that N_d is a derivation; as $N_d(X_{i_1,\ldots,i_k}) = X_{i_1,\ldots,i_k,d}$ for all $i_1, \ldots, i_k, d \in \mathcal{D}$, $\mathcal{H}^{\mathcal{D}}_{CM}$ is stable under N_d for any $d \in \mathcal{D}$. As the $X_{i_1,\ldots,i_k}$ are homogeneous of degree k,

$$X_{i_1,\ldots,i_k} \bullet 1 = kX_{i_1,\ldots,i_k}.$$

Hence, $\mathcal{H}^{\mathcal{D}}_{CM}$ is stable under the derivation $D : x \mapsto x \bullet 1$. We obtain

$$\begin{aligned}
\Delta(X_{i_1,\ldots,i_k}) &= \Delta(X_{i_1,\ldots,i_{k-1}} \bullet \bullet_{i_k}) \qquad (12)\\
&= X^{(1)}_{i_1,\ldots,i_{k-1}} \otimes X^{(2)}_{i_1,\ldots,i_{k-1}} \bullet \bullet_{i_k}\\
&+ X^{(1)}_{i_1,\ldots,i_{k-1}} \bullet \bullet_{i_k} \otimes X^{(2)}_{i_1,\ldots,i_{k-1}} + X^{(1)}_{i_1,\ldots,i_{k-1}} \bullet \emptyset \otimes X^{(2)}_{i_1,\ldots,i_{k-1}} \bullet_{i_k}.
\end{aligned}$$

An easy induction on k proves that $\Delta(X_{i_1,\ldots,_k})$ belongs to $\mathcal{H}^{\mathcal{D}}_{CM} \otimes \mathcal{H}^{\mathcal{D}}_{CM}$. $\square$

Proposition 5.9. *We assume that $\mathcal{D}$ is finite. Then $\mathcal{H}_{CM}^{\mathcal{D}}$ is the graded dual of the enveloping algebra of the augmentation ideal of the Com-PreLie algebra $T(V,f)$, where $V=\mathrm{Vect}(\mathcal{D})$ and $f=\mathrm{Id}_V$.*

Proof. We put $W=\mathrm{Vect}(X_{i_1,\ldots,i_k}\mid k\geqslant 1, i_1,\ldots,i_k\in\mathcal{D})$. As this is the case for $\mathcal{H}_{CK}^{\mathcal{D}}$, for any $x\in W$,

$$\Delta(x)-x\otimes 1+1\otimes x\in W\otimes\mathcal{H}_{CM}^{\mathcal{D}}.$$

This implies that the graded dual of $\mathcal{H}_{CM}^{\mathcal{D}}$ is the enveloping of a graded algebra $\mathfrak{g}$; as a vector space, $\mathfrak{g}$ is identified with W^* and its preLie product is dual of the bracket δ defined on W by $(\pi_W\otimes\pi_W)\circ\Delta$, where π_W is the canonical projection on W which vanishes on $(1)+(\mathcal{H}_{CM}^{\mathcal{D}})_+^2$. By (12), using Sweedler's notation $\delta(x)=x^{(1)}\otimes x^{(2)}$, we obtain

$$\begin{aligned}\delta(X_{i_1,\ldots,i_{k+1}})={}&X_{i_1,\ldots,i_k}^{(1)}\otimes X_{i_1,\ldots,i_k}^{(2)}\bullet X_{i_{k+1}}+X_{i_1,\ldots,i_k}^{(1)}\bullet X_{i_{k+1}}\otimes X_{i_1,\ldots,i_k}^{(2)}\\&+kX_{i_1,\ldots,i_k}\otimes X_{i_{k+1}}.\end{aligned}$$

We shall use the following notations. If $I\subseteq[k]$, we put

- $m(I)=\max(i\mid [i]\subseteq I)$, with the convention $m(I)=0$ if $1\notin I$.
- $X_{i_I}=X_{i_{p_1},\ldots i_{p_l}}$ if $I=\{p_1<\ldots<p_l\}$.

An easy induction proves that

$$\forall i_1,\ldots,i_k\in\mathcal{D},\qquad \delta(X_{i_1,\ldots,i_k})=\sum_{\emptyset\subsetneq I\subseteq[k]}m(I)X_{i_I}\otimes X_{i_{[k]\setminus I}}.$$

We identify W^* and $T(V)_+$ via the pairing given by

$$\forall i_1,\ldots,i_k,j_1,\ldots,j_l\in\mathcal{D},\qquad \langle X_{i_1,\ldots,i_k},j_1\ldots j_l\rangle=\delta_{(i_1,\ldots,i_k),(j_1,\ldots,j_l)}.$$

The preLie product on $T(V)_+$ induced by δ is then given by

$$i_1\ldots i_k\bullet i_{k+1}\ldots i_{k+l}=\sum_{\sigma\in\mathcal{S}h(k,l)}m_k(\sigma)i_{\sigma^{-1}(1)}\ldots i_{\sigma^{-1}(k+l)}.$$

By (9), this is precisely the preLie product of $T(V,f)$. □

Remark 5.10. The following map is a bijection:

$$\theta_{k,l}:\left\{\begin{array}{rcl}\mathcal{S}h(k,l)&\longrightarrow&\mathcal{S}h(l,k)\\ \sigma&\longrightarrow&(k+l\,k+l-1\ldots 1)\circ\sigma\circ(k+l\,k+l-1\ldots 1).\end{array}\right.$$

Moreover, for any $\sigma\in\mathcal{S}h(k,l)$,

$$m_l(\theta_{k,l}(\sigma))=\min\{i\in\{k+1,\ldots,k+l\}\mid \sigma(i)=i,\ldots,\sigma(k+l)=\sigma(k+l)\}=m'_l(\sigma),$$

with the convention $m'_l(\sigma) = 0$ if $\sigma(k+l) \neq k+l$. Then the Lie bracket associated to $\bullet$ is given by

$$\forall i_1, \ldots, i_{k+l} \in \mathcal{D},$$
$$[i_1 \ldots i_k, i_{k+1} \ldots i_{k+l}] = \sum_{\sigma \in \mathcal{S}h(k,l)} (m_k(\sigma) - m'_l(\sigma)) i_{\sigma^{-1}(1)} \ldots i_{\sigma^{-1}(k+l)}.$$

5.4. A rigidity theorem for Com-PreLie bialgebras.

Theorem 5.11. *Let $(A, m, \bullet, \Delta)$ be a connected Com-PreLie bialgebra. If f_A (defined in Proposition 2.6) is surjective, then (A, m, Δ) and $(T(\mathrm{Prim}(A)), \sqcup, \Delta)$ are isomorphic Hopf algebras.*

Proof. We put $V = \mathrm{Prim}(A)$.

First step. As f_A is surjective, there exists $g : V \longrightarrow V$ such that $f_A \circ g = \mathrm{Id}_V$. For all $x \in V$, we put

$$L_x : \begin{cases} A & \longrightarrow & A \\ y & \longrightarrow & g(x) \bullet y. \end{cases}$$

For all $y \in A$,

$$\Delta \circ L_x(y) = \emptyset \otimes g(x) \bullet y + g(x) \bullet y^{(1)} \otimes y^{(2)} = \emptyset \otimes L_x(y) + (\mathrm{Id} \otimes L_x) \circ \Delta(y).$$

Hence, L_x is a 1-cocycle of A. Moreover, $L_x(1) = g(x) \bullet 1 = f_A \circ g(x) = x$. For all $x_1, \ldots, x_n \in V$, we define $\omega(x_1, \ldots, x_n)$ inductively on n by

$$\omega(x_1, \ldots, x_n) = \begin{cases} \emptyset \text{ if } n = 0, \\ L_{x_1}(\omega(x_2, \ldots, x_{n-1})) \text{ if } n \geqslant 1. \end{cases}$$

In particular, $\omega(v) = v$ for all $v \in V$. An easy induction proves that

$$\Delta(\omega(x_1, \ldots, x_n)) = \sum_{i=0}^{n} \omega(x_1, \ldots, x_i) \otimes \omega(x_{i+1}, \ldots, x_n).$$

Hence, the following map is a coalgebra morphism:

$$\omega : \begin{cases} T(V) & \longrightarrow & A \\ x_1 \ldots x_n & \longrightarrow & \omega(x_1, \ldots, x_n). \end{cases}$$

It is injective: if $\mathrm{Ker}(\omega)$ is nonzero, then it is a nonzero coideal of $T(V)$, so it contains nonzero primitive elements of $T(V)$, that is to say nonzero elements of V. For all $v \in V$, $\omega(v) = L_v(1) = v$: contradiction. Let us prove that ω is surjective. As A is connected, for any $x \in A_+$, there exists $n \geqslant 1$ such that $\tilde{\Delta}^{(n)}(x) = 0$. Let us prove that $x \in Im(\omega)$ by induction on n. If $n = 1$, then $x \in V$, so $x = \omega(x)$. Let us assume the result at all ranks $< n$. By coassociativity of $\tilde{\Delta}$, $\tilde{\Delta}^{(n-1)}(x) \in V^{\otimes n}$. We put $\tilde{\Delta}^{(n-1)}(x) = x_1 \otimes \ldots \otimes x_n \in V^{\otimes n}$. Then $\tilde{\Delta}^{(n-1)}(x) = \tilde{\Delta}^{(n-1)}(\omega(x_1, \ldots, x_n))$. By the induction hypothesis, $x - \omega(x_1, \ldots, x_n) \in Im(\omega)$, so $x \in Im(\omega)$.

We proved that the coalgebras A and $T(V)$ are isomorphic. We now assume that $A = T(V)$ as a coalgebra.

Second step. We denote by π the canonical projection on V in $T(V)$. Let $\varpi : T_+(V) \longrightarrow V$ be any linear map. We define

$$F_\varpi : \left\{ \begin{array}{rcl} T(V) & \longrightarrow & T(V) \\ x_1 \ldots x_n & \longrightarrow & \displaystyle\sum_{k=1}^{n} \sum_{i_1+\ldots+i_k=n} \varpi(x_1 \ldots x_{i_1}) \ldots \varpi(x_{i_1+\ldots+i_{k-1}+1} \ldots x_n). \end{array} \right.$$

Let us prove that F_ϖ is the unique coalgebra endomorphism such that $\pi \circ F_\varpi = \varpi$. Firstly,

$$\begin{aligned} \Delta(F_\varpi(x_1 \ldots x_n)) &= \sum_{i_1+\ldots+i_k=n} \Delta(\varpi(x_1 \ldots x_{i_1}) \ldots \varpi(x_{i_1+\ldots+i_{k-1}+1} \ldots x_n)) \\ &= \sum_{i_1+\ldots+i_k=n} \sum_{j=0}^{k} \varpi(x_1 \ldots x_{i_1}) \ldots \varpi(x_{i_1+\ldots+i_{j-1}+1} \ldots x_{i_1+\ldots+i_j}) \\ &\otimes \varpi(x_{i_1+\ldots+i_j+1} \ldots x_{i_1+\ldots i_{j+1}}) \ldots \varpi(x_{i_1+\ldots+i_{k-1}+1} \ldots x_n)) \\ &= \sum_{i=0}^{n} F_\varpi(x_1 \ldots x_i) \otimes F_\varpi(x_{i+1} \ldots x_n) \\ &= (F_\varpi \otimes F_\varpi) \circ \Delta(x_1 \ldots x_n). \end{aligned}$$

Moreover,

$$\begin{aligned} \pi \circ F_\varpi(x_1 \ldots x_n) &= \sum_{k=1}^{n} \sum_{i_1+\ldots+i_k=n} \pi(\varpi(x_1 \ldots x_{i_1}) \ldots \varpi(x_{i_1+\ldots+i_{k-1}+1} \ldots x_n)) \\ &= \pi \circ \varpi(x_1 \ldots x_n) + 0 \\ &= \varpi(x_1 \ldots x_n). \end{aligned}$$

Let us now prove the unicity. Let F, G be two coalgebra endomorphisms such that $\pi \circ F = \pi \circ G = \varpi$. If $F \neq G$, let $x_1 \ldots x_n$ be a word of $T(V)$, such that $F(x_1 \ldots x_n) - G(x_1 \ldots x_n) \neq 0$, of minimal length. By minimality of n,

$$\tilde{\Delta}(F(x_1 \ldots x_n)) = (F \otimes F) \circ \tilde{\Delta}(x_1 \ldots x_n) = (G \otimes G) \circ \tilde{\Delta}(x_1 \ldots x_n) = \tilde{\Delta}(G(x_1 \ldots x_n)).$$

Hence, $F(x_1 \ldots x_n) - G(x_1 \ldots x_n) \in \mathrm{Prim}(T(V)) = V$, so

$$\begin{aligned} F(x_1 \ldots x_n) - G(x_1 \ldots x_n) &= \pi(F(x_1 \ldots x_n) - G(x_1 \ldots x_n)) \\ &= \varpi(x_1 \ldots x_n) - \varpi(x_1 \ldots x_n) \\ &= 0. \end{aligned}$$

This is a contradiction, so $F = G$.

Third step. Let $\varpi_1, \varpi_2 : T_+(V) \longrightarrow V$ and let $F_1 = F_{\varpi_1}$, $F_2 = F_{\varpi_2}$ be the associated coalgebra morphisms. Then

$$\pi \circ F_2 \circ F_1(x_1 \ldots x_n) = \sum_{i_1+\ldots+i_k=n} \varpi_2(\varpi_1(x_1 \ldots x_{i_1}) \ldots \varpi_1(x_{i_1+\ldots+i_{k-1}+1} \ldots x_n)).$$

We denote this map by $\varpi_2 \diamond \varpi_1$. By the unicity in the second step, $F_2 \circ F_1 = F_{\varpi_2 \diamond \varpi_1}$. It is not difficult to prove that for any $\varpi : T_+(V) \longrightarrow V$, there exists $\varpi' : T_+(V) \longrightarrow V$, such that $\varpi' \diamond \varpi = \varpi \diamond \varpi' = \pi$ if and only if $\varpi_{|V}$ is invertible. If this holds, then $F_\varpi \circ F_{\varpi'} = F_{\varpi'} \circ F_\varpi = F_\pi = \mathrm{Id}$, by the unicity in the second step. So, if $\varpi_{|V}$ is invertible, then F_ϖ is invertible.

Fourth step. We denote by $*$ the product of $T(V)$. Let us choose $\varpi : T_+(V) \longrightarrow V$ such that $\varpi(T_+(V) * T_+(V)) = (0)$. Let $F = F_\varpi$ be the associated coalgebra morphism. As $\emptyset$ is the unique group-like element of $T(V)$, the unit of $*$ is $\emptyset$. Let us prove that for all $x, y \in T(V)$, $F(x * y) = F(x) \cdot F(y)$. We proceed by induction on $length(x) + length(y) = n$. As $\emptyset$ is the unit for both $*$ and $\cdot$ and $F(\emptyset) = \emptyset$, it is obvious if x or y is equal to $\emptyset$: this observation covers the case $n = 0$. Let us assume the result at all rank $< n$. By the preceding observation on the unit, we can assume that $x, y \in T_+(V)$. We put $G = F \circ *$ and $H = \cdot \circ (F \otimes F)$. They are both coalgebra morphisms from $T(V) \otimes T(V)$ to $T(V)$. Moreover,

$$\pi \circ G(x \otimes y) = \pi \circ F(x * y) = \varpi(x * y) = 0.$$

As the shuffle product is graded for the length, $\pi \circ H(x \otimes y) = 0$. By the induction hypothesis,

$$\tilde{\Delta} \circ G(x \otimes y) = (G \otimes G) \circ \tilde{\Delta}(x \otimes y) = (F \otimes F) \circ \tilde{\Delta}(x \otimes y) = \tilde{\Delta} \circ F(x \otimes y).$$

Hence, $G(x \otimes y) - F(x \otimes y)$ is primitive, so belongs to V. This implies

$$G(x \otimes y) - F(x \otimes y) = \pi(G(x \otimes y) - F(x \otimes y)) = 0 - 0 = 0.$$

So $F(x * y) = G(x \otimes y) = F(x \otimes y) = F(x) \sqcup\!\sqcup F(y)$. Hence, F is a bialgebra morphism from $(T(V), *, \Delta)$ to $(T(V), \sqcup\!\sqcup, \Delta)$.

By the third and fourth steps, in order to prove that $(T(V), *, \Delta)$ and $(T(V), \sqcup\!\sqcup, \Delta)$ are isomorphic, it is enough to find $\varpi : T_+(V) \longrightarrow V$, such that $\varpi_{|V}$ is invertible and $\varpi(T_+(V) * T_+(V)) = (0)$; hence, it is enough to prove that $V \cap (A_+ * A_+) = (0)$.

Last step. We define $\Delta : \mathrm{End}(A) \longrightarrow \mathrm{End}(A \otimes A, A)$ by $\Delta(f)(x \otimes y) = f(x * y)$. We denote by $\star$ the convolution product of $\mathrm{End}(A)$ induced by the bialgebra $(A, *, \Delta)$. Let $f, g \in \mathrm{End}(A)$. We assume that we can write $\Delta(f) = f^{(1)} \otimes f^{(2)}$ and $\Delta(g) = g^{(1)} \otimes g^{(2)}$, that is to say, for all $x, y \in A$,

$$f(xy) = f^{(1)}(x) * f^{(2)}(y), \qquad g(xy) = g^{(1)}(x) * g^{(2)}(y).$$

Then, as $*$ is commutative,

$$\begin{aligned} f \star g(x * y) &= f(x^{(1)} * y^{(1)}) * g(x^{(2)} * y^{(2)}) \\ &= f^{(1)}(x^{(1)}) * f^{(2)}(y^{(1)}) * g^{(1)}(x^{(2)}) * g^{(2)}(y^{(2)}) \\ &= f^{(1)}(x^{(1)}) * g^{(1)}(x^{(2)}) * f^{(2)}(y^{(1)}) * g^{(2)}(y^{(2)}) \\ &= f^{(1)} \star g^{(1)}(x) * f^{(1)} \star g^{(2)}(y). \end{aligned}$$

Hence, $\Delta(f \star g) = \Delta(f) \star \Delta(g)$.

Let ρ be the canonical projection on A_+ and 1 be the unit of the convolution algebra $\mathrm{End}(V)$. Then $1 + \rho = \mathrm{Id}$. As $\Delta(\mathrm{Id}) = \mathrm{Id} \otimes \mathrm{Id}$ and $\Delta(1) = 1 \otimes 1$, this gives

$$\Delta(\rho) = \rho \otimes 1 + 1 \otimes \rho + \rho \otimes \rho.$$

We consider

$$\psi = \ln(1 + \rho) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \rho^{\star n}.$$

As A is connected, for all $x \in A$, $\rho^{\star n}(x) = 0$ if n is great enough, so ψ exists. Moreover, as Δ is compatible with the convolution product,

$$\begin{aligned} \Delta(\psi) &= \ln(1 \otimes 1 + \rho \otimes 1 + 1 \otimes \rho + \rho \otimes \rho) \\ &= \ln((1 + \rho) \otimes (1 + \rho)) \\ &= \ln(1 + \rho) \otimes 1) + \ln(1 \otimes (1 + \rho)) \\ &= \ln(1 + \rho) \otimes 1 + 1 \otimes \ln(1 + \rho) \\ &= \psi \otimes 1 + 1 \otimes \psi. \end{aligned}$$

We used $((1+\rho)\otimes 1) \star (1 \otimes (1+\rho)) = (1 \otimes (1+\rho)) \star ((1+\rho) \otimes 1) = (1+\rho) \otimes (1+\rho)$ for the third equality. Hence, for all $x, y \in A$,

$$\psi(x * y) = \psi(x)\varepsilon(y) + \varepsilon(x)\psi(y).$$

In particular, if $x, y \in A_+$, $\psi(x * y) = 0$. If $x \in V$, then $\rho^1(x) = x$ and if $n \geqslant 2$,

$$\rho^{*n}(x) = \sum_{i=1}^{n} \rho(1) * \ldots * \rho(1) * \rho(x) * \rho(1) * \ldots * \rho(1) = 0.$$

So $\psi(x) = x$. Finally, if $x \in V \cap (A_+ * A_+)$, $\psi(x) = x = 0$. So $V \cap (A_+ * A_+) = (0)$. □

The following result is proved for $\mathcal{H}_{CK}^{\mathcal{D}}$ in [2] and in [4]:

Corollary 5.12. *The Hopf algebras $CP(\mathcal{D})$ and $\mathcal{H}_{CK}^{\mathcal{D}}$ are isomorphic to shuffle algebras.*

Proof. $CP(\mathcal{D})$ is a connected Com-PreLie bialgebra. Moreover, if $x \in CP(\mathcal{D})$, homogeneous of degree n, $x \bullet \emptyset = nx$. Hence, as the homogeneous component of degree 0 of $\mathrm{Prim}(CP(\mathcal{D}))$ is zero, $f_{CP(\mathcal{D})}$ is invertible. By the rigidity theorem, $CP(\mathcal{D})$ is, as a Hopf algebra, isomorphic to a shuffle algebra. The proof is similar for $\mathcal{H}^{\mathcal{D}}_{CK}$. □

Remark 5.13. (1) This is not the case for $UCP(\mathcal{D})$. For example, if d, e are two distinct elements of $\mathcal{D}$, it is not difficult to prove that there is no element $x \in UCP(\mathcal{D})$ such that

$$\Delta(x) = x \otimes 1 + 1 \otimes x + \bullet_{(0,\,d)} \otimes \bullet_{(0,\,e)}.$$

So $UCP(\mathcal{D})$ is not cofree.

(2) $CP(\mathcal{D})$ and $\mathcal{H}^{\mathcal{D}}_{CK}$ are not isomorphic, as Com-PreLie bialgebras, to any $T(V, f)$. Indeed, in $T(V, f)$, for any $x \in V$ such that $f(x) = x$, $x \sqcup x = 2x \bullet x = 2xx$. In $CP(\mathcal{D})$ or $\mathcal{H}^{\mathcal{D}}_{CK}$, for any $d \in \mathcal{D}$, with $x = \bullet_d$, $f(x) = x$ but $x \cdot x \neq 2x \bullet x$.

5.5. Dual of $UCP(\mathcal{D})$ and $CP(\mathcal{D})$. We identify $UCP(\mathcal{D})$ and its graded dual by considering the basis of partitioned trees as orthonormal. Similarly, we identify $CP(\mathcal{D})$ and $\mathcal{H}^{D}_{CK}$ with their graded dual.

Let us consider the Hopf algebra $(UCP(\mathcal{D}), \cdot, \Delta)$. As a commutative algebra, it is freely generated by the set $\mathcal{UPT}_1(\mathcal{D})$ of partitioned trees decorated by $\mathbb{N} \times \mathcal{D}$ with one root. Moreover, if $T \in \mathcal{UPT}_1(\mathcal{D})$,

$$\Delta(T) - 1 \otimes T \in \mathrm{Vect}(\mathcal{UPT}_1(\mathcal{D})) \otimes UCP(\mathcal{D}).$$

Consequently, this is a right-sided combinatorial bialgebra in the sense of [11], and its graded dual is the enveloping algebra of a preLie algebra $\mathfrak{g}_{UCP}(\mathcal{D})$. Direct computations prove the following result:

Theorem 5.14. *The preLie algebra $\mathfrak{g}_{UCP}(\mathcal{D})$ is the linear span of $\mathcal{UPT}_1(\mathcal{D})$. For any $T, T' \in \mathcal{UPT}_1(\mathcal{D})$, the PreLie product is given by*

$$T \diamond T' = \sum_{\substack{s \in V(T),\\ b \in \mathcal{B}l(s) \sqcup \{*\}}} (T \bullet_{s,b} T')[-1]_s.$$

Example 5.15. *If $\mathcal{D} = \{1\}$, forgetting the second decoration of the vertices, in $\mathfrak{g}_{UCP}(\mathcal{D})$,*

$$\bullet_i \diamond \bullet_j = (1 - \delta_{i,0}) \,\mathbf{I}^{\,j}_{\,i\,-\,1}\,,$$

$$\mathbf{I}^{\,j}_{\,i} \diamond \bullet_k = (1 - \delta_{j,0}) \,\mathbf{I}^{\,k}_{\,j}{}^{-\,1}_{\,i} + (1 - \delta_{i,0}) \left({}^{j}\mathbf{V}^{k}_{i\,-\,1} + {}^{j}\nabla^{k}_{i\,-\,1} \right).$$

Similarly, the Hopf algebra $(CP(\mathcal{D}), \cdot, \Delta)$ is, as a commutative algebra, freely generated by the set $\mathcal{PT}_1(\mathcal{D})$ of partitioned trees decorated by $\mathcal{D}$ with one root. Moreover, if $T \in \mathcal{PT}_1(\mathcal{D})$,

$$\Delta(T) - 1 \otimes T \in \mathrm{Vect}(\mathcal{PT}_1(\mathcal{D})) \otimes CP(\mathcal{D}).$$

Consequently, its graded dual is the enveloping algebra of a preLie algebra $\mathfrak{g}_{CP}(\mathcal{D})$, described by the following theorem:

Theorem 5.16. *The preLie algebra $\mathfrak{g}_{CP}(\mathcal{D})$ is the linear span of $\mathcal{PT}_1(\mathcal{D})$. For any $T, T' \in \mathcal{PT}_1(\mathcal{D})$, the PreLie product is given by*

$$T \diamond T' = \sum_{\substack{s \in V(T),\\ b \in \mathcal{B}l(s) \sqcup \{*\}}} T \bullet_{s,b} T'.$$

Example 5.17. If $\mathcal{D} = \{1\}$, forgetting the decorations, in $\mathfrak{g}_{CP}(\mathcal{D})$,

Notations 5.18. Let $T \in \mathcal{PT}_1(\mathcal{D})$. We can write $T = \bullet_d \bullet (T_1 \times \ldots \times T_k) = B_d(T_1 \ldots T_k)$, where $T_1, \ldots, T_k \in \mathcal{PT}(\mathcal{D})$. Up to a change of indexation, we will always assume that $T_1, \ldots, T_p \in \mathcal{PT}_1(\mathcal{D})$ and $T_{p+1}, \ldots, T_k \notin \mathcal{PT}_1(\mathcal{D})$. The integer p is denoted by $\varsigma(T)$.

Proposition 5.19. *As a preLie algebra, $\mathfrak{g}_{CP}(\mathcal{D})$ is freely generated by the set of trees $T \in \mathcal{PT}_1(\mathcal{D})$ such that $\varsigma(T) = 0$.*

Proof. We define a coproduct on $\mathfrak{g}_{CP}(\mathcal{D})$ by

$$\forall T = B_d(T_1 \ldots T_k) \in \mathcal{PT}_1(\mathcal{D}), \qquad \delta(T) = \sum_{i=1}^{\varsigma(T)} B_d(T_1 \ldots \widehat{T_i} \ldots T_k) \otimes T_i.$$

This coproduct is permutative: indeed,

$$(\delta \otimes \mathrm{Id}) \circ \delta(T) = \sum_{1 \leqslant i \neq j \leqslant \varsigma(T)} B_d(T_1 \ldots \widehat{T_i} \ldots \widehat{T_j} \ldots T_k) \otimes T_i \otimes T_j,$$

so $(\delta \otimes \mathrm{Id}) \circ \delta = (23).(\delta \otimes \mathrm{Id}) \circ \delta$. Let $T = B_d(T_1 \ldots T_k), T' \in \mathcal{PT}_1(\mathcal{D})$. Then

$$T \diamond T' = B_d(T' T_1 \ldots T_k) + \sum_{i=1}^{k} B_d(T_1 \ldots (T_i \diamond T') \ldots T_k) + \sum_{i=1}^{k} B_d(T_1 \ldots (T_i \sqcup T') \ldots T_k).$$

Hence,

$$\begin{aligned}
&\delta(T\otimes T')\\
&=B_d(T_1\dots T_k)\otimes T'+\sum_{i=1}^{\varsigma(T)}B_d(T'T_1\dots\widehat{T_i}\dots T_k)\otimes T_i\\
&+\sum_{i=1}^{k}\sum_{\substack{j=1\\ j\neq i}}^{\varsigma(T)}B_d(T_1\dots\widehat{T_j}\dots(T_i\diamond T')\dots T_k)\otimes T_j+\sum_{i=1}^{\varsigma(T)}B_d(T_1\dots\widehat{T_i}\dots T_k)\otimes T_i\diamond T'\\
&+\sum_{i=1}^{k}\sum_{\substack{j=1\\ j\neq i}}^{\varsigma(T)}B_d(T_1\dots\widehat{T_j}\dots(T_i\sqcup T')\dots T_k)\otimes T_j\\
&=\sum_{j=1}^{\varsigma(T)}\left(B_d(T'T_1\dots\widehat{T_j}\dots T_k)+\sum_{\substack{i=1\\ i\neq j}}^{k}B_d(T_1\dots\widehat{T_j}\dots(T_i\diamond T'+T_i\sqcup T')\dots T_k)\right)\otimes T_j\\
&+\sum_{i=1}^{\varsigma(T)}B_d(T_1\dots\widehat{T_i}\dots T_k)\otimes T_i\diamond T'+T\otimes T'\\
&=\sum_{j=1}^{\varsigma(T)}B_d(T_1\dots\widehat{T_j}\dots T_k)\bullet T'\otimes T_j+\sum_{i=1}^{\varsigma(T)}B_d(T_1\dots\widehat{T_i}\dots T_k)\otimes T_i\diamond T'+T\otimes T'\\
&=T^{(1)}\diamond T'\otimes T^{(2)}+T^{(1)}\otimes T^{(2)}\diamond T'+T\otimes T'.
\end{aligned}$$

By Livernet's rigidity theorem [8], $\mathfrak{g}_{CP}(\mathcal{D})$ is freely generated, as a preLie algebra, by $\mathrm{Ker}(\delta)$.

We define

$$\Upsilon:\left\{\begin{array}{rcl}\mathfrak{g}_{CP}(\mathcal{D})\otimes\mathfrak{g}_{CP}(\mathcal{D})&\longrightarrow&\mathfrak{g}_{CP}(\mathcal{D})\\ T\otimes T'&\longrightarrow&T\bullet_{r(T),*}T',\end{array}\right.$$

where $r(T)$ is the root of T. In other words, $\Upsilon(B_d(T_1\dots T_k)\otimes T')=B_d(T'T_1\dots T_k)$; this implies that for any $T\in\mathcal{PT}_1(\mathcal{D})$, $\Upsilon\circ\delta(T)=\varsigma(T)T$. Hence, if $x=\sum a_T T\in\mathrm{Ker}(\delta)$, $\Upsilon\circ\delta(x)=\sum a_T\varsigma(T)T=0$, so x is a linear span of trees T such that $\varsigma(T)=0$. The converse is trivial. □

We denote by $PT_1^{(0)}(\mathcal{D})$ the set of partitioned trees $T\in\mathcal{PT}_1(\mathcal{D})$ with $\varsigma(T)=0$. The preceding Proposition implies that the Hopf algebras $(CP(\mathcal{D}),\cdot,\Delta)$ and $\left(\mathcal{H}_{CK}^{\mathcal{PT}_1^{(0)}(\mathcal{D})},m,\Delta\right)$ are isomorphic. We obtain an explicit isomorphism between them:

Definition 5.20. Let $T\in\mathcal{PT}(\mathcal{D})$ and $\pi=\{P_1,\dots,P_k\}$ be a partition of $V(T)$. We shall write $\pi\triangleleft T$ if the following condition holds:

- For all $i \in [k]$, the partitioned rooted forest $T_{|P_i}$, denoted by T_i, belongs to $\mathcal{PT}_1^{(0)}(\mathcal{D})$.

If $\pi \lhd T$, the contracted graph T/π is a rooted forest (one forgets about the blocks of T). The vertex of T/π corresponding to P_i is decorated by T_i, making T/π an element of $\mathcal{T}(\mathcal{PT}_1^{(0)}(\mathcal{D}))$.

Corollary 5.21. *The following map is a Hopf algebra isomorphism:*

$$\Theta : \begin{cases} (CP(\mathcal{D}), \cdot, \Delta) & \longrightarrow & \left(\mathcal{H}_{CK}^{\mathcal{PT}_1^{(0)}(\mathcal{D})}, \cdot, \Delta\right) \\ T \in \mathcal{PT}(\mathcal{D}) & \longrightarrow & \displaystyle\sum_{\pi \lhd T} T/\pi. \end{cases}$$

Example 5.22. If $\mathcal{D} = \{1\}$, forgetting the decorations, with $a = \bullet$ and $b =$ ,

5.6. Extension of the preLie product $\diamond$ to all partitioned trees. We now extend the preLie product $\diamond$ to the whole $CP(\mathcal{D})$:

Proposition 5.23. *We define a product on $CP(\mathcal{D})$ by*

$$\forall T, T' \in \mathcal{PT}(\mathcal{D}), \qquad T \diamond T' = \sum_{\substack{s \in V(T),\\ b \in \mathcal{B}l(s) \sqcup \{*\}}} T \bullet_{s,b} T'.$$

Then $(CP(\mathcal{D}), \diamond, \cdot)$ is a Com-PreLie algebra.

Proof. Obviously, for any $x, y, z \in \mathcal{PT}(\mathcal{D})$, $(x \cdot y) \diamond z = (x \diamond z) \cdot x + x \cdot (y \diamond z)$. Let $T_1, T_2, T_3 \in \mathcal{PT}(\mathcal{D})$. Then

$$\begin{aligned}
(T_1 \diamond T_2) \diamond T_3 &= \sum_{\substack{s_1 \in V(T_1),\\ b_1 \in \mathcal{B}l(s_1) \sqcup \{*\}}} \sum_{\substack{s_2 \in V(T_1),\\ b_2 \in \mathcal{B}l(s_2) \sqcup \{*\}}} (T_1 \bullet_{s_1,b_1} T_2) \bullet_{s_2,b_2} T_3 \\
&\quad + \sum_{\substack{s_1 \in V(T_1),\\ b_1 \in \mathcal{B}l(s_1) \sqcup \{*\}}} \sum_{\substack{s_2 \in V(T_2),\\ b_2 \in \mathcal{B}l(s_2) \sqcup \{*\}}} (T_1 \bullet_{s_1,b_1} T_2) \bullet_{s_2,b_2} T_3 \\
&= \sum_{\substack{s_1 \in V(T_1),\\ b_1 \in \mathcal{B}l(s_1) \sqcup \{*\}}} \sum_{\substack{s_2 \in V(T_1),\\ b_2 \in \mathcal{B}l(s_2) \sqcup \{*\}}} (T_1 \bullet_{s_1,b_1} T_2) \bullet_{s_2,b_2} T_3 \\
&\quad + \sum_{\substack{s_1 \in V(T_1),\\ b_1 \in \mathcal{B}l(s_1) \sqcup \{*\}}} \sum_{\substack{s_2 \in V(T_2),\\ b_2 \in \mathcal{B}l(s_2) \sqcup \{*\}}} T_1 \bullet_{s_1,b_1} (T_2 \bullet_{s_2,b_2} T_3) \\
&= \sum_{\substack{s_1 \in V(T_1),\\ b_1 \in \mathcal{B}l(s_1) \sqcup \{*\}}} \sum_{\substack{s_2 \in V(T_1),\\ b_2 \in \mathcal{B}l(s_2) \sqcup \{*\}}} (T_1 \bullet_{s_1,b_1} T_2) \bullet_{s_2,b_2} T_3 + T_1 \diamond (T_2 \diamond T_3).
\end{aligned}$$

Hence,

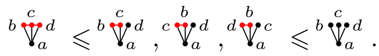
$$\begin{aligned}(T_1 \diamond T_2)\diamond T_3 - T_1 \diamond (T_2 \diamond T_3) &= \sum_{\substack{s_1\in V(T_1),\\ b_1\in \mathcal{B}l(s_1)\sqcup\{*\}}} \sum_{\substack{s_2\in V(T_1),\\ b_2\in \mathcal{B}l(s_2)\sqcup\{*\}}} (T_1 \bullet_{s_1,b_1} T_2)\bullet_{s_2,b_2} T_3\\
&= \sum_{\substack{s_1\neq s_2\in V(T_1)\\ b_1\in \mathcal{B}l(s_1)\sqcup\{*\},\\ b_2\in \mathcal{B}l(s_2)\sqcup\{*\}}} (T_1 \bullet_{s_1,b_1} T_2)\bullet_{s_2,b_2} T_3\\
&+ \sum_{\substack{s\in V(T_1),\\ b_1\neq b_2\in \mathcal{B}l(s)\sqcup\{*\}}} (T_1 \bullet_{s,b_1} T_2)\bullet_{s,b_2} T_3\\
&+ \sum_{\substack{s\in V(T_1),\\ b\in \mathcal{B}l(s)\sqcup\{*\}}} (T_1 \bullet_{s,b} T_2)\bullet_{s,b} T_3.\end{aligned}$$

The three terms of this sum are symmetric in T_2, T_3, so

$$(T_1 \diamond T_2)\diamond T_3 - T_1 \diamond (T_2 \diamond T_3) = (T_1 \diamond T_3)\diamond T_2 - T_1 \diamond (T_3 \diamond T_2).$$

Finally, $(CP(\mathcal{D}),\diamond,\cdot)$ is a Com-PreLie algebra. □

Definition 5.24. Let $T=(t,I,d)$ and $T'=(t,I',d)$ be two elements of $\mathcal{PT}(\mathcal{D})$ with the same underlying decorated rooted trees. We shall say that $T \leqslant T'$ is I' is a refinement of I. This defines a partial order on $\mathcal{PT}(\mathcal{D})$.

Example 5.25. If $a,b,c,d\in\mathcal{D}$, 

Theorem 5.26. *The following map is an isomorphism of Com-PreLie algebras:*

$$\Psi : \left\{ \begin{array}{rcl} (CP(\mathcal{D}),\circ,\cdot) & \longrightarrow & (CP(\mathcal{D}),\diamond,\cdot)\\ T\in\mathcal{PT}(\mathcal{D}) & \longrightarrow & \displaystyle\sum_{T'\leqslant T} T'. \end{array}\right.$$

Proof. As $\leqslant$ is a partial order, Ψ is bijective. Let $T_1,T_2\in\mathcal{PT}(\mathcal{D})$.

(1) If $T'\leqslant T_1\cdot T_2$, let us put $T'_1=T_1\cap T'$ and $T'_2=T_2\cap T'$. Then, obviously, $T'_1\leqslant T_1$ and $T'_2\leqslant T_2$. Moreover, $T'=T'_1\leqslant T'_2$. Conversely, if $T'_1\leqslant T_1$ and $T'_2\leqslant T_2$, then $T'_1\cdot T'_2\leqslant T_1\cdot T_2$. Hence,

$$\Psi(T_1\cdot T_2)=\sum_{T'\leqslant T_1\cdot T_2} T' = \sum_{T'_1\leqslant T_1,\, T'_2\leqslant T_2} T'_1\cdot T'_2 = \Psi(T_1)\cdot\Psi(T_2).$$

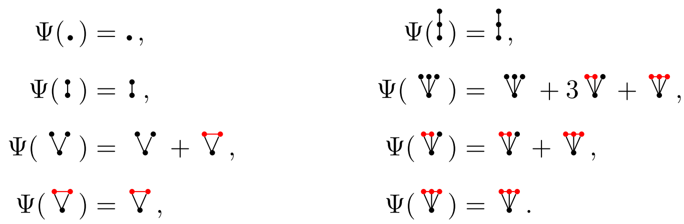
(2) Let $s\in V(T_1)$ and $T'\leqslant T_1\bullet_{s,*}T_2$. We put $T'_1=T'\cap T_1$ and $T'_2=T'\cap T_2$. Then, obviously, $T'_1\leqslant T_1$ and $T'_2\leqslant T_2$. If the block of roots of T_2 is also a block of T', then $T'=T'_1\bullet_{s,*}T'_2$. Otherwise, there exists a unique $b\in\mathcal{B}l(s)$ such that $T'=T'_1\bullet_{s,b}T'_2$. Conversely, if $T'_1\leqslant T_1$, $T'_2\leqslant T_2$, $s\in V(T'_1)$ and $b\in\mathcal{B}l(s)\sqcup\{*\}$,

then $T_1' \bullet_{s,b} T_2' \leqslant T_1 \bullet_{s,*} T_2$. Hence,

$$\begin{aligned}\Psi(T_1 \circ T_2) &= \sum_{s \in V(T_1)} \sum_{T' \leqslant T_1 \bullet_{s,*} T_2} T' \\ &= \sum_{T_1' \leqslant T_1,\, T_2' \leqslant T_2} \sum_{s \in V(T_1'), b \in \mathcal{B}l(s) \sqcup \{*\}} T_1' \bullet_{s,b} T_2' \\ &= \Psi(T_1) \diamond \psi(T_2).\end{aligned}$$

So Ψ is a Com-PreLie algebra isomorphism. $\square$

Example 5.27. In the non-decorated case,



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