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THE DEFORMATION AND CONSTRUCTION OF NIJENHUIS PAIRED MODULES

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Abstract. In this paper, we introduce the notion of Nijenhuis paired module, and give characterizations of Nijenhuis paired modules. Finally, we construct Nijenhuis paired modules from Hopf algebras, Hopf modules, dimodules and weak Hopf modules.

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1. Introduction

As is well-known, the classical bi-Hamiltonian systems have played a significant role in the research of partial differential equations [14], statistics and theoretical mechanics. The conception of Nijenhuis operator on an associative algebra was first proposed by Carinena during the investigation of bi-Hamiltonian systems [3]. However, the conception of the Nijenhuis operator on Lie algebras was introduced by Nijenhuis in the study of a pseudo-complex manifold [16], which was used to study the Poisson-Nijenhuis manifold, as well as the classical Yang-Baxter equation [5,6], etc.

The Nijenhuis operator has a close connection with the well-known Rota-Baxter operator. With the rise of Rota-Baxter algebras, the development of Nijenhuis algebras has also received significant attention. In [11], Guo et al. investigated the relationship between Nijenhuis algebras and NS algebras, as well as N-Dendriform algebras. In [18], Bai et al. studied Nijenhuis operators on pre-Lie algebras. In [15], Bibhash et al. introduced Nijenhuis Leibniz algebras, and studied the cohomology theory of Nijenhuis Leibniz algebras.

In recent years, many experts and scholars have begun to research homological theory on Rota-Baxter algebras. In [9], Guo and Lin introduced the conception

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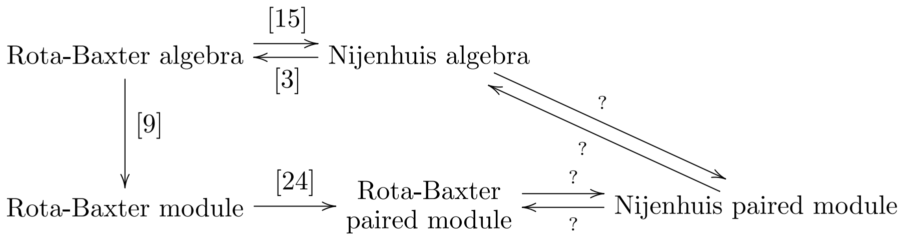
of Rota-Baxter modules on Rota-Baxter algebras and proposed a characterization of the representation theory of Rota-Baxter algebras. Subsequently, Zheng et al. introduced the conception of Rota-Baxter paired modules [24] on (general) associative algebras. They constructed and characterized Rota-Baxter paired modules using integrals and antipode in Hopf algebras. In 2021, Guo, Lang and Sheng introduced Rota-Baxter groups in [8]. Smooth Rota-Baxter operators on Lie groups were proved to be differentiable, which, in turn, led to the derivation of the factorization theorem of Semenov-Tian-Shansky for Lie groups via the factorization theorem for Rota-Baxter Lie groups. Subsequently, the concept of Rota-Baxter systems of groups [13] proposed by Zhonghua Li and Shukun Wang, and the concept of Rota-Baxter skew braces [20] proposed by Ximu Wang, Chongxia Zhang and Liangyun Zhang, were a generalization of Rota-Baxter groups.

With the development of Rota-Baxter algebras, Nijenhuis algebras have been largely developed. Therefore, it is necessary to explore the connection between Rota-Baxter paired modules and Nijenhuis paired modules to fill the research gap in the above fields. The research background and content of this paper are summarized in the diagram: the arrow with a citation above is the research background, and the arrow with a question mark above is the main research content.



This paper is arranged and organized as follows: In Section 2, we introduce the concept of Nijenhuis paired modules. In Section 3, we present the properties and characterizations of Nijenhuis paired modules. In Section 4, we construct some Nijenhuis paired modules from Hopf algebres, Hopf modules, dimodules and weak Hopf modules.

The objects discussed in this paper are all considered on the field K. Here, id denotes the identity map, and Sweedler′s notations [17] are used for algebras and modules.

2. Basic definitions

Definition 2.1. A **Rota-Baxter algebra** [7] is an algebra A with a linear operator P on A that satisfies the **Rota-Baxter identity**

$$P(x)P(y) = P(P(x)y + xP(y) + \lambda xy) \tag{1}$$

for all $x, y \in A$, where λ called the **weight**, is a fixed element in the ground field K of the algebra A.

Definition 2.2. Fix a $\lambda \in K$. Let A be an algebra and M a left A-module. A pair (P, T) of linear maps $P : A \to A$ and $T : M \to M$ is called a **Rota-Baxter paired operator** [24] of weight λ on (A, M) or simply on M if

$$P(a) \cdot T(m) = T(P(a) \cdot m) + T(a \cdot T(m)) + \lambda T(a \cdot m) \tag{2}$$

for all $a \in A, m \in M$.

Definition 2.3. A **Nijenhuis algebra** [3] is an algebra A with a linear operator N on A that satisfies the **Nijenhuis equation**

$$N(x)N(y) + N^2(xy) = N(N(x)y + xN(y)) \tag{3}$$

for all $x, y \in A$.

We give some examples of Nijenhuis algebras.

Example 2.4. (a) Let A be an associative algbera, take $a \in A$. Define two linear map as follows:

$$L_a : A \to A, L_a(x) = ax; R_a : A \to A, R_a(y) = ya,$$

for all $x, y \in A$. Then L_a and R_a are Nijenhuis operators on A.
(b) Let (A, P) be a Baxter algebra (that is, (A, P) is a Rota-Baxter algebra with weight 0). If P is idempotent, then (A, P) is a Nijenhuis algebra.
(c) Let (A, μ) be an associative algebra, (A, Δ) be a coassociative coalgebra. If the following equation:

$$\Delta(ab) = \Delta(a)b + a\Delta(b) - \mu(\Delta(a)) \otimes b \tag{4}$$

holds, for all $a, b \in A$, we call (A, μ, Δ) a **Dendriform-Nijenhuis bialgebra** [12]. Let (A, μ, Δ) be a Dendriform-Nijenhuis bialgebra. Then $(\mathrm{End}(A), \beta)$ is a Nijenhuis algebra [12, Proposition 3.2],
where

$$\beta : \mathrm{End}(A) \to \mathrm{End}(A), \beta(f) = \mathrm{id} * f.$$

(d) Let A be a bialgebra, H a Hopf algebra, and $i : H \to A, \pi : A \to H$ be two bialgebra maps satisfying $\pi \circ i = \mathrm{id}$. Then, by [10], $\Pi \equiv \mathrm{id}*(i \circ \pi \circ S)$ is an idempotent Rota-Baxter operator of weight -1 on the algebra $\mathrm{End}(A)$. Thus Π is a Nijenhuis operator on $\mathrm{End}(A)$ by Definitions 2.1 and 2.3.
Here S denotes the antipode of Hopf algebra H, and "$*$" denotes the convolutional multiplication on the algebra $\mathrm{End}(A)$.

3. Nijenhuis paired modules

Definition 3.1. Fix a $\lambda \in K$. Let A be an algebra and M a left A-module. A pair (N, T) of linear maps $N : A \to A$ and $T : M \to M$ is called a **Nijenhuis paired operator** on (A, M) or simply on M if

$$N(a) \cdot T(m) = T(N(a) \cdot m) + T(a \cdot T(m)) - T^2(a \cdot m) \tag{5}$$

for all $a \in A, m \in M$. We then call the triple (M, N, T) a **Nijenhuis paired A-module**.

For a given linear operator $T : M \to M$, if for every linear map $N : A \to A$, (M, N, T) is a Nijenhuis paired A-module, then (M, T) is called a **generic Nijenhuis paired module**.

A Nijenhuis paired A-submodule V of Nijenhuis paired A-module (M, N, T) is a submodule V of M such that $T(V) \subseteq V$.

Let (M, N, T) and (M', N', T') be Nijenhuis paired A-modules. **A Nijenhuis paired module map** $f : (M, N, T) \to (M', N', T')$ is a module map such that $f \circ T = T' \circ f$.

Let f be a Nijenhuis paired module map from (M, N, T) to (M', N', T'). Then, it is not difficult to prove that $\ker(f)$ and $\operatorname{im}(f)$ are Nijenhuis paired A-submodules of M and M', respectively.

A **right (generic) Nijenhuis paired A-module** can be similarly defined.

Example 3.2. (a) Let A be an algebra regarded as a left A-module. If (A, N) is a Nijenhuis algebra, then (A, N, N) is a Nijenhuis paired A-module.
(b) Let (M, N, T) be a Rota-Baxter paired A-module of weight -1. If Rota-Baxter paired operator T is idempotent, then (M, N, T) is a Nijenhuis paired A-module by Definitions 2.3 and 3.1.
(c) Let (M, N, T) be a Nijenhuis paired A-module. Then $(M, \mu N, \mu T)$ is also a Nijenhuis paired A-module, where $\mu \in K$.

Lemma 3.3. *Let M be a left A-module. Then (M, T) is a generic Nijenhuis paired A-module, if there exists an A-linear map $T : M \to M$.*

Proof. For every linear map $N : A \to A$, since T is an A-linear map, we have

$$\begin{aligned} N(a) \cdot T(m) &= N(a) \cdot T(m) + a \cdot T^2(m) - a \cdot T^2(m) \\ &= T(N(a) \cdot m) + T(a \cdot T(m)) - T^2(a \cdot m), \end{aligned}$$

for any $a \in A, m \in M$. Hence, (M, T) is a generic Nijenhuis paired A-module by Definition 3.1. $\square$

Lemma 3.4. *Let M be a left A-module, and $T : M \to M$ a K-linear map. Define*

$$L_M(A) := \{a \in A \mid T(a \cdot m) = a \cdot T(m), \forall m \in M\}.$$

Then $L_M(A)$ is a subalgebra of A. Thus T is $L_M(A)$-linear.

Proof. For any $a, b \in L_M(A), m \in M$, we have

$$T((ab) \cdot m) = T(a \cdot (b \cdot m)) = a \cdot T(b \cdot m) = (ab) \cdot T(m),$$

namely, $ab \in L_M(A)$. So $L_M(A)$ is a subalgebra of A, and T is $L_M(A)$-linear. □

Theorem 3.5. *Let M be a left A-module and $T : M \to M$ be a K-linear map. Then (M, T) is a generic Nijenhuis paired $L_M(A)$-module.*

Proof. It follows from the combining Lemmas 3.3 and 3.4. □

Proposition 3.6. *Let (M, N, T) be a Nijenhuis paired A-module. Then $(M, \widetilde{N}, \widetilde{T})$ is also a Nijenhuis paired A-module, where $\widetilde{N} = -k\mathrm{id} - N, \widetilde{T} = -k\mathrm{id} - T$, $k \in K$.*

Proof. In fact, for any $a \in A, m \in M$, we have

$$\begin{aligned}
\widetilde{N}(a) \cdot \widetilde{T}(m) &= (-k\mathrm{id} - N)(a) \cdot (-k\mathrm{id} - T)(m) \\
&= (-ka - N(a)) \cdot (-km - T(m)) \\
&= k^2 a \cdot m + ka \cdot T(m) + kN(a) \cdot m + N(a) \cdot T(m), \\
\widetilde{T}(\widetilde{N}(a) \cdot m) &= (-k\mathrm{id} - T)((-k\mathrm{id} - N)(a) \cdot m) \\
&= (-k\mathrm{id} - T) \cdot ((-ka - N(a)) \cdot m) \\
&= (-k\mathrm{id} - T) \cdot (-ka \cdot m - N(a) \cdot m) \\
&= k^2 a \cdot m + kT(a \cdot m) + kN(a) \cdot m + T(N(a) \cdot m), \\
\widetilde{T}(a \cdot \widetilde{T}(m)) &= (-k\mathrm{id} - T)(a \cdot (-k\mathrm{id} - T)(m)) \\
&= (-k\mathrm{id} - T)(-ka \cdot m - a \cdot T(m)) \\
&= k^2 a \cdot m + ka \cdot T(m) + kT(a \cdot m) + T(a \cdot T(m)), \\
\widetilde{T}^2(a \cdot m) &= (-k\mathrm{id} - T)^2(a \cdot m) \\
&= (-k\mathrm{id} - T)(-ka \cdot m - T(a \cdot m)) \\
&= k^2 a \cdot m + kT(a \cdot m) + kT(a \cdot m) + T^2(a \cdot m).
\end{aligned}$$

Let (M, N, T) be a Nijenhuis paired A-module. Then it is easy to see that

$$\widetilde{N}(a) \cdot \widetilde{T}(m) = \widetilde{T}(\widetilde{N}(a) \cdot m) + \widetilde{T}(a \cdot \widetilde{T}(m)) - \widetilde{T}^2(a \cdot m).$$

Therefore, $(M, \widetilde{N}, \widetilde{T})$ is a Nijenhuis paired A-module. □

Proposition 3.7. *Let (M, N, T) be a Nijenhuis paired A-module. Then $(M, N, \phi^{-1}T\phi)$ is a Nijenhuis paired A-module, if there exists an automorphism of A-module $\phi : M \to M$.*

Proof. For any $a \in A, m \in M$, we have

$$\begin{aligned}
&\phi^{-1}T\phi(N(a)\cdot m + a\cdot \phi^{-1}T\phi(m) - \phi^{-1}T\phi(a\cdot m))\\
&= \phi^{-1}(T(N(a)\cdot\phi(m) + a\cdot T\phi(m) - T(a\cdot\phi(m)))\\
&= \phi^{-1}(N(a)\cdot T(\phi(m)))\\
&= N(a)\cdot\phi^{-1}T\phi(m).
\end{aligned}$$

Thus

$$N(a)\cdot\phi^{-1}T\phi(m) = \phi^{-1}T\phi(N(a)\cdot m + a\cdot\phi^{-1}T\phi(m) - \phi^{-1}T\phi(a\cdot m)).$$

That is, $(M, N, \phi^{-1}T\phi)$ is a Nijenhuis paired A-module. □

Proposition 3.8. *Let (M_i, N, T_i) be a family of Nijenhuis paired A-modules, and $\bigoplus_{i\in I} M_i$ denote the direct sum of A-modules M_i, $i \in I$. Define*

$$\widetilde{T} : \bigoplus_{i\in I} M_i \to \bigoplus_{i\in I} M_i, (m_i) \mapsto (T_i(m_i)).$$

Then $(\bigoplus_{i\in I} M_i, N, \widetilde{T})$ is a Nijenhuis paired A-module.

Proof. For all $a \in A, (m_i) \in \bigoplus_{i\in I} M_i$, we have

$$\begin{aligned}
N(a)\cdot\widetilde{T}(m_i) &= N(a)\cdot(T_i(m_i)) = (N(a)\cdot T_i(m_i))\\
&= (T_i(N(a)\cdot m_i) + T_i(a\cdot T_i(m_i))) - T^2(a\cdot m_i)\\
&= \widetilde{T}(N(a)\cdot(m_i)) + \widetilde{T}(a\cdot\widetilde{T}(m_i)) - \widetilde{T}^2(a\cdot(m_i)).
\end{aligned}$$

Thus $(\bigoplus_{i\in I} M_i, N, \widetilde{T})$ is a Nijenhuis paired A-module. □

Proposition 3.9. *Let H be a bialgebra with a counit ε, M be a left H-module, and $M^H = \{m \in M \mid h\cdot m = \varepsilon(h)m, \forall h \in H\}$. Then (M^H, T) is a generic Nijenhuis paired H-module, if there exists a left H-module map $T : M \to M$.*

Proof. Since H is a bialgebra, ε is an algebra map. Hence M^H is a left H-module. For all $h \in H, m \in M^H$, we have

$$h\cdot T(m) = T(h\cdot m) = T(\varepsilon(h)m) = \varepsilon(h)T(m),$$

thus $T(m) \in M^H$, and

$$\begin{aligned}
&T(N(h)\cdot m + h\cdot T(m) - T(h\cdot m))\\
&= T(\varepsilon(N(h))\cdot m) + T(\varepsilon(h)T(m)) - T^2(\varepsilon(h)m)\\
&= \varepsilon(N(h))T(m) + \varepsilon(h)T^2(m) - \varepsilon(h)T^2(m)\\
&= \varepsilon(N(h))T(m).
\end{aligned}$$

Namely,

$$N(h)\cdot T(m) = T(N(h)\cdot m + h\cdot T(m) - T(h\cdot m)).$$

Hence (M^H, T) is a generic Nijenhuis paired H-module. □

Proposition 3.10. *Let (M, N, T) be a Nijenhuis paired A-module. Define a linear map as follows:*

$$\overline{T} : \mathrm{End}(M) \to \mathrm{End}(M), \overline{T}(f)(m) = f(T(m)).$$

Then $(\mathrm{End}(M), \overline{T}, \overline{T})$ *is a Nijenhuis paired* $\mathrm{End}(M)$*-module.*

Proof. By Example 3.2, it suffices to show that $(\mathrm{End}(M), \overline{T})$ is a Nijenhuis algebra. As a matter of fact, for any $f, g \in \mathrm{End}(M), m \in M$, we have

$$(\overline{T}(f)\overline{T}(g))(m) = \overline{T}(f)(g(T(m))) = f(T(g(T(m)))),$$

$$\begin{aligned}
&\overline{T}((\overline{T}(f)g) + f\overline{T}(g) - \overline{T}(fg))(m)\\
&= (\overline{T}(f)g + f\overline{T}(g) - \overline{T}(fg))T(m)\\
&= \overline{T}(f)(g(T(m))) + f\overline{T}(g)(T(m)) - \overline{T}(fg)(T(m))\\
&= f(T(g(T(m)))) + \underbrace{f(g(T^2(m))) - fg(T^2(m))}\\
&= f(T(g(T(m)))).
\end{aligned}$$

Thus $(\mathrm{End}(M), \overline{T})$ is a Nijenhuis algebra by Definition 2.3. □

Remark 3.11. In the above proposition, Nijenhuis operator can also be defined as follows:

$$\overline{T} : \mathrm{End}(M) \to \mathrm{End}(M), \overline{T}(f)(m) = T(f(m)).$$

This is since for any $f, g \in \mathrm{End}(M), m \in M$, we have

$$(\overline{T}(f)\overline{T}(g))(m) = \overline{T}(f)(T(g(m)) = T(f(T(g(m)))),$$

and

$$
\begin{aligned}
&\overline{T}((\overline{T}(f)g)+f\overline{T}(g)-\overline{T}(fg))(m)\\
&=\overline{T}((\overline{T}(f)g)+f\overline{T}(g)-\overline{T}(fg)(m))\\
&=\overline{T}(T(f(g(m)))+f(T(g(m)))-T(f(g(m))))\\
&=T^2(f(g(m)))+T(f(T(g(m))))-T^2(f(g(m)))\\
&=T(f(T(g(m)))).
\end{aligned}
$$

Namely,

$$\overline{T}(f)\overline{T}(g)=\overline{T}((\overline{T}(f)g)+f\overline{T}(g)-\overline{T}(fg)).$$

Hence $(\mathrm{End}(M),\overline{T})$ is a Nijenhuis algebra, and $(\mathrm{End}(M),\overline{T},\overline{T})$ is a Nijenhuis paired $\mathrm{End}(M)$-module.

Proposition 3.12. *Let (A,N) be a Nijenhuis algebra, and (M,N,T) be a Nijenhuis paired A-module. Define another binary operator $\star$ on A by*

$$a\star b=N(a)b+aN(b)-N(ab),$$

and another operator $\triangleright$ between A and M by

$$a\triangleright m=N(a)\cdot m+a\cdot T(m)-T(a\cdot m),$$

for $a,b\in A, m\in M$. Then the following conclusions hold.

(a) $T(a\triangleright m)=N(a)\cdot T(m)$.
(b) $(M,\triangleright)$ *is an* $(L_M(A),\star)$*-module.*
(c) $(M,N,T,\triangleright)$ *is a Nijenhuis paired* $(L_M(A),\star)$*-module.*

Proof. (a) Since (M,N,T) is a Nijenhuis paired A-module, we have

$$
\begin{aligned}
T(a\triangleright m)&=T(N(a)\cdot m+a\cdot T(m)-T(a\cdot m))\\
&=N(a)\cdot T(m)
\end{aligned}
$$

for any $a,b\in A, m\in M$.
(b) In fact, for any $a,b\in L_M(A), m\in M$, it suffices to show that $(a\star b)\triangleright m=$

$a \triangleright (b \triangleright m)$:

$$
\begin{aligned}
(a \star b) \triangleright m &= N(a \star b) \cdot m + (a \star b) \cdot T(m) - T((a \star b) \cdot m) \\
&= (N(a)N(b)) \cdot m + (N(a)b + aN(b) - N(ab)) \cdot T(m) \\
&- T((N(a)b + aN(b) - N(ab)) \cdot m \\
&= (N(a)N(b) \cdot m + (N(a)b) \cdot T(m) + (aN(b)) \cdot T(m) \\
&- N(ab) \cdot T(m)) - T((N(a)b + aN(b) - N(ab)) \cdot m) \\
&= (N(a)N(b) \cdot m + (N(a)b) \cdot T(m) + (aN(b)) \cdot T(m) \\
&- N(ab) \cdot T(m) - (N(a)b) \cdot T(m)) - (aN(b)) \cdot T(m) \\
&+ N(ab) \cdot T(m) = (N(a)N(b)) \cdot m,
\end{aligned}
$$

$$
\begin{aligned}
a \triangleright (b \triangleright m) &= a \triangleright (N(b) \cdot m + b \cdot T(m) - T(b \cdot m)) \\
&= (N(a)N(b)) \cdot m + (N(a)b) \cdot T(m) - N(a) \cdot T(b \cdot m) \\
&+ a \cdot T(N(b) \cdot m) + a \cdot T(b \cdot m) - a \cdot T^2(b \cdot m) - T((aN(b) \cdot m) \\
&- T(ab \cdot T(m)) + T(a \cdot T(b \cdot m)) \\
&= N(a)N(b) \cdot m + N(a)b \cdot m - N(a)b \cdot T(m) + aN(b) \cdot T(m) \\
&+ ab \cdot T^2(m) - ab \cdot T^2(m) - aN(b) \cdot T(m) - ab \cdot T^2(m) \\
&+ ab \cdot T^2(m) = (N(a)N(b)) \cdot m.
\end{aligned}
$$

(c) It follows that $(M, \triangleright)$ is an $(L_M(A), \star)$-module by (b), and for any $a \in L_M(A), m \in M$, we can obtain the following equation by (a):

$$
\begin{aligned}
N(a) \triangleright T(m) &= N^2(a) \cdot T(m) + N(a) \cdot T^2(m) - T(N(a) \cdot T(m)) \\
&= N^2(a) \cdot T(m) + N(a) \cdot T^2(m) - T^2(a \triangleright m) \\
&= T(N(a) \triangleright m) + T(a \triangleright T(m)) - T^2(a \triangleright m).
\end{aligned}
$$

Hence $(M, N, T, \triangleright)$ is a Nijenhuis paired $(L_M(A), \star)$-module. □

Proposition 3.13. *Let M be a left A-module, $0 \neq \xi \in A$. Define a linear map as follows:*

$$N_\xi : A \to A, N_\xi(a) = \xi a, T_\xi : M \to M, T_\xi(m) = \xi \cdot m.$$

Then (M, N_ξ, T_ξ) is a Nijenhuis paired A-module.
Furthermore, if A is a commutative algebra, then (M, T_ξ) is a generic Nijenhuis paired A-module.

Proof. In fact, for any $a \in A, m \in M$, we have

$$N_\xi(a) \cdot T_\xi(m) = \xi a \cdot \xi \cdot m = \xi a \xi \cdot m,$$

$$\begin{aligned} T_\xi(N_\xi(a)\cdot m)+T_\xi(a\cdot T_\xi(m)) &= T_\xi(\xi a\cdot m)+T_\xi(a\xi\cdot m)\\ &= \xi^2 a\cdot m+\xi a\xi\cdot m, \end{aligned}$$

$$T_\xi^2(a\cdot m)=T_\xi(\xi a\cdot m)=\xi^2 a\cdot m,$$

so

$$N_\xi(a)\cdot T_\xi(m)+T_\xi^2(a\cdot m)=T_\xi(N_\xi(a)\cdot m)+T_\xi(a\cdot T_\xi(m)).$$

Namely, (M, N_ξ, T_ξ) is a Nijenhuis paired A-module.
When A is commutative, it is easy to see that T_ξ is A-linear. So, according to Lemma 3.3, we know that (M, T_ξ) is a generic Nijenhuis paired A-module. □

Remark 3.14. Let H be a weak bialgebra (it is both an algebra and a coalgebra satisfying the weak condition "$\Delta(xy)=\Delta(x)\Delta(y)$ but $\Delta(1_H)\neq 1_H\otimes 1_H$ for $x,y\in H$", see [1]). Then, H^* and H^R are algebras, where H^* denotes the linear dual algebra of H, and H^R the source algebra of H in [1] (that is, $H^R=\mathrm{Im}\sqcap^R$, where $\sqcap^R: H\to H$ is given by $\sqcap^R(h)=1_1\varepsilon(h1_2)$). Moreover, we have the following conclusions.

(1) H is a left H^*-module by the defined action: $f\rightharpoonup h=<f,h_2>h_1$, for $h\in H, f\in H^*$.

Therefore, by the above proposition, for any nonzero element $\xi\in H^*$, we know that (H, N_ξ, T_ξ) is a Nijenhuis paired H^*-module.

Here $N_\xi: H^*\to H^*, N_\xi(f)=\xi f, T_\xi: H\to H, T_\xi(h)=\xi\rightharpoonup h$.

(2) H is a left H^R-module by the multiplication of H. Thus, by the above proposition, for any nonzero element $\chi\in H^R$, we know that (H, N_χ, T_χ) is a Nijenhuis paired H^R-module.

Here $N_\chi: H^R\to H^R, N_\chi(x)=\chi x, T_\chi: H\to H, T_\chi(h)=\chi h$.

In the following, we establish the relations between Nijenhuis paired modules and Rota-Baxter paired modules.

Proposition 3.15. *Let M be a left A-module, and $N: A\to A, T: M\to M$ two linear maps. Then the following conclusions hold.*

(a) *If $T^2=0$, then (N,T) is a Nijenhuis paired operator on the A-module M if and only if (N,T) is a Rota-Baxter paired operator of weight 0 on the A-module M.*

(b) *If $T^2=T$, then (N,T) is a Nijenhuis paired operator on the A-module M if and only if (N,T) is a Rota-Baxter paired operator of weight -1 on the A-module M.*

(c) *If* $T^2 = \mathrm{id}$*, then* (N, T) *is a Nijenhuis paired operator on the* A*-module* M *if and only if* $(N + \mathrm{id}, T + \mathrm{id})$ *is a Rota-Baxter paired operator of weight* -2 *on the* A*-module* M*.*

(d) *If* $T^2 = \mathrm{id}$*, then* (N, T) *is a Nijenhuis paired operator on the* A*-module* M *if and only if* (N, T) *is a modified Rota-Baxter paired operator* [4] *of weight* -1 *on the* A*-module* M*.*

Proof. (a) If $T^2 = 0$, then the conclusion holds by Definitions 2.2 and 3.1.
(b) If $T^2 = T$, and (M, T) is a Nijenhuis paired operator on the A-module M, we have

$$N(a) \cdot T(m) = T(N(a) \cdot m) + T(a \cdot T(m)) - T(a \cdot m)$$

for all $a \in A, m \in M$. Namely, (N, T) is a Rota-Baxter paired operator of weight -1 on the A-module M.

Similarly, the converse holds.

(c) In fact, for any $a \in A, m \in M$, we have

$$(N + \mathrm{id})(a) \cdot (T + \mathrm{id})(m) = N(a) \cdot T(m) + N(a) \cdot m + a \cdot T(m) + a \cdot m,$$

and

$$\begin{aligned}
&(T + \mathrm{id})((N + \mathrm{id})(a) \cdot m) + (T + \mathrm{id})(a \cdot (T + \mathrm{id})(m)) - 2(T + \mathrm{id})(a \cdot m) \\
&= (T + \mathrm{id})(N(a) \cdot m + a \cdot m) \\
&+ (T + \mathrm{id})(a \cdot T(m) + a \cdot m) - 2(T(a \cdot m) + a \cdot m) \\
&= T(N(a) \cdot m) + T(a \cdot m) + N(a) \cdot m + a \cdot m + T(a \cdot T(m)) \\
&+ T(a \cdot m) + a \cdot T(m) + a \cdot m - 2T(a \cdot m) - 2a \cdot m \\
&= T(N(a) \cdot m) + a \cdot T(m) + N(a) \cdot m + T(a \cdot T(m)),
\end{aligned}$$

if $(N + \mathrm{id}, T + \mathrm{id})$ is a Rota-Baxter paired operator of weight -2 on the A-module M, then, according to $T^2 = \mathrm{id}$, we have

$$N(a) \cdot T(m) + T^2(a \cdot m) = T(N(a) \cdot m) + T(a \cdot T(m)).$$

Namely, (N, T) is a Nijenhuis paired operator on the A-module M.

Similarly, the converse holds.

(d) If (N, T) is a Nijenhuis paired operator on the A-module M, then, by $T^2 = \mathrm{id}$, we have

$$N(a) \cdot T(m) = T(N(a) \cdot m) + T(a \cdot T(m)) - a \cdot m$$

for any $a \in A, m \in M$. Namely, (N, T) is a modified Rota-Baxter paired operator of weight -1.

The converse can be similarly proved. □

4. Constructions of Nijenhuis paired modules

In this section, we construct some Nijenhuis paired modules from Hopf algebres, Hopf modules, dimodules and weak Hopf modules.

4.1. The constructions on Hopf algebras.

Definition 4.1. (a) Let H be a bialgebra with a counit ε, and A an algebra. If $(A, \cdot)$ is a left H-module, which satisfies the following conditions:
(i) $h \cdot (ab) = (h_1 \cdot a)(h_2 \cdot b)$,
(ii) $h \cdot (1_A) = \varepsilon(h)1_A$,
then A is called **left H-module algebra**.

(b) Let A be a left H-module algebra, and define a **smash product** $A\#H = A \otimes H$ as a vector space with the following product:

$$(a\#h)(b\#g) = a(h_1 \cdot b)\#h_2g.$$

Then $A\#H$ is an associative algebra with unity $1_A\#1_H$ by [17].
Note that A and H are subalgebras of $A\#H$.

For a given bialgebra H, if there exists an element $x \in H$ satisfying $hx = \varepsilon(h)x$ for any $x \in H$, then x is called a **left integral** [17].

Furthermore, if H is a finite dimensional semisimple Hopf algebra, then there exists a left integral e, such that $\varepsilon(e) = 1$ by [17, Theorem 5.18].

The following lemma can be seen in [24, Corollary 3.2].

Lemma 4.2. *Let H be a finite dimensional semisimple Hopf algebra with a left integral e, and M a left H-module. Define a linear map as follows:*

$$T : M \to M, T(m) = e \cdot m.$$

Then $T^2 = T$, and (M, T) is a generic Rota-Baxter paired H-module of weight -1.

Proposition 4.3. *Let H be a finite dimensional semisimple Hopf algebra with a left integral e, and M a left H-module. Define a linear map as follows:*

$$T : M \to M, T(m) = e \cdot m.$$

Then $(T(M), T)$ is a generic Nijenhuis paired H-module.

Proof. It follows that T is idempotent by Lemma 4.2, thus $T(T(M)) \subseteq T(M)$. Moreover for any $h \in H, m \in M$, and for every linear map $N : H \to H$, we have

$$\begin{aligned} & T(N(h) \cdot T(m)) + T(h \cdot T^2(m)) - T^2(h \cdot T(m)) \\ &= T(N(h)e \cdot m) + T(he \cdot m) - T(he \cdot m) \\ &= eN(h)e \cdot m + ehe \cdot m - ehe \cdot m \\ &= eN(h)e \cdot m = \varepsilon(N(h))e \cdot m \\ &= N(h)e \cdot m = N(h) \cdot T(m) \\ &= N(h) \cdot T^2(m). \end{aligned}$$

Thus, $(T(M), T)$ is a generic Nijenhuis paired H-module. □

Proposition 4.4. (1) *Let H be a finite dimensional semisimple Hopf algebra with a left integral e. Then the following conclusions hold.*

(a) *Let M be a left H-module. Define a linear map as follows*

$$T : M \to M, T(m) = e \cdot m.$$

Then (M, T) is a generic Nijenhuis paired H-module.

(b) *Let $A\#H$ be a smash product. Define the following linear map:*

$$\widetilde{T} : A\#H \to A\#H, (a\#h) \mapsto e_1 \cdot a\#e_2h.$$

Then $(A\#H, \widetilde{T})$ is a generic Nijenhuis paired H-module, whose module structure is given by its multiplication.

(2) *Let $A\#H$ be a smash product. Define the following linear map:*

$$\widehat{T} : A\#H \to A\#H, (a\#h) \mapsto a\#he.$$

Then $(A\#H, \widehat{T})$ is a generic Nijenhuis paired A-module, whose module structure is given by its multiplication.

Proof. (a) It follows by Lemma 4.2 and Example 3.2.
(b) It is easy to see that $\widetilde{T}(a\#h) = (1\#e)(a\#h) = e(a\#h)$, so (b) holds by (a).
(2) It is easy to find that $\widehat{T}$ is a left A-module map, so the conclusion (c) holds by Lemma 3.3. □

4.2. The constructions on Hopf modules.

Definition 4.5. Let H be a bialgebra. Then M is called a left **H-Hopf module** [17] if M is a left H-module and a left H-comodule satisfying the following condition:

$$\rho(h \cdot m) = h_1 m_{(-1)} \otimes h_2 \cdot m_{(0)} \tag{6}$$

for all $h \in H, m \in M$.

Proposition 4.6. *Let H be a Hopf algebra with antipode S, M a left H-Hopf module, and $N : H \to H$ an algebra map. Define a linear map as follows:*

$$T : M \to M, m \mapsto S(m_{(-1)}) \cdot m_{(0)}.$$

Then (M, N, T) is a Nijenhuis paired H-module if and only if T is a left $N(H)$-module map.

Proof. In fact, for any $h \in H, m \in M$, we have

$$\begin{aligned} T(h \cdot m) &= S(h_1 m_{(-1)}) \cdot (h_2 \cdot m_{(0)}) = S(m_{(-1)})S(h_1)h_2 \cdot m_{(0)} \\ &= \varepsilon(h)S(m_{(-1)}) \cdot m_{(0)} = \varepsilon(h)T(m). \end{aligned}$$

Thus, we have

$$\begin{aligned} &T(N(h) \cdot m + h \cdot T(m) - T(h \cdot m)) \\ &= \varepsilon(N(h))T(m) + \varepsilon(h)T^2(m) - \varepsilon(h)T^2(m) \\ &= \varepsilon(N(h))T(m). \end{aligned}$$

Assume that (M, N, T) is a Nijenhuis paired H-module. Then, by the above proof, we obtain that

$$N(h) \cdot T(m) = \varepsilon(N(h))T(m) = T(N(h) \cdot m)$$

for any $h \in H, m \in M$, so T is a left $N(H)$-module map.
Conversely, the proof is obvious. □

Remark 4.7. Assume that M is a left H-Hopf module algebra. Then, by [10, Theorem 2.4], (M, T) is a Rota-Baxter algebra of weight -1, where T is given as in the above proposition. Since M is a Hopf module, we easily prove that $T^2 = T$. Thus (M, T, T) is a Nijenhuis paired M-module by Example 3.2.

4.3. The constructions on dimodules.

Definition 4.8. Let H be a bialgebra, and M a left H-module as well as a right H-comodule. Then M is called a left, right H**-dimodule** [2] if the following condition is satisfied:

$$\rho(h \cdot m) = h \cdot m_{(0)} \otimes m_{(1)} \tag{7}$$

for all $h \in H, m \in M$.

Proposition 4.9. *Let H be a bialgebra, M a left, right H-dimodule, $0 \neq \tau \in H^*$. Define a linear map as follows:*

$$T : M \to M, T(m) = m_{(0)}\tau(m_{(1)}).$$

Then (M,T) is a generic Nijenhuis paired H-module.

Proof. Since M is a left, right H-dimodule, we have

$$T(h\cdot m)=(h\cdot m)_{(0)}\tau((h\cdot m)_{(1)})=h\cdot m_{(0)}\tau(m_{(1)})=h\cdot T(m)$$

for any $h\in H, m\in M$. Namely, T is a left H-module map, so (M,T) is a generic Nijenhuis paired H-module by Lemma 3.3. □

Remark 4.10. (1) Let (H,σ) be a Long skew bialgebra. Then, by [21], $(H,\rightharpoonup,\Delta)$ is a left, right H-dimodule by the defined action: $x\rightharpoonup h=\sigma(h_2,x)h_1$, for $x,h\in H$.

Therefore, by the above proposition, for any nonzero element $\tau\in H^*$, we know that every Long skew bialgebra (H,σ) can induce a generic Nijenhuis paired H-module (H,T) naturally, where $T(h)=h_1\tau(h_2)$, for $h\in H$.

(2) Let (H,σ) be a coquasitriangular Hopf algebra. Then, by [23], $(H,\twoheadrightarrow,\Delta)$ is a left, right H-dimodule by the defined action: $x\twoheadrightarrow h=\sigma(x,h_1)h_2$, for $x,h\in H$.

Therefore, by the above proposition, for any nonzero element $\tau\in H^*$, we know that every coquasitriangular Hopf algebra (H,σ) can induce a generic Nijenhuis paired H-module (H,T), where $T(h)=h_1\tau(h_2)$, for $h\in H$.

4.4. The constructions on weak Doi-Hopf modules.

Definition 4.11. Let H be a weak bialgebra, and A a weak right H-comodule algebra. We call M a weak right (A,H)**-Doi-Hopf module** [24] if M is a right A-module and a right H-comodule satisfying the following condition:

$$\rho(m\cdot a)=m_{(0)}\cdot a_{(0)}\otimes m_{(1)}a_{(1)} \tag{8}$$

for all $m\in M, a\in A$.

The following lemma can be found in [24].

Lemma 4.12. *Let H be a weak Hopf algebra, A a weak right H-comodule algebra, and M a weak right (A,H)-Doi-Hopf module. If there exist a weak right H-comodule algebra map $\phi: H\to A$ and two linear maps defined as follows:*

$$E_A: A\to A, a\mapsto a_{(0)}\phi(S((a_{(1)})), \tag{9}$$

$$E_M: M\to M, m\mapsto m_{(0)}\cdot\phi(S(m_{(1)})), \tag{10}$$

then we have the following conclusions:

(a) (M,E_A,E_M) *is a Rota-Baxter paired right A-module of weight -1,*
(b) (A,E_A) *is a Rota-Baxter algebra of weight -1.*

Proposition 4.13. *Let H be a weak Hopf algebra, A a weak right H-comodule algebra, and M a weak right (A, H)-Doi-Hopf module. If there exists a weak right H-comodule algebra map $\phi : H \to A$, then the following conclusions hold by the above defined maps E_A, E_M.*

(a) *(M, E_A, E_M) is a Nijenhuis paired A-module.*

(b) *(A, E_A) is a Nijenhuis algebra.*

Proof. (a) By the proof of [24, Theorem 3.15], we have

$$E_M(E_M(m) \cdot a) = E_M(m) \cdot E_A(a)$$

for all $m \in M, a \in A$. Moreover we have $E_A(1_A) = 1_A$ by [22, Lemma 3.2], so $E_M^2 = E_M$. Hence, by Lemma 4.12 and Example 3.2, we can prove (M, E_A, E_M) is a Nijenhuis paired A-module.

(b) Obviously, every given weak right H-comodule algebra A is a weak right (A, H)-Doi-Hopf module, whose module structure is given by its multiplication. Again by the above proof, we know that E_A is idempotent. Therefore, (A, E_A) is a Nijenhuis algebra by Lemma 4.12 and Example 2.4. □

Remark 4.14. Let H be a weak Hopf algebra. Then H is a weak right H-comodule algebra, whose module structure is given by its multiplication and comodule structure is given by its comultiplication.

Obviously, id : $H \to H$ is a weak right H-comodule algebra map, so (H, E_H) is a Nijenhuis algebra by Proposition 4.13.

In fact, the map E_H is defined by $E_H(h) = h_1 S(h_2)$, for $h \in H$, which is exactly a source map given in [1] and [19].

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