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(m,n)-$C2$ MODULES AND (m,n)-$D2$ MODULES

Phan Hong Tin and Nguyen Quoc Tien

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Abstract. We study the concept of (m,n)-$C2$ modules with m,n positive integers, which unifies strongly $C2$, n-$C2$ and $GC2$ modules. Several characterizations are obtained. It is shown that $R^{(\mathbb{N})}$ is (m,n)-$C2$ as a right R-module if and only if R is right perfect and right strongly $C2$. Connections between an (m,n)-$C2$ module and its endomorphism ring are also studied. We prove that if the endomorphism ring of an R-module M is a right (m,n)-$C2$ ring, then M is an (m,n)-$C2$ module. Also we obtain some dual statements of (m,n)-$D2$ modules. Some characterizations of (semi)perfect and (semi)regular rings are studied. We show that $S = \text{End}(M_R)$ is a regular ring if and only if M is a dual Rickart module and (m,n)-$D2$ with $m > n$.

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1. Introduction

Throughout this paper, R is a ring with unity and M is a unital right R-module. For a submodule N of M, we use $N \leq M$ and $N \ll M$ to mean that N is a submodule of M and N is a small submodule of M, respectively. For a subset X of R, let $r(X)$ (respectively, $l(X)$) denote the right (respectively, left) annihilator of X in R. Homomorphisms of modules are written on the left of their arguments. For a right R-module M, $S = \text{End}(M_R)$ will be denoted the endomorphism ring of M. Let k, n, m be positive integers. We denote the set of all $1 \times n$ (resp. $n \times 1$) matrices over M_R (resp. ${}_RM$) by M^n (resp. M_n) and the set of all $n \times k$ (resp. $n \times n$) matrices over S by $\mathbb{M}_{n\times k}(S)$ (resp. $\mathbb{M}_n(S)$). Let $s = (x_1, x_2, \ldots, x_n) \in S^n$ and $m = (m_1, \ldots, m_k)^T \in M_n$. We write $sm = \sum_{i=1}^{n} x_i(m_i)$. Assume that $s_1, s_2, \ldots, s_m \in S^n$ and $A \in \mathbb{M}_{n\times k}(S)$, we write

$$\mathbf{r}_{M_n}(s_1, s_2, \ldots, s_m) = \{x = (m_1, \ldots, m_k)^T \in M_n \mid s_1x = s_2x = \cdots = s_mx = 0\}$$

and $\mathbf{r}_{M_k}(A) = \{x = (m_1, \ldots, m_k)^T \in M_k \mid Ax = 0\}$ (see [1], [13] and [20]).

Recall that a module M_R is called a $C2$ module if every submodule of M_R that is isomorphic to a direct summand of M_R is itself a direct summand of M_R, and M_R is called a $D2$-module if every submodule A of M_R is a direct summand of M_R whenever M/A is isomorphic to a direct summand of M_R (see [11]). Recently, many authors have shown interest in and studied extensions of $C2$ and $D2$ modules along with related modules. They have presented numerous results regarding the structure of rings and modules through these modules ([2,3,4,6]). Modules invariant under automorphisms of their injective hull are an important class of modules satisfying the $C2$ condition, which have been extensively studied in recent years ([5,14,15,16,17]).

In [7], Kourki introduced the notion of strongly $C2$ modules motivated by a need to put the notion of strongly $C2$ rings in the general module theoretic setting by utilizing this representation. A module M_R is called a strongly $C2$ module if M_R^n is a $C2$ module for every positive integer n. A ring R is called right $C2$ (respectively, strongly right $C2$) if R_R is a $C2$ (respectively, strongly $C2$) module (see [12]). As a continuation of the strongly $C2$ property, Li-Chen-Kourki introduced the notions of n-$C2$ modules. M_R is called an n-$C2$ module if the annihilator $\mathbf{r}_M(s_1, s_2, \ldots, s_n) \neq 0$ for any $s_1, s_2, \ldots, s_n$ in S satisfying $Ss_1 + Ss_2 + \cdots + Ss_n \neq S$ ([10]). Clearly, $GC2$ modules (every submodule of M that is isomorphic to M is itself a direct summand of M) are 1-$C2$, and 2-$C2$ modules are $C2$ by [10, Proposition 23.8].

In Section 2 of the present paper, we introduce the notion of (m,n)-$C2$ modules and provide some characterizations and investigate its properties. Clearly, n-$C2$ modules are just $(n,1)$-$C2$. It is shown that every direct summand of an (m,n)-$C2$ module inherits the property. We also obtained some connections between an (m,n)-$C2$ module and its endomorphism ring. We prove that if $S = \mathrm{End}(M_R)$ is a right (m,n)-$C2$ ring, then M_R is an (m,n)-$C2$ module. A ring R is called (von Neumann) regular if for every $a \in R$, there exists some $b \in R$ such that $a = aba$. We show that the endomorphism ring S is regular if and only if M_R is an (m,n)-$C2$ module with $m > n$ and $\mathrm{Ker}(s)$ is a direct summand of M for all $s \in S$.

In Section 3, we introduced the notion of (m,n)-$D2$ modules and obtained some dual statements of n-$D2$ modules and strongly $D2$ modules. We prove that if M is (m,n)-$D2$ (respectively, $GD2$), then every direct summand of M is an (m,n)-$D2$ (respectively, $GD2$) module. Similar to (m,n)-$C2$ modules, we show that $S = \mathrm{End}(M_R)$ is a regular ring if and only if M is a dual Rickart module and (m,n)-$D2$ with $m > n$.

2. (m,n)-$C2$ modules

Let R be a ring and M_R be a right R-module, $S = \mathrm{End}(M_R)$ be the endomorphism ring of M_R and m, n be positive integers. M_R is called an (m,n)-$C2$ module if the annihilator $\mathbf{r}_{M_n}(s_1, s_2, \ldots, s_m) \neq 0$ for any $s_1, s_2, \ldots, s_m \in S^n$ satisfying $Ss_1 + Ss_2 + \cdots + Ss_m \neq S^n$.

A ring R is called a right (m,n)-$C2$ ring if R_R is an (m,n)-$C2$ module.

Example 2.1. (1) M is n-$C2$ if and only if M is $(n,1)$-$C2$.

(2) A module M_R is called $GC2$ if every submodule of M that is isomorphic to M is itself a direct summand of M [13]. One can check that M is $GC2$ if and only if M is $(1,1)$-$C2$.

The following theorem extends Li-Chen-Kourki [10, Theorem 2.2].

Theorem 2.2. *Let R be a ring, M be a right R-module, $S = \mathrm{End}(M_R)$ and m, n be positive integers. The following conditions are equivalent:*

(1) *M is (m,n)-$C2$.*

(2) *For every $A \in \mathbb{M}_{m\times n}(S)$, if $\mathbf{r}_{M_n}(A) = 0$, then there exists a matrix B in $\mathbb{M}_{n\times m}(S)$ such that $BA = I_n$, where I_n is the identity matrix in $\mathbb{M}_n(S)$.*

(3) *Any monomorphism $\alpha : M_n \to M_m$ splits.*

Proof. (1) $\Rightarrow$ (2) Suppose $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \in \mathbb{M}_{m\times n}(S)$ and $\mathbf{r}_{M_n}(A) = 0$. For every $i = 1, 2, \ldots, m$, we denote $s_i = (a_{i1}, a_{i2}, \ldots, a_{in})$. Then $\mathbf{r}_{M_n}(s_1, s_2, \ldots, s_m) = 0$. By (1), we have $Ss_1 + Ss_2 + \cdots + Ss_m = S^n$. There exist $b_{ij} \in S$ for all $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$ such that

$$\begin{array}{rcl} (1,0,\ldots,0) & = & b_{11}s_1 + b_{12}s_2 + \cdots + b_{1m}s_m \\ (0,1,\ldots,0) & = & b_{21}s_1 + b_{22}s_2 + \cdots + b_{2m}s_m \\ \vdots & & \vdots \\ (0,0,\ldots,1) & = & b_{n1}s_1 + b_{n2}s_2 + \cdots + b_{nm}s_m \end{array}$$

If we take $B = (b_{ij})_{n\times m} \in \mathbb{M}_{n\times m}(S)$, we get $BA = I_n$ as desired.

(2) $\Rightarrow$ (3) We recall that any homomorphism $\alpha : M_n \to M_m$ can be seen as an $m \times n$ matrix, say A, over S. Since α is a monomorphism, $\mathbf{r}_{M_n}(A) = 0$. There exists a matrix $B \in \mathbb{M}_{n\times m}(S)$ such that $BA = I_n$ by (2) and a homomorphism $\beta : M_m \to M_n$ such that $\beta\alpha = 1_{M_n}$. Hence $\alpha : M_n \to M_m$ splits.

(3) $\Rightarrow$ (1) Suppose $\mathbf{r}_{M_n}(s_1, s_2, \ldots, s_m) = 0$, where $s_1, s_2, \ldots, s_m \in S^n$. It is

sufficient to prove $Ss_1 + Ss_2 + \cdots + Ss_m = S^n$. For every $i = 1, 2, \ldots, m$, let $s_i = (s_{i1}, s_{i2}, \ldots, s_{in})$. Define the map $\alpha : M_n \to M_m$ via

$$(m_1, m_2, \ldots, m_n)^T \mapsto (\sum_{j=1}^{n} s_{1j}(m_j), \sum_{j=1}^{n} s_{2j}(m_j), \ldots, \sum_{j=1}^{n} s_{mj}(m_j))^T.$$

Then α is a right R-module monomorphism and it can be seen as an $m \times n$ matrix over S, denoted by A. Therefore $A = \begin{pmatrix} s_{11} & s_{12} & \cdots & s_{1n} \\ s_{21} & s_{22} & \cdots & s_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ s_{m1} & s_{m2} & \cdots & s_{mn} \end{pmatrix}$. So α splits by (3). Then there exists a homomorphism $\beta : M_m \to M_n$ such that $\beta\alpha = 1_{M_n}$. Now, there exists $B = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1m} \\ b_{21} & b_{22} & \cdots & b_{2m} \\ \cdots & \cdots & \cdots & \cdots \\ b_{n1} & b_{n2} & \cdots & b_{nm} \end{pmatrix} \in \mathbb{M}_{n\times m}(S)$ such that $BA = I_n$. So $Ss_1 + Ss_2 + \cdots + Ss_n = S^n$ as required. □

We have the following corollaries.

Corollary 2.3. *Let R be a ring, M be a right R-module and n be a positive integer. The following conditions are equivalent:*

(1) *M is an n-$C2$ module.*

(2) *M is an (n,k)-$C2$ module for every positive integer k.*

Corollary 2.4. *Let M_R be a right R-module with $S = \mathrm{End}(M_R)$. The following conditions are equivalent:*

(1) *M is a strongly $C2$ module.*

(2) *For every positive integers m and n, M is an (m,n)-$C2$ module.*

The properties n-$C2$ and $GC2$ are inherited by direct summands (see [10, Proposition 2.3] and [21, Theorem 7], respectively).

Proposition 2.5. *Let M be a right R-module and m, n be positive integers with $m \geq n$. If M is (m,n)-$C2$, then every direct summand of M is an (m,n)-$C2$ module.*

Proof. Assume that M is an (m,n)-C2 module and $N = e(M)$, where $e^2 = e \in \mathrm{End}(M)$. Let $S = \mathrm{End}(M)$ and $S' = \mathrm{End}(N)$. Let $A' = (a'_{ij})_{m\times n} \in \mathbb{M}_{m\times n}(S')$ with $\mathbf{r}_{N_n}(A') = 0$. For every $i \in \{1, 2, \ldots, m\}$ and $j \in \{1, 2, \ldots, n\}$, denote

$$a_{ij} = \begin{cases} \iota a'_{ij} e & \text{if } i \neq j \\ \iota a'_{ij} e + (1-e) & \text{if } i = j \end{cases}$$

where $\iota : N \to M$ is the inclusion map. Then $a_{ij} \in S$. Let $A = (a_{ij})_{m\times n} \in \mathbb{M}_{m\times n}(S)$. It is easy to see that $\mathbf{r}_{M_n}(A) = 0$. Since M is an (m,n)-C2 module, there exists $B = (b_{ij})_{n\times m} \in \mathbb{M}_{n\times m}(S)$ such that $BA = I_n$. For each $i \in \{1, 2, \ldots, n\}$, $j \in \{1, 2, \ldots, m\}$, let $b'_{ij} = eb_{ij}\iota$. Then $b'_{ij} \in S'$. Let $B' = (b'_{ij})_{n\times m} \in \mathbb{M}_{n\times m}(S')$. It follows that $B'A' = I_n$. Thus N is an (m,n)-C2 right R-module. □

Let M_R be a right R-module with $S = \mathrm{End}(M_R)$, and n be a positive integer. In [10, Proposition 2.5], it is shown that if S is a right n-$C2$ ring, then M is an n-$C2$ module.

Proposition 2.6. *Let M_R be a right R-module with $S = \mathrm{End}(M_R)$ and m, n be positive integers. If S is a right (m,n)-$C2$ ring, then M is an (m,n)-$C2$ module.*

Proof. Suppose S is a right (m,n)-C2 ring and $\mathbf{r}_{M_n}(s_1, \ldots, s_m) = 0$ for some $s_1, \ldots, s_m \in S^n$. Assume that $s \in \mathbf{r}_{S_n}(s_1, \ldots, s_m)$. It is easy to see that $s = 0$. So $Ss_1 + \cdots + Ss_m = S^n$ and M is an (m,n)-$C2$ module. □

The following fact shows that the concept of (m,n)-$C2$ modules unifies also the concept of $C2$ modules.

Proposition 2.7. *Let M be a right R-module and m, n be positive integers with $m > n$. If M is (m,n)-$C2$, then M is a $C2$ module.*

Proof. Let $M = A \oplus B$ and $f : A \to M$ be a monomorphism. Then the map $\varphi : M_n \to M_m$ via $\varphi(a_i + b_i)_n^T = (f(a_1), b_1, a_2 + b_2, \cdots, a_n + b_n, 0, \cdots, 0)^T$ ($a_i \in A, b_i \in B$) is a monomorphism, hence it splits by Theorem 2.2. Thus, $f(A) \oplus B$ is a direct summand of M_m, hence $f(A)$ is a direct summand of M. Therefore M is a $C2$ module. □

Corollary 2.8. [10, Proposition 2.8] *Every 2-$C2$ module is a $C2$ module.*

The next example shows that there exist $(m,2)$-$C2$ modules but not $(m,1)$-$C2$.

Example 2.9. Let R be a triangle matrix ring over a field K. Then R is right Artinian. It follows that R_R is $(2,2)$-$C2$, but R_R is not $(2,1)$-$C2$. In fact, if R_R is $(2,1)$-$C2$, then R_R is 2-$C2$ by Example 2.1. Thus, R_R is a $C2$-module by Proposition 2.7, a contradiction.

It is well-known that for every right R-module M, $S = \mathrm{End}(M_R)$ is regular if and only if $\mathrm{Ker}(s)$ and $\mathrm{Im}(s)$ are direct summands of M for all $s \in S$.

Corollary 2.10. *Let M_R be a right R-module with $S = \mathrm{End}(M_R)$ and m, n be positive integers.*

(1) *S is regular if and only if M is (m,n)-$C2$ with $m > n$ and* $\mathrm{Ker}(s)$ *is a direct summand of M for all $s \in S$.*

(2) *S is regular if and only if M is strongly $C2$ and* $\mathrm{Ker}(s)$ *is a direct summand of M for all $s \in S$.*

We conclude this section by giving some characterizations of right strongly $C2$ rings.

Theorem 2.11. *Let M_R be a right R-module with $S = \mathrm{End}(M_R)$. Then the following conditions are equivalent:*

(1) *R is a right strongly $C2$ ring.*

(2) *For every $k \geq 1$, $\mathbb{M}_{k\times k}(R)$ is a right (m,n)-$C2$ ring for some positive integers m,n with $m > n$.*

Proof. By Propositions 2.5, 2.6 and 2.7. □

In the following theorem, we follow some notations which are used in the proof of [19, Theorem 2.13].

Theorem 2.12. *Let m,n be positive integers. The following conditions are equivalent:*

(1) *$R^{(\mathbb{N})}$ is (m,n)-$C2$ as a right R-module.*

(2) *$R^{(\mathbb{N})}$ is $C2$ as a right R-module.*

(3) *$R^{(\mathbb{N})}$ is $GC2$ as a right R-module.*

(4) *R is a right perfect and right strongly $C2$ ring.*

Proof. (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) is obvious as $(R^{(\mathbb{N})})^k \cong R^{(\mathbb{N})}$ for all positive integers k and Proposition 2.7.

(3) $\Rightarrow$ (4) For every $k \in \mathbb{N}$, we note that $R^k = A \oplus B$ and $f : A \to R^k$ is a monomorphism. Hence we can assume R^k is a direct summand of $R^{(\mathbb{N})}$. Write $R^{(\mathbb{N})} = A \oplus C$ for some $C \leq R^{(\mathbb{N})}$. Define $\varphi : R^{(\mathbb{N})} \to R^{(\mathbb{N})}$ via $\varphi(a + c) = (f(a), c)$ for all $a \in A,\ c \in C$. Clearly, φ is a monomorphism. Since $R^{(\mathbb{N})}$ is $GC2$, φ splits. That means $\mathrm{Im}(\varphi)$ is a direct summand of $R^{(\mathbb{N})}$ or $f(A)$ is a direct summand of $R^{(\mathbb{N})}$. Therefore $f(A)$ is a direct summand of R^k. Thus R^k is a $C2$ module. It follows that R is right strongly $C2$.

Now we show that R is a right perfect ring. By [1, Theorem 28.4], we only need to show that R satisfies DCC on principal left ideals of R. Let $Ra_1 \geq Ra_2a_1 \geq \cdots$ be any descending chain of principal left ideals of R. Let $F = R^{(\mathbb{N})}$ be a free module with a basis $\{x_1, x_2, \dots\}$ and G be the submodule of F generated by $\{y_i = x_i - x_{i+1}a_i,\ i \in \mathbb{N}\}$. By [1, Lemma 28.1], G is free with a basis $\{y_1, y_2, \dots\}$. Thus

$G \cong F$. Since F is a $GC2$ module, G is a direct summand of F. Then the chain $Ra_1 \geq Ra_2a_1 \geq \cdots$ terminates by [1, Lemma 28.2].
(4) $\Rightarrow$ (3) Let K be a submodule of $F = R^{(\mathbb{N})}$ and $\varphi : K \to R^{(\mathbb{N})}$ be an isomorphism. In order to show that K is also a direct summand of F, we only need to prove that F/K is a projective R-module. Since R is right perfect, by [1, Theorem 28.4], every flat right R-module is projective. Thus, we just need to show that F/K is flat. Let

$$\mathfrak{U} = \{L(k) = R^{n_1} \oplus R^{n_2} \oplus \cdots \oplus R^{n_k} \mid k \in \mathbb{N}, n_j \in \mathbb{N}\}.$$

Then $F = \cup_{k\in\mathbb{N}} L(k)$ and $FI = \cup_{k\in\mathbb{N}} L(k)I$ for any left ideal I of R. Let

$$\mathfrak{B} = \{K(k) = \varphi^{-1}(L(k)) \mid k \in \mathbb{N}\}.$$

It follows that $K = \cup_{k\in\mathbb{N}} K(k)$ and $KI = \cup_{k\in\mathbb{N}} K(k)I$ for any left ideal I of R. Since R is right strongly C2 and $K(k) \cong L(k)$ for each $L(k) \in \mathfrak{U}$, it is easy to see that $K(k)$ is a direct summand of F for each $K(k) \in \mathfrak{B}$. It shows that $F/K(k)$ is a flat module for each $K(k) \in \mathfrak{B}$. Let I be any left ideal of R, $K(k) \cap FI = K(k)I$ and $K(k) \in \mathfrak{B}$. Then

$$K \cap FI = (\cup_{k\in\mathbb{N}} K(k)) \cap FI = \cup_{k\in\mathbb{N}} (K(k) \cap FI) = \cup_{k\in\mathbb{N}} K(k)I = KI.$$

Thus, F/K is flat. □

Corollary 2.13. *The following conditions are equivalent for a ring R with $J(R) = Z(R_R)$:*

(1) $R^{(\mathbb{N})}$ *is a $GC2$ right R-module.*
(2) *R is right perfect.*

3. (m, n)-$D2$ modules

Let R be a ring and M_R be a right R-module, $S = \text{End}(M_R)$ be the endomorphism ring of M_R and m, n be positive integers. M_R is called an (m, n)-$D2$ module if $s_1M + s_2M + \cdots + s_mM \neq M^n$ for any $s_1, s_2, \ldots, s_m \in S^n$ satisfying $s_1S + s_2S + \cdots + s_mS \neq S^n$.

Example 3.1. Let R be a ring.

(1) M is an n-$D2$ module if and only if M is an $(n, 1)$-$D2$ module.
(2) A module M is called $GD2$ if for any submodule A of M for which M/A is isomorphic to M, then A is a direct summand of M. It is easy to see that a module M is $GD2$ if and only if M is $(1, 1)$-$D2$.

The following theorem extends Li-Chen-Kourki [10, Theorem 4.2].

Theorem 3.2. *Let M_R be a right R-module with $S = \mathrm{End}(M_R)$ and m, n be positive integers. Then the following conditions are equivalent:*

(1) *M is an (m,n)-$D2$ module.*
(2) *For every $A \in \mathbb{M}_{n\times m}(S)$, if $AM_m = M_n$, there exists a matrix $B \in \mathbb{M}_{m\times n}(S)$ such that $AB = I_n$.*
(3) *Any epimorphism $\alpha : M_m \to M_n$ splits.*

Proof. (1) ⇒ (2) Suppose $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{pmatrix} \in \mathbb{M}_{n\times m}(S)$ and $AM_m = M_n$. For every $i = 1, 2, \ldots, m$, we denote $s_i = (a_{1i}, a_{2i}, \ldots, a_{ni})$. We can get $s_1M + s_2M + \cdots + s_mM = M^n$. By (1), $s_1S + s_2S + \cdots + s_mS = S^n$. There exist $b_{ij} \in S$ for all $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$ such that

$$\begin{array}{rcl} (1, 0, \ldots, 0) & = & s_1b_{11} + s_2b_{21} + \cdots + s_mb_{m1} \\ (0, 1, \ldots, 0) & = & s_1b_{12} + s_2b_{22} + \cdots + s_mb_{m2} \\ \cdots & \cdots & \cdots \\ (0, 0, \ldots, 1) & = & s_1b_{1n} + s_2b_{2n} + \cdots + s_mb_{mn}. \end{array}$$

If we take $B = (b_{ij})_{m\times n} \in \mathbb{M}_{m\times n}(S)$, then $AB = I_n$ as desired.
(2) ⇒ (3) Remark that any homomorphism $\alpha : M_m \to M_n$ can be seen as an $n \times m$ matrix, say A, over S. Now, since α is an epimorphism, we get $AM_m = M_n$. There exists a matrix $B \in \mathbb{M}_{m\times n}(S)$ such that $AB = I_n$ by (2) and there exists a homomorphism $\beta : M_n \to M_m$ such that $\alpha\beta = 1_{M_n}$. Hence $\alpha : M_m \to M_n$ splits.
(3) ⇒ (1) Suppose $s_1M + s_2M + \cdots + s_mM = M^n$, where $s_1, s_2, \ldots, s_m \in S^n$. It is sufficient to prove $s_1S + s_2S + \cdots + s_mS = S^n$. For every $i = 1, 2, \ldots, m$, let $s_i = (s_{1i}, s_{2i}, \ldots, s_{ni})$ and $A = (s_{ij})_{n\times m}$. Define a map $\alpha : M_m \to M_n$ via $(x_1, x_2, \ldots, x_m)^T \mapsto A(x_1, x_2, \ldots, x_m)^T$. It is easy to see that α is an epimorphism. By (3), we have that α splits and there is a homomorphism $\beta : M_n \to M_m$ such that $\alpha\beta = 1_{M_n}$. Now there exists $B = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ & \cdots & \cdots & \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{pmatrix} \in \mathbb{M}_{m\times n}(S)$ such that $AB = I_n$. Hence $s_1S + s_2S + \cdots + s_nS = S^n$ as required. □

We have the following corollaries.

Corollary 3.3. *Let R be a ring, M be a right R-module and n be a positive integer. The following conditions are equivalent:*

(1) *M is an n-$D2$ module.*
(2) *M is an (n,k)-$D2$ module for every positive integer k.*

Corollary 3.4. *Let M_R be a right R-module with $S = \text{End}(M_R)$. The following conditions are equivalent:*

(1) *M is a strongly $D2$ module.*
(2) *For every positive integers m and n, then M is an (m,n)-$D2$ module.*

The property n-$D2$ is inherited by direct summands by [10, Proposition 4.3].

Proposition 3.5. *Let M be a right R-module and m, n be positive integers with $m \geq n$. If M is an (m,n)-$D2$ module, then every direct summand of M is an (m,n)-$D2$ module.*

Proof. Assume M is an (m,n)-$D2$ module and $N = eM$, where $e^2 = e \in \text{End}(M_R)$. Let $S = \text{End}(M_R)$ and $S' = \text{End}(N_R)$. Assume that $A' = (a'_{ij})_{n\times m} \in \mathbb{M}_{n\times m}(S')$ with $A'N_m = N_n$. For every $i \in \{1,2,\dots,n\}$, $j \in \{1,2,\dots,m\}$, we denote

$$a_{ij} = \begin{cases} \iota a'_{ij}e & \text{if } i \neq j \\ \iota a'_{ij}e + (1-e) & \text{if } i = j \end{cases}$$

where $\iota : N \to M$ is the inclusion map. Then $a_{ij} \in \text{End}(M_R)$. Let $A = (a_{ij})_{n\times m} \in \mathbb{M}_{n\times m}(S)$. It is easy to see that $AM_m = M_n$. Since M is an (m,n)-$D2$ module, there exists $B = (b_{ij})_{m\times n} \in \mathbb{M}_{m\times n}(S)$ such that $AB = I_n$. For each $i \in \{1,2,\dots,n\}$, $j \in \{1,2,\dots,m\}$, let $b'_{ij} = eb_{ij}\iota$. Then $b'_{ij} \in S'$. If we take $B' = (b'_{ij})_{m\times n} \in \mathbb{M}_{m\times n}(S')$, we get $A'B' = I_n$. Thus N is an (m,n)-$D2$ module. □

Corollary 3.6. *If M is a $GD2$ module, then every direct summand of M is also a $GD2$ module.*

An R-module M satisfies $(D3)$ if for any two direct summands A, B of M with $A + B = M$, the sum $A \cap B$ is a direct summand of M.

Recall that

- a right R-module M_R is $D2$ if and only if for every direct summand N of M, every epimorphism $M \to N$ splits (see [20]),
- $D2$ implies $D3$ (see [11, Proposition 4.6]).

Theorem 3.7. *If M^2 is a $D3$ module, then M is a $D2$ module.*

Proof. Let $K \leq M$ and $\varphi : L \to M/K$ be an isomorphism with $M = L \oplus L'$. Let $K' = \{(l,x) \mid \varphi(l) = x + K\}$, $M' = M \oplus 0$ and $H = L' \oplus M$. Then $K' \oplus M' = M^2$

and H is a direct summand of M^2. On the other hand, $M^2 = K' + H$ and $K' \cap H = 0 \oplus K$. Since M^2 is $D3$, $K' \cap H = 0 \oplus K$ is a direct summand of M^2. It follows that K is a direct summand of M. □

The following fact shows that the concept of (m,n)-$D2$ modules unifies also the concept of $D2$ modules.

Proposition 3.8. *Let M be a right R-module and m, n be positive integers with $m > n$. If M is an (m,n)-$D2$ module, then M is a $D2$ module.*

Proof. Let $M = A \oplus B$ and $f: M \to A$ be an epimorphism. Then the map $\varphi: M_m \to M_n$ via $\varphi(a_i + b_i)_m^T = (f(a_1 + b_1) + b_2, a_3 + b_3, \cdots, a_{n+1} + b_{n+1})^T$ ($a_i \in A, b_i \in B$) is an epimorphism, hence φ splits. It follows that $\mathrm{Ker}(f) \oplus A$ is a direct summand of M_m, and so $\mathrm{Ker}(f)$ is a direct summand of M. Thus M is a $D2$ module. □

Corollary 3.9. [10, Proposition 4.5] *If M is a 2-$D2$ right R-module, then M is a $D2$ module.*

According to Rizvi and Roman ([18] and [8]), a module M is said to be Rickart if for any $f \in \mathrm{End}(M_R)$, $\mathrm{Ker}(f) = \mathbf{r}_M(f) = eM$ for some $e^2 = e \in \mathrm{End}(M_R)$. A module M is said to be dual Rickart if for any $f \in \mathrm{End}(M_R)$, $\mathrm{Im}(f) = eM$ for some $e^2 = e \in \mathrm{End}(M_R)$ ([9]).

Corollary 3.10. *Let M_R be a right R-module with $S = \mathrm{End}(M_R)$ and m, n be positive integers.*

(1) *S is a regular ring if and only if M is a dual Rickart module and (m,n)-$D2$ with $m > n$.*
(2) *S is a regular ring if and only if M is a dual Rickart and strongly $D2$ module.*

It is well known that the direct sum of two $D2$ modules need not be $D2$. For instance, let p be prime and $M_1 = \mathbb{Z}_p$ and M_2 an infinite direct sum of copies of Z_{p^2}. Then M_1 and M_2 are $D2$. But $M = M_1 \oplus M_2$ is not $D2$ as a $\mathbb{Z}$-module.

Theorem 3.11. *The following conditions are equivalent for a ring R.*

(1) *R is semisimple.*
(2) *Every $GD2$ module is projective.*
(3) *Every direct sum of any family of $GD2$ modules is projective.*
(4) *The direct sum of two $GD2$ modules is projective.*

Proof. (1) $\Rightarrow$ (2) This follows from [1, Exercise 16.9].
(2) $\Rightarrow$ (3) $\Rightarrow$ (4) This is clear.
(4) $\Rightarrow$ (1) Assume that the direct sum of any two $GD2$ modules is $GD2$. Let M be a simple right R-module. Hence M is a $GD2$ module. By our assumption, $M \oplus R_R$ is a projective module since R_R is also $GD2$. Hence M is projective. By [1, Exercise 16.9], R is semisimple. □

It is well-known that a ring R is right perfect if and only if every right R-module has a projective cover. We also have a similar result for $D2$ modules.

Theorem 3.12. *The following conditions are equivalent for a ring R:*

(1) *R is right perfect.*
(2) *For any right R-module M, there exists an epimorphism $f : N \to M$ such that N is $D2$ and $Ker(f) \ll N$.*

Proof. (1) $\Rightarrow$ (2) This is clear.
(2) $\Rightarrow$ (1) Let M be a right R-module. There exists a free module F and an epimorphism $\psi : F \to M$. By (2), there exists an epimorphism $\phi : X \to F \oplus M$ such that X is $D2$ and $Ker(\phi) \ll X$. Consider the natural projections $p_1 : F \oplus M \to F$ and $p_2 : F \oplus M \to M$. Then $p_1\phi : X \to F$ is an epimorphism. By the projectivity of F, $X = \mathrm{Ker}(p_1\phi) \oplus T$ with $T \leq X$. Let $M' = \mathrm{Ker}(p_1\phi)$. We get $X/M' \cong F$ and $X/M' \cong T$ and so $F \cong T$. Hence, we can regard $X = M' \oplus F$. Clearly, $f = \phi|_{M'} : M' \to M$ is an epimorphism. Now we will show that M' is a projective cover of M. Assume that $A + Ker(f) = M'$. Since $Ker(f) \leq Ker(\phi)$, we have $F + A + Ker(\phi) = M' + F = X$ whence $F + A = F + M'$. Hence $A = M'$ or $Ker(f) \ll M'$.

On the other hand, since F is projective, there exists $\overline{\psi} : F \to M'$ such that $f\overline{\psi} = \psi$. But $Ker(f) \ll M'$ and so $\overline{\psi}$ is an epimorphism. Consider the natural projections $\pi_1 : X \to F$, $\pi_2 : X \to M'$. Then $\overline{\psi}\pi_1 : X \to M'$ is an epimorphism. Since M' is a direct summand of X and X is $D2$, we have $\mathrm{Ker}(\overline{\psi}\pi_1)$ is a direct summand of X. Then there exists $k : M' \to X$ such that $(\overline{\psi}\pi_1)k = id_{M'}$. It follows that $(\overline{\psi}\pi_1)k\pi_2 = \pi_2$. Let $h = k\pi_2 : X \to X$. Then $\overline{\psi}\pi_1 h = \pi_2$. Let $g = \pi_1 h\iota$ where $\iota : M' \to X$ is the natural inclusion. Then $\overline{\psi}g = id$, and M' is isomorphic to a direct summand of F and hence M' is projective. Thus M' is the projective cover of M. □

Corollary 3.13. *The following conditions are equivalent for a ring R:*

(1) *R is semiperfect.*

(2) *For any finitely generated right R-module M, there exists an epimorphism $f : N \to M$ such that N is $D2$ and $Ker(f) \ll N$.*

We conclude this paper by giving a characterization of semiregular rings.

Corollary 3.14. *The following conditions are equivalent for a ring R:*

(1) *R is semiregular.*

(2) *For any finitely presented right R-module M, there exists an epimorphism $f : N \to M$ such that N is $D2$ and $Ker(f) \ll N$.*

Proof. By the proof of Theorem 3.12, if M is finitely presented and $M \cong F/K$, where F is free and both F and K are finitely generated, then $F \oplus M$ is also finitely presented. Thus M has a projective cover. It follows that R is semiregular by [13, Theorem B.56]. □

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Phan Hong Tin (Corresponding Author)
Faculty of Information Technology
Industrial University of Ho Chi Minh City
Ho Chi Minh City, Vietnam
e-mail: phanhongtin@iuh.edu.vn

Nguyen Quoc Tien
Faculty of Applied Sciences
Ho Chi Minh City University of Industry and Trade
Ho Chi Minh City, Vietnam
e-mail: tiennq@huit.edu.vn