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IDEALS IN HOM-ASSOCIATIVE WEYL ALGEBRAS

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Abstract. We introduce hom-associative versions of the higher order Weyl algebras, generalizing the construction of the first hom-associative Weyl algebras. We then show that the higher order hom-associative Weyl algebras are simple, and that all their one-sided ideals are principal.

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1. Introduction

Dixmier [5] has shown that every left (right) ideal of the first Weyl algebra A_1 over a field K of characteristic zero can be generated by two elements. Later, and more generally, Stafford [9] has shown that every left (right) ideal of a simple left (right) Noetherian ring with Krull dimension n can be generated by $n+1$ elements; in particular, this result applies to the nth Weyl algebra A_n over K. Stafford [10] has further improved this result for A_n and shown that every left (right) ideal of A_n over K can be generated by two elements, a classical result today more commonly known as *Stafford's theorem*.

In this article, we introduce *higher order hom-associative Weyl algebras* as hom-associative deformations of the higher order Weyl algebras over K and consider what a hom-associative version of Stafford's theorem would look like. We prove that, subject to a non-triviality condition on the deformation, the higher order hom-associative Weyl algebras are simple (Corollary 3.7) and that all their one-sided ideals are principal (Theorem 3.11).

2. Preliminaries

Throughout this article, we denote by $\mathbb{N}$ the set of non-negative integers. By a *non-associative algebra* over an associative, commutative, and unital ring R, we mean an R-algebra A which is not necessarily associative and not necessarily unital.

2.1. Hom-associative algebras. *Hom-associative algebras* were introduced in [8] as non-associative algebras with a "twisted" associativity condition. In particular, by using the commutator as a bracket, any hom-associative algebra gives rise to a *hom-Lie algebra*; the latter introduced in [7] as a generalization of a Lie algebra, now with a twisted Jacobi identity.

Definition 2.1. (Hom-associative algebra) A *hom-associative algebra* over an associative, commutative, and unital ring R, is a non-associative R-algebra A with an R-linear map α, where for all $a, b, c \in A$, the *hom-associative condition* holds,

$$\alpha(a)(bc) = (ab)\alpha(c).$$

Since α in the above definition "twists" the associativity condition, it is referred to as a *twisting map.*

For hom-associative algebras it is usually too restrictive to expect them to be unital. Instead, a related condition, called *weak unitality*, is of interest.

Definition 2.2. (Weak unitality) Let A be a hom-associative algebra. If for all $a \in A$, $ea = ae = \alpha(a)$ for some $e \in A$, we say that A is *weakly unital* with *weak identity element* e.

The so-called *Yau twist* gives a way of constructing (weakly unital) hom-associative algebras from (unital) associative algebras.

Proposition 2.3. ([6,11]) *Let A be an associative algebra and let α be an algebra endomorphism on A. Define a new product $*$ on A by $a*b := \alpha(ab)$ for any $a, b \in A$. Then A with product $*$, called the* Yau twist *of A, is a hom-associative algebra with twisting map α. If A is unital with identity element 1_A, then the Yau twist of A is weakly unital with weak identity element 1_A.*

By a left (right) *hom-ideal* in a hom-associative algebra, we mean a left (right) ideal that is also invariant under the twisting map. If the algebra is weakly unital, then all ideals, one-sided and two-sided, are automatically invariant under the twisting map.

2.2. The nth Weyl algebra. The *nth Weyl algebra*, A_n, over a field K of characteristic zero is the free, associative, and unital algebra with generators $x_1, x_2, \ldots, x_n$

and $y_1, y_2, \ldots, y_n$, $K\langle x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n\rangle$, modulo the commutation relations

$$x_i x_j = x_j x_i \text{ for all } i, j \in \{1, 2, \ldots, n\},$$
$$y_i y_j = y_j y_i \text{ for all } i, j \in \{1, 2, \ldots, n\},$$
$$x_i y_j = y_j x_i \text{ for all } i, j \in \{1, 2, \ldots, n\} \text{ such that } i \neq j,$$
$$x_i y_i = y_i x_i + 1 \text{ for all } i \in \{1, 2, \ldots, n\}.$$

2.3. The first hom-associative Weyl algebras. In [3], a family of *hom-associative Weyl algebras* $\{A_1^k\}_{k\in K}$ was constructed as a generalization of A_1 to the hom-associative setting (see also [2] for the case when K has prime characteristic), including A_1 as the member corresponding to $k = 0$. The definition of A_1^k is as follows:

Definition 2.4. (The first hom-associative Weyl algebra) Let α_k be the K-automorphism on A_1 defined by $\alpha_k(x) := x$, $\alpha_k(y) := y + k$, and $\alpha_k(1_{A_1}) := 1_{A_1}$ for any $k \in K$. The *first hom-associative Weyl algebra* A_1^k is the Yau twist of A_1 by α_k.

For each $k \in K$, we thus get a hom-associative Weyl algebra A_1^k which is weakly unital with weak identity element 1_{A_1}. In [3], it was proven that A_1^k is simple for all $k \in K$. In [1], the study of A_1^k was continued. The morphisms and derivations on A_1^k were characterized, and an analogue of the famous *Dixmier conjecture*, first introduced by Dixmier [4], was proven. It was also shown that A_1^k is a formal deformation of A_1 with k as deformation parameter, this in contrast to the associative setting where A_1 is formally rigid and thus cannot be formally deformed.

2.4. Monomial orderings. We introduce an ordering, the so-called *graded lexicographic ordering* on $\mathbb{N}^n$, where a vector is larger than another vector if it has larger sum of all its elements. In case of a tie, we apply *lexicographic ordering*, that is, $(1, 0, 0, \ldots, 0) > (0, 1, 0, \ldots, 0) > \cdots > (0, 0, 0, \ldots, 1)$. For example, $(0, 0, 3) > (1, 1, 0) > (0, 2, 0) > (0, 1, 0) > (0, 0, 0)$. Note that this is a total ordering on $\mathbb{N}^n$ and that any subset has a smallest element. Furthermore, it is impossible to find an infinite decreasing sequence in $\mathbb{N}^n$. Note that this gives an ordering of the monomials in $K[y_1, y_2, \ldots, y_n]$.

Any $p \in A_n$ can be written as $\sum_{l\in\mathbb{N}^n} p_l x_1^{l_1} x_2^{l_2} \cdots x_n^{l_n}$ where $p_l \in K[y_1, y_2, \ldots, y_n]$, $l = (l_1, l_2, \ldots, l_n) \in \mathbb{N}^n$, and only finitely many of the p_l are non-zero. We define $\deg_x(p)$ as the largest l in graded lexicographic order such that p_l is non-zero, and $L(p) = p_{\deg_x(p)}$. We also define $\deg_y$ in a similar way. We will often write $\deg_y(p) = 0$, where 0 should be understood as the zero vector of appropriate dimension.

3. Ideals in higher order hom-associative Weyl algebras

We define the *nth hom-associative Weyl algebra* in anaology with how the first hom-associative Weyl algebra is defined.

Definition 3.1. (The nth hom-associative Weyl algebra) Let K be a field of characteristic zero and let $k = (k_1, k_2, \ldots, k_n) \in K^n$. Define the K-automorphism α_k on A_n by $\alpha_k(x_i) := x_i$, $\alpha_k(y_i) := y_i + k_i$, and $\alpha_k(1_{A_n}) := 1_{A_n}$ for $1 \le i \le n$. The *nth hom-associative Weyl algebra* A_n^k is the Yau twist of A_n by α_k.

We will suppose $k_1 k_2 \ldots k_n \neq 0$.

Proposition 3.2. *If I is a left (right) ideal of A_n, then I is a left (right) ideal of A_n^k if and only if $\alpha_k(I) \subseteq I$ if and only if $\alpha_k(I) = I$.*

Proof. We show the left case; the right case is similar. To this end, let I be a left ideal of A_n. If $\alpha_k(I) \subseteq I$, $p \in A_n$, and $q \in I$, then $p * q = \alpha_k(pq) \in \alpha_k(I) \subseteq I$.

If I is a left ideal of A_n^k and $q \in I$, then $\alpha_k(q) = 1_{A_n} * q \in I$, so $\alpha_k(I) \subseteq I$. □

Example 3.3. Let I be the left ideal of A_1 generated by x^n for some $n \in \mathbb{N}_{>0}$. Then I is a non-trivial left ideal (for example, $y \notin I$). Any element in I may be written as px^n for some $p \in A_1$. We have $\alpha_k(px^n) = \alpha_k(p)\alpha_k(x^n) = \alpha_k(p)x^n \in I$, so $\alpha_k(I) \subseteq I$. Similarly, if I is the right ideal of A_1 generated by x^n, then I is a non-trivial right ideal such that $\alpha_k(I) \subseteq I$. By Proposition 3.2, I is a non-trivial left (right) ideal of A_1^k.

By the next example, not all left (right) ideals of A_1 are left (right) ideals of A_1^k when $k \neq 0$.

Example 3.4. Let I be the left (right) ideal of A_1 generated by y. Then $x \notin I$, so $I \neq A_1$. Assume that $k \neq 0$ and $\alpha_k(I) \subseteq I$. Then $y + k = \alpha_k(y) \in I$, so $k = (y + k) - y \in I$. Hence $1_{A_1} \in I$, which implies $I = A_1$; a contradiction. By Proposition 3.2, I is not a left (right) ideal of A_1^k.

Lemma 3.5. *If I is a left (right) ideal of A_n^k where $k_1 k_2 \cdots k_n \neq 0$, then $\alpha_k(I) = I$.*

Proof. Since any left (right) ideal I of A_n^k is also a left (right) hom-ideal, $\alpha_k(I) \subseteq I$.

Now, let I be a left ideal of A_n^k. If $0 \neq p \in I$, we claim that we can find an element $p' \in I$ such that $\deg_x(p') = \deg_x(p)$ and $L(p') = 1$. If $L(p) = c \in K$, we can take $p' = c^{-1} * p$. Otherwise, we note that $\deg_x(\alpha_k(p) - p) = \deg_x(p)$, and that $L(\alpha_k(p))$ and $L(p)$ have the same leading term using our monomial ordering on $K[y_1, y_2, \ldots, y_n]$. Thus, $L(\alpha_k(p) - p)$ has lower degree than $L(p)$ using our monomial ordering. We can repeat this process until we get an element in I with a constant as leading coefficient w.r.t. $x_1, x_2, \ldots, x_n$, and with the same degree in $x_1, x_2, \ldots, x_n$ as p.

Now suppose $I \not\subseteq \alpha_k(I)$. Then there is at least one element in I that does not belong to $\alpha_k(I)$. Pick such an element, q, of lowest possible degree w.r.t. $x_1, x_2, \ldots, x_n$. Find an element $q' \in I$ such that $\deg_x(q') = \deg_x(q)$ and $L(q') = 1$. Set $r := \alpha_k^2(\alpha_k^{-2}(L(q))q') = \alpha_k(\alpha_k^{-2}(L(q)) * q')$. Note that $\deg_x(r) = \deg_x(q)$ and that $L(r) = L(q)$. Since $\alpha_k^{-2}(L(q)) * q' \in I$, we have $r \in \alpha_k(I) \subseteq I$. Hence $q - r \in I$, and by the minimality of q, we must have $q - r \in \alpha_k(I)$. However, this would imply that also $q \in \alpha_k(I)$, which is a contradiction.

Now let I be a right ideal of A_n^k. If $0 \neq p \in I$, we can find an element $p' \in I$ such that $\deg_x(p') = \deg_x(p)$ and $L(p') = 1$. If $L(p) = c \in K$, we can take $p' = p * c^{-1}$. Otherwise we proceed like in the left case.

Now suppose $I \not\subseteq \alpha_k(I)$. Then there is at least one element in I that does not belong to $\alpha_k(I)$. Pick such an element, q, of lowest possible degree w.r.t. $x_1, x_2, \ldots, x_n$. Find an element $q' \in I$ such that $\deg_x(q') = \deg_x(q)$ and $L(q') = 1$. Set $r := \alpha_k^2(q'\alpha_k^{-2}(L(q))) = \alpha_k(q' * \alpha_k^{-2}(L(q)))$. As in the left case, we get $r \in \alpha_k(I)$ and $q - r \in I$. By the minimality of q, we get $q - r \in \alpha_k(I)$, which gives the contradiction $q \in \alpha_k(I)$. □

Proposition 3.6. *Any left (right) ideal of A_n^k for $k_1k_2\cdots k_n \neq 0$ is a left (right) ideal of A_n.*

Proof. Let us prove the left case; the right case is similar. To this end, suppose I is a left ideal of A_n^k. To show that I is also a left ideal of A_n, it is enough to show that $A_nI \subseteq I$. Since I is a left ideal of A_n^k, we know that $A_n * I \subseteq I$. Moreover, $A_n * I = \alpha_k(A_nI)$, so $A_nI \subseteq \alpha_k^{-1}(I)$. By Lemma 3.5, $\alpha_k(I) = I$, so $\alpha_k^{-1}(I) = I$. □

Corollary 3.7. *A_n^k is simple for $k_1k_2\cdots k_n \neq 0$.*

Proof. Let I be an ideal of A_n^k for $k_1k_2\cdots k_n \neq 0$. By Proposition 3.6, I is also an ideal of A_n. Since it is well known that A_n is simple, I must be trivial. □

Corollary 3.8. *Any left (right) ideal of A_n^k for $k_1k_2\cdots k_n \neq 0$ is generated by two elements.*

Proof. Let I be a left ideal of A_n^k. Since I is also a left ideal of A_n, we know that it is generated as an ideal of A_n by two elements, say p and q. We want to show that $\alpha^{-1}(p)$ and $\alpha^{-1}(q)$ generate I as a left ideal of A_n^k. If $r \in I$, then there are $a, b \in A_n$ such that $r = ap + bq = \alpha(\alpha^{-1}(a)\alpha^{-1}(p) + \alpha^{-1}(b)\alpha^{-1}(q)) = \alpha^{-1}(a) *$

$\alpha^{-1}(p) + \alpha^{-1}(b) * \alpha^{-1}(q)$. Clearly this shows that $\alpha^{-1}(p)$ and $\alpha^{-1}(q)$ generate I as a left ideal of A_n^k. (That they are elements of I follows from Lemma 3.5.)

The right case is similar. □

Lemma 3.9. *Any left (right) ideal, I, of A_n generated by elements $p_1, p_2, \ldots, p_m$ with $\deg_y(p_1) = \deg_y(p_2) = \cdots = \deg_y(p_m) = 0$ is a principal left (right) ideal of A_n^k if $k \neq 0$.*

Proof. Assume, without loss of generality, that $k_1 \neq 0$. We first show that I is a left ideal of A_n^k. If $q \in I$, then we can write $q = \sum_{i=1}^{m} r_i p_i$ for some $r_i \in A_n$ and $\alpha_k(q) = \sum_{i=1}^{m} \alpha_k(r_i) p_i \in I$. Hence, by Proposition 3.2, I is a left ideal of A_n^k. It remains to show it is a principal left ideal of A_n^k.

We will proceed by induction, and in fact we will show that I of A_n is generated as a left ideal of A_n^k by $t = p_1 + y_1 p_2 + \cdots + y_1^{m-1} p_m$. To handle the case $m = 2$ set $t := p_1 + y_1 p_2$. Let J be the left ideal of A_n^k generated by t. Then $\alpha_k(t) = p_1 + y_1 p_2 + k_1 p_2 \in J$, so $p_2 = k_1^{-1} * (\alpha_k(t) - t) \in J$, and hence $p_1 = t - (y_1 - k_1) * p_2 \in J$. Hence $I = J$.

Now assume we have proven the result for m and wish to prove it for $m + 1$. Set $t := p_1 + y_1 p_2 + \cdots + y_1^m p_{m+1}$. Let J be the left ideal of A_n^k generated by t. Obviously, $J \subseteq I$. Note that $t_1 := \alpha_k(t) - t = r_{1,2} p_2 + r_{1,3} p_3 + \cdots + r_{1,m+1} p_{m+1}$, where $r_{1,i} \in K[y_1]$ and $\deg_{y_1}(r_{1,i}) = i - 2$ for all $i \in \{2, 3, \ldots, m+1\}$. Also note that $t_1 \in J$. We can then set $t_2 := \alpha_k(t_1) - t_1$ and note that $t_2 = r_{2,3} p_3 + \cdots + r_{2,m+1} p_{m+1}$ where $r_{2,i} \in K[y_1]$ and $\deg_{y_1}(r_{2,i}) = i - 3$ for all $i \in \{3, 4, \ldots, m+1\}$. Also $t_2 \in J$. Proceeding in a similar way, we get an element $t_m \in J$ such that $t_m = r_{m,m+1} p_{m+1}$, where $r_{m,m+1} \in K$ and $r_{m,m+1} \neq 0$. Thus $p_{m+1} \in J$ and $p_1 + y_1 p_2 + \cdots + y_1^{m-1} p_m \in J$, so by the induction assumption, $J = I$.

The right case is similar; one sets $t := p_1 + p_2 y_1 + \cdots p_m y_1^{m-1}$ instead. □

Lemma 3.10. *For any $p \in A_n$, there are $q_1, q_2, \ldots, q_m \in A_n$ with $\deg_y(q_1) = \deg_y(q_2) = \cdots = \deg_y(q_m) = 0$, such that the left (right) ideal of A_n^k for $k_1 k_2 \cdots k_n \neq 0$ generated by p equals the left (right) ideal of A_n^k generated by $q_1, q_2, \ldots, q_m$.*

Proof. Let I be the left ideal of A_n^k generated by p. Set $p = \sum_{a \in \mathbb{N}^n} p_a x_1^{a_1} x_2^{a_2} \ldots x_n^{a_n}$, where each $p_a \in K[y_1, y_2, \ldots, y_n]$, and let $E(p) = \{a \in \mathbb{N}^n \mid p_a \neq 0\}$. We prove the lemma by induction over $|E(p)|$.

We begin with the case when $|E(p)| = 1$. If $\deg_y(p) = 0$, we are done. Otherwise, set $p' = p - \alpha_k(p)$. Then $E(p') = E(p)$ and $\deg_y(p') < \deg_y(p)$. Repeat as necessary until we find $q_1 \in I$ such that $E(q_1) = E(p)$ and $\deg_y(q_1) = 0$. Then there is $r \in K[y_1, y_2, \ldots, y_n]$ such that $p = r q_1$, so p is in the left ideal of A_n generated by

p, and thus also in the left ideal of A_n^k generated by q_1. Hence the left ideal of A_n^k generated by q_1 is I.

Now suppose we have proven the lemma when $|E(p)| \leq \ell$. Assume $|E(p)| = \ell+1$. If $\deg_y(p) = 0$, we are done, so let $\deg_y(p) > 0$. Set $p' = p - \alpha_k(p)$. Note that $E(p') \subseteq E(p)$, $p' \neq 0$, and that if $E(p') = E(p)$, then $\deg_y(p') < \deg_y(p)$. Repeat as necessary until we find $q_1 \in I$ such that $E(q_1) \subseteq E(p), q_1 \neq 0$, and $\deg_y(q_1) = 0$. We can then find $r \in K[y_1, y_2, \ldots, y_n]$ such that $E(p - rq_1) \subsetneq E(p)$. By the induction assumption, there are $q_2, q_3, \ldots q_m$ with $\deg_y(q_2) = \deg_y(q_3) = \cdots = \deg_y(q_m) = 0$ that generate the same left ideal of A_n^k as $p - rq_1$. Then $q_1, q_2, \ldots, q_m$ are elements that generate I as a left ideal of A_n^k and $\deg_y(q_1) = \deg_y(q_2) = \cdots = \deg_y(q_m) = 0$.

The right case is similar. □

Theorem 3.11. *Any left (right) ideal of A_n^k for $k_1 k_2 \cdots k_n \neq 0$ is principal.*

Proof. Let I be a left ideal of A_n^k. We know it is generated by elements p, q as a left ideal of A_n. By Lemma 3.10, we can find $p_1, p_2, \ldots, p_\ell$ and $q_1, q_2, \ldots, q_m$ such that the p_i generate the same left ideal of A_n^k as p, the q_i generate the same left ideal of A_n^k as q, and $\deg_y(p_1) = \deg_y(p_2) = \cdots = \deg_y(p_\ell) = \deg_y(q_1) = \deg_y(q_2) = \cdots = \deg_y(q_m) = 0$. Clearly, $p_1, p_2, \ldots, p_\ell, q_1, q_2, \ldots, q_m$ generate I as a left ideal of A_n^k. By Lemma 3.9 we are done.

The right case is similar. □

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