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# ON SEMI-INJECTIVE LATTICES

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Abstract. In a previous paper, we explored, in the context of the category $\mathcal{L}_{\mathcal{M}}$ of complete modular lattices and linear morphisms introduced by T. Albu and M. Iosif, the lattice-theoretic counterparts of semi-projective retractable modules and their ring of endomorphisms. In this work, we investigate the dual situation. That is, we introduce the concept of semi-injective coretractable lattices, and we study their relation to their monoid of endomorphisms.

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## 1. Introduction

In [1], T. Albu and M. Iosif introduced the category $\mathcal{L}_{\mathcal{M}}$ of bounded modular lattices and linear morphisms.

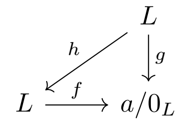
The class of bounded modular lattices becomes a category when equipped with the usual lattice homomorphisms. However, these homomorphisms fail to express important module-theoretic properties. In contrast, linear lattice morphisms, or linear morphisms for short, which will be defined in the next section, summon the notions of kernel and image of module homomorphisms, so the First Isomorphism Theorem for modules holds for bounded modular lattices. This property motivated us to explore lattice-theoretic counterparts of module-theoretic results, restricting ourselves to complete modular lattices (see [7], [8], [12], [13], and [14]).

To be precise, in [8], we defined a *semi-projective* lattice as a lattice $L \in \mathcal{L}_{\mathcal{M}}$ such that for any initial interval $a/0_L$ of $L$ and any diagram

$$\begin{array}{ccccc} & & L & & \\ & & \big\downarrow{\scriptstyle g} & & \\ L & \xrightarrow{f} & a/0_L & \longrightarrow & 0 \end{array}$$

with exact row[1], there exists a linear morphism $h : L \longrightarrow L$ that makes

[1]Exactness of a sequence of linear morphisms is defined in [8]. This notion parallels the corresponding one for modules.




a commutative diagram; that is, $f \circ h = g$. (The concept of semi-projectivity for lattices is inspired by the works of Haghany and Vedadi [10] and of M. K. Patel [15].) Building on this definition, we found some properties of retractable[2] semi-projective lattices and their relation to their monoid of endomorphisms.

In Section 3 of this paper, we introduce the dual notion of semi-injective lattices. Shortly thereafter, we prove that the concepts of semi-projectivity and semi-injectivity are indeed dual to each other, in the formal sense that a lattice $L \in \mathcal{L}_{\mathcal{M}}$ is semi-projective if and only if its opposite lattice is semi-injective. This result establishes a bridge that allows us to prove the dual propositions of several results in [8].

## 2. Preliminaries

This section presents fundamental concepts and definitions related to bounded lattices and to the category $\mathcal{L}_{\mathcal{M}}$ of linear modular lattices and linear morphisms.

For a bounded lattice $L$, we write $0_L$ (resp., $1_L$) for the least (resp., greatest) element of $L$. Also, given elements $a, b \in L$ with $a \leq b$, we define the *interval*

$$b/a = \{x \in L | a \leq x \leq b\}.$$

Special cases are the *initial interval* $a/0_L$, where $a \in L$, and the *quotient interval* $1_L/b$, where $b \in L$.

We write $L^{op}$ to denote the opposite lattice of $L$. Let us write $\wedge_{op}$ and $\vee_{op}$ for the meet and join operations in $L^{op}$, respectively. Of course, $0_{L^{op}} = 1_L$ and $1_{L^{op}} = 0_L$. When there is no room for ambiguity, we use $(b/a)_{op}$ to denote the interval $a/b$ of $L^{op}$. Note that the opposite of an initial interval of $L$ is a quotient interval of $L^{op}$, and vice versa.

Denote as $\mathcal{L}$ the collection of all bounded modular lattices.

**Definition 2.1.** [1, Definition 1.1] Let $L, L' \in \mathcal{L}$. The mapping $f : L \longrightarrow L'$ is called a *linear morphism* if there exists $k_f \in L$, referred to as the kernel of $f$, and $a' \in L'$ such that the following two conditions hold:

1) $f(x) = f(x \vee k_f)$ for all $x \in L$.
2) The function $f$ induces a lattice isomorphism $\overline{f} : 1_L/k_f \longrightarrow a'/0_{L'}$ such that $\overline{f}(x) = f(x)$ for all $x \in 1/k_f$.

[2]Also defined in [8].

For basic properties of linear morphisms, we refer the reader to [1, Section 1].

In [1, Proposition 2.2(1)], the authors introduce the *category* $\mathcal{L}_{\mathcal{M}}$ *of linear modular lattices*, whose objects are bounded modular lattices and whose morphisms are linear morphisms.

As proved in [1, Proposition 2.2(2)-(4)], in this category monomorphisms are precisely injective linear morphisms (which are precisely linear morphisms with kernel zero), epimorphisms are precisely surjective linear morphisms, and isomorphisms are precisely lattice isomorphisms.

Throughout this work, the class of objects of $\mathcal{L}_{\mathcal{M}}$ will be the subclass of $\mathcal{L}$ consisting of all *complete* modular lattices. Thus, also [3, Lemma 0.6] is relevant.

We observe that the category $\mathcal{L}_{\mathcal{M}}$ is not abelian, as it fails to be preadditive, among other conditions (see [9, Theorem 6.5.5(c)] for a proof). However, as illustrated in [8], it possesses a rich structure that brings it closer to this property. For instance, it has a *zero object*, which is the zero lattice (that is, the lattice with a single element, denoted as 0), and a unique *zero morphism* in each hom-set (namely, the morphism that factors through the zero object). Further, every morphism $L \xrightarrow{f} L'$ has a kernel[3] given by the inclusion mapping $k_f/0_L \overset{i}{\hookrightarrow} L$, a cokernel[3], given by the canonical linear morphism $L' \xrightarrow{\_\vee f(1_L)} {}^{1_{L'}}/_{f(1_L)}$, and an image[3], given by the inclusion mapping ${}^{f(1_L)}/_{0_{L'}} \overset{i}{\hookrightarrow} L'$ (for more details on these categorical constructions, see [8, Section 2]). Moreover, $\mathcal{L}_{\mathcal{M}}$ is an exact category in the following sense:

**Definition 2.2.** [11, Chapter I, Section 15] A category is *exact* if it satisfies the following three conditions:

(1) Each morphism has a kernel and a cokernel.
(2) Every monomorphism is the kernel of some morphism, and every epimorphism is a cokernel.
(3) Any morphism $f$ can be expressed as $f = m \circ e$, where $m$ is a monomorphism and $e$ is an epimorphism.

Categories as the above have also been called *p-exact* categories after Puppe (see [5]).

**Theorem 2.3.** *$\mathcal{L}_{\mathcal{M}}$ is an exact category.*

**Proof.** Let $L \xrightarrow{f} L'$ be a linear morphism. Then, by [8, Theorem 2.3] and [8, Theorem 2.5], $f$ has a kernel and a cokernel, respectively. Further, by [8, Remark

[3]In the categorical sense.

2.9], $f$ can be decomposed as $f = m_f \circ e_f$, where $m_f$ is an injective linear morphism and $e_f$ is a surjective linear morphism.

Lastly, let us assume that $f : L \longrightarrow M$ is a monomorphism. Then, $f$ is the kernel of the linear morphism $M \overset{f(1_L)\vee(-)}{\longrightarrow} {}^{1_M}/_{f(1_L)}$. Likewise, if $M \overset{g}{\longrightarrow} L$ is an epimorphism, then $g$ is the cokernel of the inclusion mapping $k_g/0_M \overset{\iota}{\hookrightarrow} M$. $\square$

Given a lattice $L \in \mathcal{L}_{\mathcal{M}}$, the set

$$S = \left\{L \overset{f}{\longrightarrow} L \mid f \text{ is a linear morphism}\right\} = End_{\mathcal{L}_{\mathcal{M}}}(L)$$

becomes a monoid whose binary operation is composition. The identity element for this operation is $Id_L$. Note that this is a monoid with zero: the zero morphism $0_{L,L}$.

**Definition 2.4.** Let $L \in \mathcal{L}_{\mathcal{M}}$ and let $S = End_{\mathcal{L}_{\mathcal{M}}}(L)$. We say that $H \subseteq S$ is a *right ideal of* $S$ if $H$ is non-empty and closed under right composition with elements of $S$; that is, for $h \in H$ and $f \in S$, we have that $h \circ f \in H$. Accordingly, we say that $H \subseteq S$ is a *left ideal of* $S$ if $H$ is non-empty and is closed under left composition, that is, $f \circ h \in H$ for any $h \in H$ and $f \in S$.

Clearly, $H$ is a right ideal of $S$ if and only if $0_{L,L} \in H$ and $H$ is closed under right composition with elements of $S$. Further, the set $\mathcal{R}(S)$ of right ideals in $S$ is partially ordered by inclusion. Hence, $(\mathcal{R}(S), \subseteq)$ is a lattice whose meet and join operations are intersection and union of sets, respectively. Moreover, $\mathcal{R}(S) \in \mathcal{L}_{\mathcal{M}}$ as every distributive lattice is modular. Symmetrical statements hold for the set $\mathcal{L}(S)$ of left ideals of $S$.

Recall that an element $a$ of a lattice $L$ with zero is said to be *essential* (in $L$) if for every $0_L \neq b \in L$, it happens that $a \wedge b \neq 0_L$.

**Definition 2.5.** Let $I$ and $J$ be two right (left) ideals of $S = End_{\mathcal{L}_{\mathcal{M}}}(L)$, with $I \subseteq J$. We say that $I$ is essential in $J$ if $I$ is an essential element of the initial interval ${}^{J}/_{\{0\}}$ of $\mathcal{R}(S)$ $(\mathcal{L}(S))$.

**Definition 2.6.** Let $L$ be a bounded lattice. We say that $a \in L$ is *uniform* (in $L$) if every nonzero $b \in L$ such that $b \leq a$ is essential in $a/0_L$. Furthermore, we say that the lattice $L$ is uniform if the element $1_L$ is uniform in $L$.

Thus, for $L \in \mathcal{L}_{\mathcal{M}}$, a right (left) ideal $J$ of $S = End_{\mathcal{L}_{\mathcal{M}}}(L)$ is uniform if every right (left) ideal $I$ contained in $J$ is essential in $J$.

We close this section with two lemmas required for subsequent proofs.

**Lemma 2.7.** [8, Lemma 2.4] *If $L, L' \in \mathcal{L}_{\mathcal{M}}$ and $f, g : L \longrightarrow L'$ are linear morphisms with respective kernels $k_f$, $k_g$, such that $k_f = k_g$ and that the induced lattice isomorphisms $\overline{f}$ and $\overline{g}$ coincide, then $f = g$.*

**Lemma 2.8.** *Given the linear morphisms $L \xrightarrow{h} M$ and $M \xrightarrow{f} N$, if $x \in (f \circ h)(L)$, then*

$$\overline{(f \circ h)}^{-1}(x) = \overline{h}^{-1}\big(h(1_L) \wedge \overline{f}^{-1}(x)\big).$$

**Proof.** By the proof of [1, Lemma 2.1],

$$k_{f \circ h} = \overline{h}^{-1}(h(1_L) \wedge k_f) \leq \overline{h}^{-1}(h(1_L) \wedge \overline{f}^{-1}(x)),$$

for all $x \in {}^{f(1_M)}/_{0_N}$.

Now, as $x \in (f \circ h)(L) = f\big({}^{h(1_L)}/_{0_M}\big)$,

$$\overline{f}^{-1}(x) \leq \overline{f}^{-1}\big((f \circ h)(1_L)\big) = h(1_L) \vee k_f.$$

Thus, by modularity,

$$(f \circ h)\big(\overline{h}^{-1}\big(h(1_L) \wedge \overline{f}^{-1}(x)\big)\big) = f\big(h(1_L) \wedge \overline{f}^{-1}(x)\big) = f\big(\big(h(1_L) \wedge \overline{f}^{-1}(x)\big) \vee k_f\big)$$

$$= f\big(\big(h(1_L) \vee k_f\big) \wedge \overline{f}^{-1}(x)\big) = f\big(\overline{f}^{-1}(x)\big) = x.$$

Since the restriction of the linear morphism $f \circ h$ to $1_L/k_{f \circ h}$ is injective,

$$\overline{(f \circ h)}^{-1}(x) = \overline{h}^{-1}\big(h(1_L) \wedge \overline{f}^{-1}(x)\big).$$ □

## 3. Semi-injective lattices

We start this section by translating the definition of some well-known module properties into lattice-theoretic language.

**Definition 3.1.** Let $L \in \mathcal{L}_{\mathcal{M}}$. An initial interval $a/0_L$ of $L$ is *$L$-cyclic* if it is isomorphic to a quotient interval of $L$. Symmetrically, a quotient interval $1_L/b$ of $L \in \mathcal{L}_{\mathcal{M}}$ is *$L$-cocyclic* if it is isomorphic to an initial interval of $L$.

**Remark 3.2.** For a lattice $L \in \mathcal{L}_{\mathcal{M}}$, the set of isomorphism classes of $L$-cyclic initial intervals is in a one-to-one correspondence with the set of isomorphism classes of $L$-cocyclic quotient intervals.

**Definition 3.3.** A lattice $L \in \mathcal{L}_{\mathcal{M}}$ is *coretractable* if for any non-trivial quotient interval $1_L/b$ of $L$, one has that

$$Hom_{\mathcal{L}_{\mathcal{M}}}(1_L/b, L) \neq 0.$$

In other words, $L$ is coretractable if every non-trivial quotient interval of $L$ has a non-trivial $L$-cocyclic quotient interval. The following two examples display a coretractable and a non-coretractable lattice, respectively.

**Example 3.4.** Every complemented lattice $L \in \mathcal{L}_{\mathcal{M}}$ is coretractable. Indeed, let $1_L/b$ be a non-trivial quotient interval of the complemented lattice $L$. If $a \in L$ is a complement of $b$, then the mapping

$$1_L/b = {}^{a \vee b}/_{b} \xrightarrow{\_\wedge a} {}^{a}/_{a \wedge b} = a/0_L$$

is a lattice isomorphism, by modularity. Hence, the composite $\iota \circ (\_ \bigwedge a)$ is a non-trivial linear morphism in $Hom_{\mathcal{L}_{\mathcal{M}}}(1_L/b, L)$.

**Example 3.5.** Let us consider the lattice $L = \{\frac{1}{n}\}_{n \in \mathbb{N} \setminus \{0\}} \cup \{0\}$, with order induced by the rational numbers. We claim that $L$ is not coretractable. Indeed, $L$ does not have finite non-trivial initial intervals. Thus, the simple quotient interval $1/\frac{1}{2}$ cannot be isomorphic to any initial interval.

**Definition 3.6.** A lattice $L \in \mathcal{L}_{\mathcal{M}}$ is *semi-injective* if for any quotient interval $1_L/b$ of $L$, and any diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & 1_L/b & \xrightarrow{k} & L \\ & & \downarrow{\scriptstyle g} & & \\ & & L & & \end{array}$$

with exact row, there exists a linear morphism $h : L \longrightarrow L$ that makes

$$\begin{array}{ccc} 1_L/b & \xrightarrow{k} & L \\ \downarrow{\scriptstyle g} & \swarrow{\scriptstyle h} & \\ L & & \end{array}$$

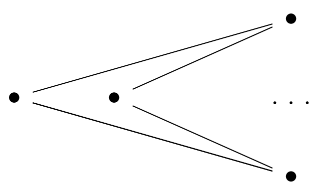
a commutative diagram; that is, $h \circ k = g$.

The following two examples are of both semi-injective and coretractable lattices.

**Example 3.7.** The simple lattice $\{0, 1\}$ is semi-injective and coretractable, because it is a complemented lattice whose only non-trivial initial interval is the lattice $\{0, 1\}$.

Recall that the *length* of a chain $C$ is $|C| - 1$, and that the length of a lattice $L$ is the greatest length of a chain in $L$.

**Example 3.8.** Every lattice $L \in \mathcal{L}_{\mathcal{M}}$ of length 2 is semi-injective and coretractable. Indeed, any such lattice has the form



Clearly, if $L$ has only three elements, then it is coretractable, and if $L$ has more than three elements, then it is complemented. Thus, these lattices are all coretractable.

Let now $1_L/a$ be a quotient interval of a lattice $L \in \mathcal{L}_{\mathcal{M}}$ of length 2, and consider the diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & 1_L/a & \xrightarrow{f} & L \\ & & \Big\downarrow{\scriptstyle g} & & \\ & & L & & \end{array}$$

with exact row. Note that if $g = 0$, then $0_{L,L}$ makes the diagram

$$\begin{array}{ccc} 1_L/a & \xrightarrow{f} & L \\ \Big\downarrow{\scriptstyle g} & \swarrow{\scriptstyle 0_{L,L}} & \\ L & & \end{array}$$

commutative.

We now assume that $g \neq 0$. In particular, $a < 1_L$. If $a = 0_L$, exactness of the top row implies that $f$ is a lattice isomorphism, so that $g \circ f^{-1}$ is a linear morphism that makes the diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & 1_L/a & \xrightarrow{f} & L \\ & & \Big\downarrow{\scriptstyle g} & \swarrow{\scriptstyle g\circ f^{-1}} & \\ & & L & & \end{array}$$

commutative. Suppose now that $a \in L$ is a coatom, so the quotient interval $1_L/a$ is a simple lattice. In this case, any non-trivial linear morphism $1_L/a \xrightarrow{\alpha} L$ is uniquely determined by the image $\alpha(1_L)$. With this in mind, set the mapping $h : L \longrightarrow L$ such that $h(f(1_L)) = g(1_L)$, $h(g(1_L)) = f(1_L)$, and $h(x) = x$ for all $x \in L$ with $x \notin \{f(1_L), g(1_L)\}$. Note that $h$ is a lattice isomorphism and consequently a linear morphism. Furthermore, the composite $h \circ f$ is a non-trivial linear morphism and $(h \circ f)(1_L) = g(1_L)$, so that

$$\begin{array}{ccccc} 0 & \longrightarrow & 1_L/a & \xrightarrow{f} & L \\ & & \Big\downarrow{\scriptstyle g} & \swarrow{\scriptstyle h} & \\ & & L & & \end{array}$$

is a commutative diagram. Therefore, $L$ is a semi-injective lattice.

**Remark 3.9.** Given $f \in End_{\mathcal{L}_{\mathcal{M}}}(L)$, the induced lattice isomorphism $1_L/k_f \xrightarrow{\overline{f}} {}^{f(1_L)}/0_L$ gives rise to a lattice isomorphism $\varphi : ({}^{f(1_L)}/0_L)^{op} \longrightarrow (1_L/k_f)^{op}$ such that $\varphi(x) = \overline{f}^{-1}(x)$. Note that $k_\varphi = f(1_L)$. Further, since $\varphi$ is an isomorphism from a quotient interval to an initial interval, it induces the linear endomorphism $f^{op} \in End_{\mathcal{L}_{\mathcal{M}}}(L^{op})$ such that

$$f^{op}(x) = \varphi\big(x \vee_{op} f(1_L)\big) = \overline{f}^{-1}\big(x \wedge f(1_L)\big),$$

so that $\varphi = \overline{f^{op}}$.

Henceforward, we will refer to $f^{op}$ as the *opposite linear morphism of* $f$. Note that $k_{f^{op}} = f(1_L)$, that $f^{op}(1_{L^{op}}) = k_f$, and that $(f^{op})^{op} = f$.

Clearly, for any linear morphism $L \xrightarrow{f} H$, the above construction yields the opposite linear morphism $H^{op} \xrightarrow{f^{op}} L^{op}$.

**Lemma 3.10.** *Let* $f, h \in End_{\mathcal{L_M}}(L)$*. Then,* $(h \circ f)^{op} = f^{op} \circ h^{op}$.

**Proof.** By the proof of [1, Lemma 2.1], for the linear morphism $g = h \circ f$, we have that

$$k_{f^{op} \circ h^{op}} = \overline{h^{op}}^{-1}(k_{f^{op}} \wedge_{op} h^{op}(1_{L^{op}})) = \overline{h^{op}}^{-1}(f(1_L) \wedge_{op} k_h) = \overline{h}(f(1_L) \vee k_h) =$$
$$h(f(1_L)) = g(1_L) = k_{g^{op}}.$$

Furthermore, by Lemma 2.8, for $x \in \big({}^{g(1_L)}/{}_{0_L}\big)^{op}$ one has that

$$g^{op}(x) = \overline{g}^{-1}(x) = \overline{h \circ f}^{-1}(x) = \overline{f}^{-1}\big(\overline{h}^{-1}(x) \wedge f(1_L)\big)$$
$$= \overline{f^{op}}\big(\overline{h}^{-1}(x) \vee_{op} f(1_L)\big) = f^{op}\big(\overline{h}^{-1}(x)\big) = f^{op}\big(h^{op}(x)\big) = (f^{op} \circ h^{op})(x).$$

Therefore, by Lemma 2.7, $g^{op} = f^{op} \circ h^{op}$, that is, $(h \circ f)^{op} = f^{op} \circ h^{op}$. □

Note that the above lemma holds for any $L, L', L'' \in \mathcal{L_M}$ and any linear morphisms $L \xrightarrow{f} L' \xrightarrow{h} L''$.

The next result shows that the duality between semi-injective and semi-projective lattices comes from the dualities between lattices and opposite lattices, and morphisms and opposite morphisms.

Observe that if a linear morphism $f$ is injective, then $f^{op}$ is surjective, and vice versa.

**Theorem 3.11.** *A lattice* $L \in \mathcal{L_M}$ *is semi-projective if and only if* $L^{op}$ *is semi-injective.*

**Proof.** $(\Rightarrow)$ Let $1_{L^{op}}/b = (b/0_L)^{op}$ be a quotient interval of $L^{op}$. Consider the diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & (b/0_L)^{op} & \xrightarrow{f} & L^{op} \\ & & \big\downarrow{\scriptstyle g} & & \\ & & L^{op} & & \end{array}$$

with exact row. Then, for the opposite linear morphisms $f^{op}$ and $g^{op}$, we get the solid part of the diagram

$$\begin{array}{ccccc} & & L & & \\ & \overset{h}{\swarrow} & \downarrow{\scriptstyle g^{op}} & & \\ L & \xrightarrow[f^{op}]{} & b/0_L & \longrightarrow & 0 \end{array}$$

again with exact row. Since $L$ is semi-projective, there exists $h$ such that $f^{op} \circ h = g^{op}$. By Lemma 3.10,

$$h^{op} \circ f = (f^{op} \circ h)^{op} = (g^{op})^{op} = g.$$

Therefore, $L^{op}$ is semi-injective.

($\Leftarrow$) Consider the diagram

$$\begin{array}{ccccc} & & L & & \\ & & \downarrow{\scriptstyle g} & & \\ L & \xrightarrow[f]{} & a/0_L & \longrightarrow & 0 \end{array}$$

with exact row. Taking the opposite linear morphisms, we obtain the solid part of the diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & (a/0_L)^{op} & \xrightarrow{f^{op}} & L^{op} \\ & & \downarrow{\scriptstyle g^{op}} & \overset{h}{\swarrow} & \\ & & L^{op} & & \end{array}$$

with exact row. This time, $h$ exists because $L^{op}$ is semi-injective, and it satisfies that $(h \circ f_{op}) = g_{op}$. Thus, by Lemma 3.10,

$$f \circ h^{op} = (h \circ f^{op})^{op} = (g^{op})^{op} = g.$$

Therefore, $L$ is semi-projective. $\square$

Furthermore, the duality between left ideals in $End_{\mathcal{L}_\mathcal{M}}(L)$ and right ideals in $End_{\mathcal{L}_\mathcal{M}}(L^{op})$ follows from the composition of linear morphisms in the monoid of endomorphisms, as we show in the next result.

**Lemma 3.12.** *A subset $I \subseteq End_{\mathcal{L}_\mathcal{M}}(L)$ is a left ideal if and only if $I^{op} = \{f^{op} | f \in I\}$ is a right ideal of $End_{\mathcal{L}_\mathcal{M}}(L^{op})$.*

**Proof.** ($\Rightarrow$) Let $f^{op} \in I^{op}$ and $g \in End_{\mathcal{L}_\mathcal{M}}(L^{op})$. By Lemma 3.10, $(f^{op} \circ g)^{op} = g^{op} \circ f \in I$, and thus,

$$f^{op} \circ g = ((f^{op} \circ g)^{op})^{op} \in I^{op}.$$

Therefore, $I^{op}$ is a right ideal of $End_{\mathcal{L}_\mathcal{M}}(L^{op})$.

($\Leftarrow$) Let $f \in I$ and $g \in End_{\mathcal{L}_\mathcal{M}}(L)$. By Lemma 3.10, $(g \circ f)^{op} = f^{op} \circ g^{op} \in I^{op}$, so that

$$g \circ f = ((g \circ f)^{op})^{op} \in I.$$

Hence, $I$ is a left ideal of $End_{\mathcal{L}_{\mathcal{M}}}(L)$. □

**Corollary 3.13.** *Let $L \in \mathcal{L}_{\mathcal{M}}$. The lattice of left ideals of $End_{\mathcal{L}_{\mathcal{M}}}(L)$ is isomorphic to the lattice of right ideals of $End_{\mathcal{L}_{\mathcal{M}}}(L^{op})$.*

Clearly, symmetrical versions of the last two results hold.

**Definition 3.14.** For a lattice $L \in \mathcal{L}_{\mathcal{M}}$ and an element $a \in L$, we call the element $a$ *strongly invariant* (in $L$) if $f(a) \leq a$ for any linear endomorphism $f \in End_{\mathcal{L}_{\mathcal{M}}}(L)$.

**Theorem 3.15.** *Let $L \in \mathcal{L}_{\mathcal{M}}$. If $a \in L$ is strongly invariant, then $a$ is strongly invariant in $L^{op}$.*

**Proof.** Let $L \xrightarrow{f} L$ be a linear morphism, so that $L^{op} \xrightarrow{f^{op}} L^{op}$. By hypothesis, $f(a) \leq a$ in $L$. Thus, in $L^{op}$,

$$f^{op}(a) \leq_{op} f^{op}(f(a)) = \overline{f}^{-1}(f(a)) = a \vee k_f \leq_{op} a.$$

Therefore, $a$ is strongly invariant in $L^{op}$. □

**Proposition 3.16.** *Let $L \in \mathcal{L}_{\mathcal{M}}$. Then, $L$ is semi-injective if and only if*

$$End_{\mathcal{L}_{\mathcal{M}}}(L) \circ g \subseteq End_{\mathcal{L}_{\mathcal{M}}}(L) \circ f$$

*for any $f, g \in End_{\mathcal{L}_{\mathcal{M}}}(L)$ such that $k_g \geq k_f$.*

**Proof.** By Theorem 3.11, $L$ is semi-injective if and only if $L^{op}$ is semi-projective. Also, according to [8, Proposition 3.6], a lattice $L \in \mathcal{L}_{\mathcal{M}}$ is semi-projective if and only if

$$g \circ End_{\mathcal{L}_{\mathcal{M}}}(L) \subseteq f \circ End_{\mathcal{L}_{\mathcal{M}}}(L)$$

for any $f, g \in End_{\mathcal{L}_{\mathcal{M}}}(L)$ such that $g(1_L) \leq f(1_L)$.

Thus, a lattice $L \in \mathcal{L}_{\mathcal{M}}$ is semi-injective if and only if

$$g^{op} \circ End_{\mathcal{L}_{\mathcal{M}}}(L^{op}) \subseteq f^{op} \circ End_{\mathcal{L}_{\mathcal{M}}}(L^{op})$$

for any $f^{op}, g^{op} \in End_{\mathcal{L}_{\mathcal{M}}}(L^{op})$ such that $g^{op}(1_{L^{op}}) \leq_{op} f^{op}(1_{L^{op}})$. However, by Lemma 3.10, the latter can be formulated as

$$End_{\mathcal{L}_{\mathcal{M}}}(L) \circ g \subseteq End_{\mathcal{L}_{\mathcal{M}}}(L) \circ f,$$

for any $f, g \in End_{\mathcal{L}_{\mathcal{M}}}(L)$ such that $k_g \geq k_f$, which concludes the proof. □

**Definition 3.17.** For a lattice $L \in \mathcal{L}_{\mathcal{M}}$ and an element $b \in L$, we write

$$S_b^{k\geq} = \{f \in S | k_f \geqslant b\}.$$

We claim that the set $S_b^{k\geq}$ is a left ideal of $S = End_{\mathcal{L}_\mathcal{M}}(L)$. Indeed, for $f \in S$, $f \in S_b^{k\geq}$ if and only if $f(b) = 0_L$ (see [1, Proposition 1.3(2)]).

**Lemma 3.18.** *Let $L \in \mathcal{L}_\mathcal{M}$ and let $b \in L$. Then, a bijection exists between the sets $Hom_{\mathcal{L}_\mathcal{M}}(1_L/b, L)$ and $S_b^{k\geq}$.*

**Proof.** Note that, for $f \in Hom_{\mathcal{L}_\mathcal{M}}(1_L/b, L)$, the composite $\overline{f} \circ (\_ \vee k_f) \in S = End_{\mathcal{L}_\mathcal{M}}(L)$. Furthermore, $\overline{f} \circ (\_ \vee k_f) \in S_b^{k\geq}$.

We claim that the mapping $Hom_{\mathcal{L}_\mathcal{M}}(1_L/b, L) \xrightarrow{\mathcal{F}} S_b^{k\geq}$ such that

$$f \longmapsto \overline{f} \circ (\_ \vee k_f)$$

is a bijection. On the one hand, if $f, g \in Hom_{\mathcal{L}_\mathcal{M}}(1_L/b, L)$ are such that

$$\overline{f} \circ (\_ \vee k_f) = \mathcal{F}(f) = \mathcal{F}(g) = \overline{g} \circ (\_ \bigvee k_g),$$

then

$$k_f = k_{\overline{f} \circ (\_ \vee k_f)} = k_{\overline{g} \circ (\_ \vee k_g)} = k_g,$$

so that $\overline{f} = \overline{g}$. Thus, by Lemma 2.7, $f = g$. Therefore, $\mathcal{F}$ is injective. On the other hand, if $f \in S_b^{k\geq}$, then the linear morphism $f|_{1/b}$ lies in the preimage of $f$ under $\mathcal{F}$, so that $\mathcal{F}$ is surjective. □

**Theorem 3.19.** *Let $L \in \mathcal{L}_\mathcal{M}$, and let $S = End_{\mathcal{L}_\mathcal{M}}(L)$. Then, $L$ is semi-injective if and only if*

$$S \circ f = S_{k_f}^{k\geq}$$

*for any $f \in S$.*

**Proof.** Note first that, by Theorem 3.11, $L$ is semi-injective if and only if $L^{op}$ is semi-projective, and that this last claim is equivalent, by [8, Theorem 3.11], to

$$f^{op} \circ End_{\mathcal{L}_\mathcal{M}}(L^{op}) = Hom_{\mathcal{L}_\mathcal{M}}(L^{op}, f^{op}(L^{op}))$$

for any $f^{op} \in End_{\mathcal{L}_\mathcal{M}}(L^{op})$.

Now, for necessity, let $f \in S$. For any $g \in S_{k_f}^{k\geq}$, $k_g \geq k_f$, that is, $g^{op}(1_{L^{op}}) \leq_{op} f^{op}(1_{L^{op}})$. This means that $g^{op}(L^{op}) \subseteq f^{op}(L^{op})$, and thus, one can write $g^{op} : L^{op} \longrightarrow f^{op}(L^{op})$. Consequently, there exists some $h^{op} \in End_{\mathcal{L}_\mathcal{M}}(L^{op})$ such that $f^{op} \circ h^{op} = g^{op}$, that is, $h \circ f = g$ with $h \in S$. Hence, $g \in S \circ f$, and so, $S_{k_f}^{k\geq} \subseteq S \circ f$. The reverse inclusion is clear.

For sufficiency, let $f^{op} \in End_{\mathcal{L}_\mathcal{M}}(L^{op})$. As, clearly,

$$f^{op} \circ End_{\mathcal{L}_\mathcal{M}}(L^{op}) \subseteq Hom_{\mathcal{L}_\mathcal{M}}(L^{op}, f^{op}(L^{op})),$$

it suffices to prove the reverse inclusion. Since $f^{op}(L^{op}) = (1_L/k_f)^{op}$, each $g^{op} \in Hom_{\mathcal{L}_\mathcal{M}}(L^{op}, f^{op}(L^{op}))$ can be written as $g^{op} : L^{op} \longrightarrow (1_L/k_f)^{op}$. Then, $g : (1_L/k_f) \longrightarrow L$, so that $k_g \geq k_f$, and thus, $g \in S^{k\geq}_{k_f} = S \circ f$. It follows that there exists $h \in S$ such that $g = h \circ f$, that is, $g^{op} = f^{op} \circ h^{op}$. Therefore, $g^{op} \in f^{op} \circ End_{\mathcal{L}_\mathcal{M}}(L^{op})$, which ends the proof. □

**Theorem 3.20.** *Let $L \in \mathcal{L}_\mathcal{M}$ be a semi-injective lattice. Then, a bijection exists between the set of $L$-cocyclic quotient intervals of $L$ and the set of principal left ideals of $S = End_{\mathcal{L}_\mathcal{M}}(L)$.*

**Proof.** Note first that, as $L \in \mathcal{L}_\mathcal{M}$ is semi-injective, $L^{op}$ is semi-projective, by Theorem 3.11. Thus, by [8, Theorem 3.14], there exists a bijection between the set of $L^{op}$-cyclic initial intervals of $L^{op}$ and the set of principal right ideals of $End_{\mathcal{L}_\mathcal{M}}(L^{op})$.

Now, there is an obvious bijection between the set of $L^{op}$-cyclic initial intervals of $L^{op}$ and the set of $L$-cocyclic quotient intervals of $L$. Also, by Lemma 3.12, there is a one-to-one correspondence between the set of principal right ideals of $End_{\mathcal{L}_\mathcal{M}}(L^{op})$ and the set of principal left ideals of $S$. The result follows. □

Recall that an element $a$ of a lattice $L$ with a greatest element $1_L$ is said to be *superfluous* (in $L$) if for every $1_L \neq b \in L$, it happens that $a \vee b \neq 1_L$.

**Definition 3.21.** Let $L \in \mathcal{L}_\mathcal{M}$. We say that $a \in L$ is *hollow* (in $L$) if every element of $1_L/a$ is superfluous in $1_L/a$. Furthermore, we say that the lattice $L$ is hollow if the element $0_L$ is hollow in $L$.

**Definition 3.22.** Let $L \in \mathcal{L}_\mathcal{M}$, and let $1_L/m$ be a quotient interval of $L$. We say that $L$ *co-generates* $1_L/m$ if there exists a family of linear morphisms $\{f_t\}_{t\in T} \subseteq Hom_{\mathcal{L}_\mathcal{M}}(1_L/m, L)$ such that

$$m = \bigwedge_{t\in T} k_{f_t}.$$

**Theorem 3.23.** *Let $L \in \mathcal{L}_\mathcal{M}$ be a coretractable lattice and let $S = End_{\mathcal{L}_\mathcal{M}}(L)$. For $I, J \in \mathcal{L}(S)$ such that $I \subseteq J$ and $n, m \in L$ such that $n \leq m$, the following statements hold:*

(a) *If $S^{k\geq}_m$ is essential in $S^{k\geq}_n$, then $m$ is superfluous in $1_L/n$. If $L$ is semi-injective, then the converse holds.*
(b) *If $I$ is essential in $S^{k\geq}_{\bigwedge_{g\in J} k_g}$, then $\bigwedge_{f\in I} k_f$ is superfluous in $1_L/\bigwedge_{g\in J} k_g$.*
(c) *Suppose that $L$ is semi-injective. If $I = S^{k\geq}_{\bigwedge_{f\in I} k_f}$ and $\bigwedge_{f\in I} k_f$ is superfluous in $1_L/\bigwedge_{g\in J} k_g$, then $I$ is essential in $J$.*

(d) *If $S_m^{k\geq}$ is uniform as a left ideal of $S$, then $m$ is hollow in $L$. If $L$ is semi-injective, then the converse holds.*

(e) *Suppose that $L$ is semi-injective. If $\bigwedge_{f\in I} k_f$ is hollow in $L$, then $I$ is a uniform left ideal of $S$.*

(f) *Consider the following statements:*

(i) *$S_m^{k\geq}$ is simple (that is, an atom of $\mathcal{L}(S)$).*

(ii) *$1_L/m$ is the simple lattice.*

*If $L$ co-generates $1_L/m$, then $(i)$ implies $(ii)$. If $L$ is semi-injective, then $(ii)$ implies $(i)$.*

(g) *If $I = S^{k\geq}_{\bigwedge_{f\in I} k_f}$ is simple, then ${}^{1_L}/\bigwedge_{f\in I} k_f$ is the simple lattice. If $L$ is semi-injective, then the converse holds.*

**Proof.** (a) Let $t \in 1_L/n$ such that $m \vee t = 1_L$. Note that $S_t^{k\geq} \subseteq S_n^{k\geq}$ because $t \geq n$. Now, if $f \in S_m^{k\geq} \cap S_t^{k\geq}$, then $k_f \geq m \vee t = 1_L$, so $f = 0$. Thus, as $S_m^{k\geq}$ is essential in $S_n^{k\geq}$ by hypothesis, $S_t^{k\geq} = \{0\}$. Then, as $L$ is coretractable, $t = 1_L$, so that $m$ is superfluous in $1_L/n$.

For the converse, let $L$ be semi-injective, and assume that $S_m^{k\geq} \cap (S \circ f) = \{0\}$ for $f \in S_n^{k\geq}$. Then, by Theorem 3.19, $S \circ f = S_{k_f}^{k\geq}$, and thus, $S_m^{k\geq} \cap S_{k_f}^{k\geq} = \{0\}$. Since $S_m^{k\geq} \cap S_{k_f}^{k\geq} = S_{m\vee k_f}^{k\geq}$ and $L$ is coretractable, it follows that $m \vee k_f = 1_L$. Hence, $k_f = 1_L$, that is, $f = 0$. Therefore, $S_m^{k\geq}$ is essential in $S_n^{k\geq}$.

(b) Since $I \subseteq S^{k\geq}_{\bigwedge_{f\in I} k_f} \subseteq S^{k\geq}_{\bigwedge_{f\in J} k_f}$ and $I$ is essential in $S^{k\geq}_{\bigwedge_{g\in J} k_g}$, clearly $S^{k\geq}_{\bigwedge_{f\in I} k_f}$ is also essential in $S^{k\geq}_{\bigwedge_{g\in J} k_g}$. Hence, by part (a), it follows that $\bigwedge_{f\in I} k_f$ is superfluous in ${}^{1_L}/\bigwedge_{g\in J} k_g$.

(c) By part $(a)$, $S^{k\geq}_{\bigwedge_{f\in I} k_f}$ is essential in $S^{k\geq}_{\bigwedge_{g\in J} k_g}$. Now, let $K \in \mathcal{L}(S)$ such that $K \subseteq J$ and $I \cap K = \{0\}$. Since $K \subseteq J \subseteq S^{k\geq}_{\bigwedge_{g\in J} k_g}$ and $I = S^{k\geq}_{\bigwedge_{f\in I} k_f}$ is essential in $S^{k\geq}_{\bigwedge_{g\in J} k_g}$, $K = \{0\}$. Therefore, $I$ is essential in $J$.

(d) Let $x, y \in 1_L/m$, with $x, y < 1_L$. As $L$ is coretractable, both $S_x^{k\geq}$ and $S_y^{k\geq}$ are non-trivial. Since $S_x^{k\geq}, S_y^{k\geq} \subseteq S_m^{k\geq}$, and $S_m^{k\geq}$ is uniform by hypothesis, we obtain that

$$S_{x\vee y}^{k\geq} = S_x^{k\geq} \cap S_y^{k\geq} \neq \{0\}.$$

Therefore, $x \vee y \neq 1_L$, so that $m$ is hollow in $L$.

For the converse, note first that, by Theorem 3.11, $L^{op}$ is semi-projective. Also, since $m$ is hollow in $L$, $m$ is uniform in $L^{op}$. Thus, [8, Theorem 3.17(e)] provides that $Hom_{\mathcal{L_M}}\left(L^{op}, (1_L/m)^{op}\right)$ is a uniform right ideal of $End_{\mathcal{L_M}}(L^{op})$. The result follows from noting that $\left(Hom_{\mathcal{L_M}}\left(L^{op}, (1_L/m)^{op}\right)\right)^{op} = S_m^{k\geq}$ and that $End_{\mathcal{L_M}}((L^{op})^{op}) = S$.

(e) By (d), $S^{k\geq}_{\bigwedge_{f\in I} k_f}$ is a uniform left ideal of $S$. The fact that $I \subseteq S^{k\geq}_{\bigwedge_{f\in I} k_f}$ yields the result.

(f) Note first that $(i)$ implies that $m \neq 1_L$. Let $m \leq k < 1_L$. Then, since $L$ is coretractable, $S_k^{k\geq} \neq \{0\}$. As $S_m^{k\geq}$ is simple and $S_k^{k\geq} \subseteq S_m^{k\geq}$, it follows that $S_k^{k\geq} = S_m^{k\geq}$. Now, as $L$ co-generates $1_L/m$, there exists a family of linear morphisms $\{f_t\}_{t\in T} \subseteq S_m^{k\geq}$ such that $m = \bigwedge_{t\in T} k_{f_t}$. It follows that

$$m \leq k \leq \bigwedge_{t\in T} k_{f_t} = m.$$

Hence, $k = m$, and thus, the lattice $1_L/m$ is simple.

Suppose now that $(ii)$ holds. Then, by coretractability, $S_m^{k\geq} \neq \{0\}$. Let $f \in S_m^{k\geq}$ such that $f \neq 0$. By $(ii)$, $k_f = m$. Since $L$ is semi-injective, by Theorem 3.19

$$S_m^{k\geq} = S_{k_f}^{k\geq} = S \circ f.$$

Therefore, $S_m^{k\geq}$ is a simple left ideal.

(g) Necessity follows directly from $(f)$.

For sufficiency, note first that, by (f), $S^{k\geq}_{\bigwedge_{f\in I} k_f}$ is simple. Now, since $\bigwedge_{f\in I} k_f \neq 1_L$, it happens that $\{0\} \neq I \subseteq S^{k\geq}_{\bigwedge_{f\in I} k_f}$, so that $I = S^{k\geq}_{\bigwedge_{f\in I} k_f}$. $\square$

**Definition 3.24.** Let $L \in \mathcal{L_M}$, and let $S = End_{\mathcal{L_M}}(L)$. For $I \in \mathcal{L}(S)$, we call a function $\psi : I \longrightarrow S$ a *left morphism* if for any $g \in I$ and $f \in S$

$$\psi(f \circ g) = f \circ \psi(g).$$

(Note that, in this situation, $\psi(I)$ is necessarily a left ideal of $S$.)

In case $I = S$, we shall call $\psi$ a *left endomorphism*.

**Remark 3.25.** A left endomorphism $\psi : S \longrightarrow S$ satisfies that

$$\psi(f) = \psi(f \circ Id_L) = f \circ \psi(Id_L)$$

for any $f \in S$. Hence, the left endomorphism $\psi$ is completely determined by its effect on $Id_L$.

Note that, for a given $L$, the set of left endomorphisms from $S$ to $S$ is closed under composition. A left endomorphism will be called a *left monomorphism* if, with respect to this operation, it is cancellable on the left.

**Lemma 3.26.** *Let $L \in \mathcal{L}_{\mathcal{M}}$, and $S = End_{\mathcal{L}_{\mathcal{M}}}(L)$. For any left monomorphism $\varphi : S \longrightarrow S$,*

$$\{f \in S | \varphi(f) = 0\} = \{0\}.$$

**Proof.** Let $g \in S$ such that $\varphi(g) = 0$. Set $\psi_g : S \longrightarrow S$, such that $\psi_g(h) = h \circ g$. Note that $\psi_g$ is a left endomorphism. Then, for $h \in S$,

$$(\varphi \circ \psi_g)(h) = \varphi(\psi_g(h)) = \varphi(h \circ g) = h \circ \varphi(g) = h \circ 0 = 0.$$

Thus, $\varphi \circ \psi_g = 0 = \varphi \circ 0$, so that $\psi_g = 0$ because $\varphi$ is a left monomorphism. But then,

$$0 = \psi_g(Id_L) = Id_L \circ g = g. \qquad \square$$

**Lemma 3.27.** *Let $L \in \mathcal{L}_{\mathcal{M}}$, and let $S = End_{\mathcal{L}_{\mathcal{M}}}(L)$. For $f \in S$, if $f$ is an epimorphism, then $S \xrightarrow{\_\circ f} S$ is a left monomorphism. The converse is true for coretractable $L$.*

**Proof.** ($\Rightarrow$) Since $L \xrightarrow{f} L$ is a linear epimorphism, $S \xrightarrow{\_\circ f} S$ is injective. And, of course, every injective left endomorphism is a left monomorphism.

($\Leftarrow$) If $f(1_L) < 1_L$, since $L$ is coretractable, there exists $0 \neq g \in Hom_{\mathcal{L}_{\mathcal{M}}}({}^{1_L}/{}_{f(1_L)}, L)$. Set as $h$ the composite

$$L \xrightarrow{\vee f(1_L)} {}^{1_L}/{}_{f(1_L)} \xrightarrow{g} L.$$

Since $g \neq 0$ and $\_ \vee f(1_L)$ is an epimorphism, $0 \neq h \in S$. However, $h \circ f = 0$, so, by Lemma 3.26, $\_ \circ f$ cannot be a left monomorphism. $\square$

**Remark 3.28.** A left endomorphism $\psi : S \longrightarrow S$ is surjective if and only if there exists $f \in S$ such that $\psi(f) = Id_L$. Indeed, if $f \in S$ is such that $\psi(f) = Id_L$, then

$$\psi(g \circ f) = g \circ \psi(f) = g \circ Id_L = g$$

for any $g \in S$.

**Definition 3.29.** Let $L \in \mathcal{L}_{\mathcal{M}}$. We say that $L$ is *Hopfian* if every linear epimorphism $f : L \longrightarrow L$ is a monomorphism.

**Definition 3.30.** Let $L \in \mathcal{L}_{\mathcal{M}}$, and let $S = End_{\mathcal{L}_{\mathcal{M}}}(L)$. We call the monoid $S$ *Hopfian* if there does not exist a bijective left morphism between $S$ and a proper left ideal of $S$.

**Definition 3.31.** Let $L \in \mathcal{L}_{\mathcal{M}}$, and let $S = End_{\mathcal{L}_{\mathcal{M}}}(L)$. We say that $f \in S$ is *left regular* if $g \circ f = 0$ with $g \in S$ implies that $g = 0$.

**Lemma 3.32.** *For a lattice $L \in \mathcal{L}_{\mathcal{M}}$, the monoid $S = End_{\mathcal{L}_{\mathcal{M}}}(L)$ is Hopfian if every left regular element in $S$ is a unit.*

**Proof.** Let $I \in \mathcal{L}(S)$ and let $\psi : I \longrightarrow S$ be a bijective left morphism. Let $g \in I$ such that $\psi(g) = Id_L$. We claim that $g$ is left regular. Indeed, if $h \in S$ is such that $h \circ g = 0$, then

$$0 = \psi(0) = \psi(h \circ g) = h \circ \psi(g) = h \circ Id_L = h.$$

Thus, by hypothesis, $g$ is a unit, so that there exists $f \in S$ such that $Id_L = f \circ g \in I$. It follows that $I = S$. Therefore, $S$ is Hopfian. □

**Definition 3.33.** For $L \in \mathcal{L}_{\mathcal{M}}$ and $f \in S = End_{\mathcal{L}_{\mathcal{M}}}(L)$, the *left annihilator* of $f$ is

$$Ann_\ell(f) = \{g \in S|\ g \circ f = 0\}.$$

**Remark 3.34.** $Ann_\ell(f)$ is a left ideal of the monoid $S$ for any $L \in \mathcal{L}_{\mathcal{M}}$.

**Definition 3.35.** Let $L \in \mathcal{L}_{\mathcal{M}}$ and let $S = End_{\mathcal{L}_{\mathcal{M}}}(L)$. The *left singular ideal* of $S$ is

$$Z_\ell(S) = \{f \in S|\ Ann_\ell(f) \text{ is essential in } \mathcal{L}(S)\}.$$

**Remark 3.36.** $Z_\ell(S)$ is a two-sided ideal of the monoid $S$. The proof mirrors that in [8, Remark 3.32].

**Lemma 3.37.** *Let $L \in \mathcal{L}_{\mathcal{M}}$ and let $S = End_{\mathcal{L}_{\mathcal{M}}}(L)$. For $f \in S$, if $f$ is an epimorphism, then $f$ is left regular. The converse holds if $L$ is coretractable.*

**Proof.** Note first that

$$Ann_\ell(f) = \{g \in S|\ g \circ f = 0\} = S^{k\geq}_{f(1_L)}.$$

Also, a morphism $f \in S$ is an epimorphism if and only if $f(1_L) = 1_L$. Now, if $f \in S$ is an epimorphism, then

$$Ann_\ell(f) = S^{k\geq}_{f(1_L)} = S^{k\geq}_{1_L} = \{0\}.$$

However, $Ann_\ell(f) = \{0\}$ if and only if $f$ is left regular.

For coretractable $L$, observe that if $S^{k\geq}_{f(1_L)} = \{0\}$, then $f(1_L) = 1_L$. □

**Definition 3.38.** Let $L \in \mathcal{L}_{\mathcal{M}}$, and let $\mathcal{C}_L$ be the set of all coatoms in $L$. Then, the *Jacobson radical* of $L$ is

$$Jac(L) = \bigwedge_{x \in \mathcal{C}_L} x.$$

For a lattice $L \in \mathcal{L}_{\mathcal{M}}$, a left ideal $I$ of $S = End_{\mathcal{L}_{\mathcal{M}}}(L)$, and an initial interval $K$ of $L$, we denote by $(I)(K)$ (or just $IK$, when there is no room for ambiguity) the initial interval of $L$ determined by $\bigvee_{f \in I} f(1_K)$ (which, by [3, Lemma 0.6(1)], is a strongly invariant element of $L$). That is,

$$(I)(K) = \big(\bigvee_{f\in I} f(1_K)\big)/0_L.$$

**Definition 3.39.** Let $L \in \mathcal{L}_{\mathcal{M}}$ and $S = End_{\mathcal{L}_{\mathcal{M}}}(L)$. We say that $L$ is *quasi-projective* if for any linear epimorphism $L \xrightarrow{g} N$ and any linear morphism $L \xrightarrow{f} N$, there exists $f' \in S$ such that the following diagram is commutative.

$$\begin{array}{ccc} & & L \\ & \swarrow^{f'} & \downarrow f \\ L & \xrightarrow[g]{} & N \end{array}$$

In module-theoretic language, the above notion can be rendered as $L$ is $L$-projective, as $L$ belongs to its own projectivity class, or as $L$ belongs to its own projectivity domain.

**Theorem 3.40.** *Let $L \in \mathcal{L}_{\mathcal{M}}$ be coretractable, and let $S = End_{\mathcal{L}_{\mathcal{M}}}(L)$. Then, the following statements hold:*

(a) *$L$ is Hopfian if and only if $S$ is Hopfian.*

(b) *If $L$ is quasi-projective, then each left regular element in $S$ has a right inverse in $S$.*

(c) *$Z_\ell(S) \subseteq \{f \in S|\ f(1_L)$ is superfluous in $L\}$, and further, $(Z_\ell(S))(L) \subseteq {}^{Jac(L)}/_{0_L}$.*

**Proof.** (a) ($\Rightarrow$) Let $f \in S$ be left regular. By Lemma 3.37, $f$ is an epimorphism. Since $L$ is Hopfian, $f$ is also a monomorphism and, consequently, a unit. Therefore, $S$ is Hopfian by Lemma 3.32.

($\Leftarrow$) Assume that $L$ is not Hopfian. Then, there exists a linear epimorphism $f : L \longrightarrow L$ that is not a linear monomorphism. Then, $Id_L \notin S^{k\geq}_{k_f}$, because $k_f \neq 0_L$. Therefore, $S^{k\geq}_{k_f}$ is a proper left ideal of $S$. Consider now the induced isomorphism $\overline{f} : 1_L/k_f \longrightarrow L$. Set as $\psi$ the composite

$$S^{k\geq}_{k_f} \xrightarrow{\alpha} Hom_{\mathcal{L}_{\mathcal{M}}}(1_L/k_f, L) \xrightarrow{-\circ\overline{f}^{-1}} S,$$

where $\alpha$ is the bijection provided by Lemma 3.18.

We claim that $\psi : S^{k\geq}_{k_f} \to S$ is a bijective left morphism. Indeed, given $g \in S^{k\geq}_{k_f}$ and $h \in S$,

$$\psi(h\circ g) = \alpha(h\circ g)\circ\overline{f}^{-1} = (h\circ g)|_{1_L/k_f}\circ\overline{f}^{-1} = h\circ(g|_{1_L/k_f})\circ\overline{f}^{-1} = h\circ\alpha(g)\circ\overline{f}^{-1} = h\circ\psi(g),$$

so that $\psi$ is a left morphism. Also, since $\overline{f}^{-1}$ is surjective, $\_\circ\overline{f}^{-1}$ is injective, and then so is $\psi$. For surjectivity, it suffices to verify that $\_\circ\overline{f}^{-1}$ is surjective. Let then $g \in S$, and note that $g\circ\overline{f} \in Hom_{\mathcal{L}_{\mathcal{M}}}(1_L/k_f, L)$ is such that $(g\circ\overline{f})\circ\overline{f}^{-1} = g$.

Therefore, the monoid $S$ is not Hopfian.

(b) Let $f \in S$ be left regular. By Lemma 3.37, $f$ is an epimorphism. Thus, $f$ induces the lattice isomorphism $\overline{f} : 1_L/k_f \to L$. Now, the following diagram has an exact row, so by quasi-projectivity, there is $g \in S$ that makes it commutative:

$$\begin{array}{ccccc} & & L & & \\ & {\scriptstyle g} \swarrow & \downarrow {\scriptstyle \overline{f}^{-1}} & & \\ L & \xrightarrow[\_\vee k_f]{} & 1_L/k_f & \longrightarrow & 0. \end{array}$$

Let $x \in L$. As $f$ is a linear morphism,

$$(f \circ g)(x) = f(g(x)) = f(g(x) \vee k_f) = f(\overline{f}^{-1}(x)) = x.$$

Therefore, $g$ is a right inverse of $f$ in $S$.

(c) As noted in the proof of Lemma 3.37, $Ann_\ell(f) = S^{k\geq}_{f(1_L)}$ for each $f \in S$. In particular, when $f \in Z_\ell(S)$, $S^{k\geq}_{f(1_L)}$ is essential in $S$. Thus, by Theorem 3.23(a), $f(1_L)$ is superfluous in $L$.

Now, for the second statement, let $x \in L$ be a coatom and let $f \in Z_\ell(S)$. Then, since $f(1_L)$ is superfluous in $L$, $f(1_L) \vee x = x$, that is, $f(1_L) \leq x$. It follows that

$$f(1_L) \leq \bigwedge_{x \in \mathcal{C}_L} x = Jac(L),$$

and hence,

$$\bigvee_{f \in Z_\ell(S)} f(1_L) \leq Jac(L). \qquad \square$$

**Definition 3.41.** Let $L \in \mathcal{L}_\mathcal{M}$ and let $S = End_{\mathcal{L}_\mathcal{M}}(L)$. We call the set

$$Soc_\ell(S) = \bigcup \{ I \mid I \text{ is a simple left ideal of } S \}$$

the *left socle* of $S$.

**Lemma 3.42.** *Let $L \in \mathcal{L}_\mathcal{M}$ and $S = End_{\mathcal{L}_\mathcal{M}}(L)$. Then, $L$ is semi-injective if and only if $I = S^{k\geq}_{\bigwedge_{f\in I} k_f}$ for any cyclic left ideal $I$ of $S$.*

**Proof.** Let $I = S \circ g$ for some $g \in S$. For any $h \in S$, it is clear that $k_g \leq k_{h\circ g}$, and hence,

$$k_g \leq \bigwedge_{h \in S} k_{h \circ g} = \bigwedge_{f \in I} k_f \leq k_g.$$

Therefore, $\bigwedge_{f\in I} k_f = k_g$, hence the result follows from Theorem 3.19. $\square$

**Proposition 3.43.** *Let $L \in \mathcal{L}_\mathcal{M}$ be coretractable and semi-injective, and let $S = End_{\mathcal{L}_\mathcal{M}}(L)$. Then,*

(a) $Z_\ell(S) = \{f \in S | f(1_L)$ *is superfluous in* $L\}$.

(b) $\bigwedge_{f \in Soc_\ell(S)} k_f = Jac(L)$.

(c) $Soc_\ell(S) \subseteq S^{k\geq}_{Jac(L)}$.

**Proof.** (a) By Theorem 3.40($c$),

$$Z_\ell(S) \subseteq \{f \in S | f(1_L) \text{ is superfluous in } L\}.$$

For the reverse inclusion, let $f \in S$ be such that $f(1_L)$ is superfluous in $L$. Since $L$ is semi-injective, by Theorem 3.23(a), $Ann_\ell(f) = S^{k\geq}_{f(1_L)}$ is essential in $S$. Therefore, $f \in Z_\ell(S)$, and so,

$$\{f \in S | f(1_L) \text{ is superfluous in } L\} \subseteq Z_\ell(S).$$

(b) On the one hand, given a coatom $a \in L$, $S^{k\geq}_a \neq \{0\}$ because $L$ is coretractable. Further, since $1_L/a$ is a simple lattice, for any nonzero $f \in S^{k\geq}_a$, it must happen that $k_f = a$. Also, as $L$ is semi-injective, by Theorem 3.23(f), $S^{k\geq}_a$ is a simple left ideal. Thus,

$$\bigwedge_{f \in Soc_\ell(S)} k_f \leq \bigwedge_{f \in S^{k\geq}_a} k_f = a,$$

so that

$$\bigwedge_{f \in Soc_\ell(S)} k_f \leq Jac(L).$$

On the other hand, since $L$ is semi-injective, any simple left ideal $I$ of $S$ can be written as $I = S^{k\geq}_{\bigwedge_{f\in I} k_f}$, by Lemma 3.42. Hence, by Theorem 3.23(g), ${}^{1_L}/{\bigwedge_{f\in I} k_f}$ is a simple lattice, that is, $\bigwedge_{f\in I} k_f$ is a coatom of $L$. Therefore,

$$Jac(L) \leq \bigwedge_{f \in Soc_\ell(S)} k_f.$$

(c) Let $I$ be a simple left ideal of $S$. Since $L$ is semi-injective, by Lemma 3.42, $I = S^{k\geq}_{\bigwedge_{f\in I} k_f}$. Further, by Theorem 3.23(g), ${}^{1_L}/{\bigwedge_{f\in I} k_f}$ is a simple lattice, that is, $\bigwedge_{f\in I} k_f$ is a coatom of $L$. Thus, for any $f \in I$,

$$Jac(L) \leq \bigwedge_{f\in I} k_f \leq k_f,$$

so that $f \in S^{k\geq}_{Jac(L)}$. Consequently, $Soc_\ell(S) \subseteq S^{k\geq}_{Jac(L)}$. □

**Definition 3.44.** Let $L \in \mathcal{L}_\mathcal{M}$ and $S = End_{\mathcal{L}_\mathcal{M}}(L)$. We say that $L$ is *weakly Hopfian* if for each linear epimorphism $f \in S$, one has that $k_f$ is a superfluous element in $L$.
Regarding the monoid $S$, we say that it is *left weakly co-Hopfian* if for each left monomorphism $\varphi : S \longrightarrow S$, one has that $\varphi(S)$ is an essential left ideal of $S$.

**Example 3.45.** Let $\mathcal{P}(\mathbb{N})$ denote the power set of the set $\mathbb{N}$ of natural numbers. As any power set, $\mathcal{P}(\mathbb{N})$ is partially ordered by inclusion, and furthermore, $\mathcal{P}(\mathbb{N})$ is a complete modular lattice whose join and meet operations are the union and intersection of sets, respectively. Clearly, $0_{\mathcal{P}(\mathbb{N})} = \emptyset$ and $1_{\mathcal{P}(\mathbb{N})} = \mathbb{N}$.

Write $2\mathbb{N}$ for the set of even natural numbers, and $2\mathbb{N}+1$ for the set of odd natural numbers. The mapping $h : \mathcal{P}(\mathbb{N}) \longrightarrow {}^{2\mathbb{N}}/_{\emptyset}$ such that

$$X \mapsto 2X = \{2x \mid x \in X\}$$

is a lattice isomorphism. Also, by modularity, we have an isomorphism

$$^{\mathbb{N}}/_{2\mathbb{N}+1} = {}^{2\mathbb{N}\cup(2\mathbb{N}+1)}/_{2\mathbb{N}+1} \overset{k}{\cong} {}^{2\mathbb{N}}/_{2\mathbb{N}\cap(2\mathbb{N}+1)} = {}^{2\mathbb{N}}/_{\emptyset}.$$

In this way, we obtain a lattice isomorphism $f' = h^{-1} \circ k : {}^{\mathbb{N}}/_{2\mathbb{N}+1} \longrightarrow \mathcal{P}(\mathbb{N})$.

Let us now define the mapping $f : \mathcal{P}(\mathbb{N}) \longrightarrow \mathcal{P}(\mathbb{N})$ by $f(X) = f'(X \cup (2\mathbb{N}+1))$. Note that $f$ is a linear epimorphism in $S = End_{\mathcal{L}_{\mathcal{M}}}(\mathcal{P}(\mathbb{N}))$. Thus, by Lemma 3.27, the mapping $\_ \circ f : S \longrightarrow S$ is a left monomorphism. However, note that $S \circ f \subseteq S^{k\geq}_{2\mathbb{N}+1}$, and that $S^{k\geq}_{2\mathbb{N}+1}$ is not an essential left ideal of $S$ because $S^{k\geq}_{2\mathbb{N}+1} \cap S^{k\geq}_{2\mathbb{N}} = \{0\}$. (The fact that $S^{k\geq}_{2\mathbb{N}} \neq \{0\}$ can be verified analogously to our construction of $0 \neq f \in S^{k\geq}_{2\mathbb{N}+1}$.) Therefore, $S \circ f$ is not an essential left ideal of $S$, so that $S$ is not left weakly co-Hopfian.

Let $L \in \mathcal{L}_{\mathcal{M}}$ and $S = End_{\mathcal{L}_{\mathcal{M}}}(L)$. For a left endomorphism $\varphi : S \longrightarrow S$, let us call the set

$$\{g \in S \mid \varphi(g) = 0\}$$

the *kernel* of $\varphi$. (It is a left ideal of $S$.)

**Lemma 3.46.** *Let $L \in \mathcal{L}_{\mathcal{M}}$ and $S = End_{\mathcal{L}_{\mathcal{M}}}(L)$, and let $\varphi : S \longrightarrow S$ be a left endomorphism. If $\varphi$ is a left monomorphism, then the kernel of $\varphi$ is trivial. The converse holds for coretractable $L$.*

**Proof.** The necessity is Lemma 3.26.

For the sufficiency, note first that, by Remark 3.25, $\varphi = \_ \circ f$ where $f = \varphi(Id_L)$. We may then suppose that $f$ is left regular. Since $L$ is coretractable, by Lemma 3.37, $f$ is an epimorphism, hence, by Lemma 3.27, $\varphi$ is a left monomorphism. $\square$

**Definition 3.47.** We say that a lattice $L \in \mathcal{L}_{\mathcal{M}}$ is *weakly co-compressible* if for all $a \in L$ with $a \neq 1_L$, there exists a linear morphism $f \in S^{k\geq}_a$ such that $f^2 \neq 0$.

**Definition 3.48.** A lattice $L \in \mathcal{L}_{\mathcal{M}}$ is *cyclic* if it has a superfluous coatom.

**Definition 3.49.** We say that a lattice $L \in \mathcal{L}_{\mathcal{M}}$ is *finitely generated* if for any subset $\{x_a\}_{a\in A} \subseteq L$ such that $\bigvee_{a\in A} x_a = 1_L$, there exists a finite subset $F \subseteq A$ with

$$\bigvee_{a\in F} x_a = 1_L.$$

Similarly, we say that the monoid $S = End_{\mathcal{L}_\mathcal{M}}(L)$ is *finitely cogenerated* if for any family of left ideals $\{I_a\}_{a\in A}$ of $S$ such that $\bigcap_{a\in A} I_a = \{0\}$, there exists a finite subset $F$ of $A$ with

$$\bigcap_{a\in F} I_a = \{0\}.$$

We say that $S$ is *semiprime* if for every left ideal $I$ and $0 \neq n \in \mathbb{N}$,

$$I \neq \{0\} \Rightarrow I^n \neq \{0\},$$

where $I^n = \{f_1 \circ \ldots \circ f_n \mid f_i \in I \text{ for each } 1 \leq i \leq n\}$.

**Definition 3.50.** We say that the monoid $S = End_{\mathcal{L}_\mathcal{M}}(L)$ is *cocyclic* if there exists a non-trivial endomorphism $f \in S$ that lies in every nonzero left ideal of $S$.

**Theorem 3.51.** *Let $L \in \mathcal{L}_\mathcal{M}$ be coretractable and semi-injective, and let $S = End_{\mathcal{L}_\mathcal{M}}(L)$. Then, the following statements are true:*

(a) *$L$ is weakly Hopfian if and only if $S$ is left weakly co-Hopfian.*

(b) *$Hom_{\mathcal{L}_\mathcal{M}}(L, a/0_L) = \{0\}$ for every superfluous element $a \in L$ if and only if $Z_\ell(S) = \{0\}$.*

(c) *If $L$ is weakly co-compressible, then $S$ is semiprime.*

(d) *$L$ is cyclic if and only if $S$ is cocyclic.*

(e) *If $S$ is finitely cogenerated, then $L$ is finitely generated.*

**Proof.** (a) ($\Rightarrow$) Let $\varphi : S \longrightarrow S$ be a left monomorphism. Note that, by Remark 3.25, $\varphi = \_ \circ f$ where $f = \varphi(Id_L)$. As $L$ is coretractable, by Lemma 3.27, $f$ is a linear epimorphism. Now, as $L$ is weakly Hopfian, the element $k_f$ is superfluous in $L$, and thus also in ${}^{1_L}/_{\bigwedge_{g\in S} k_g}$ (seeing as $\bigwedge_{g\in S} k_g \leq k_f$). Set $I = \varphi(S) = S \circ f$. By Lemma 3.42 (and its proof), $I = S^{k\geq}_{\bigwedge_{f\in I} k_f}$ and $\bigwedge_{h\in I} k_h = k_f$, so Theorem 3.23(c) (making $J = S$) provides that $I = \varphi(S)$ is essential in $S$. Therefore, $S$ is left weakly co-Hopfian.

($\Leftarrow$) Let $f \in S$ be a linear epimorphism. By Lemma 3.27, $\_ \circ f$ is a left monomorphism. Since $S$ is weakly co-Hopfian (and bearing in mind Theorem 3.19), $S \circ f = S^{k\geq}_{k_f}$ is essential in $S$. Therefore, by Theorem 3.23(a), $k_f$ is superfluous in $L$.

(b) Note first that, since $L$ is coretractable and semi-injective,

$$Z_\ell(S) = \{f \in S | f(1_L) \text{ is superfluous in } L\}$$

due to Proposition 3.43($a$).

($\Rightarrow$) Let $f \in S$ be such that $f(1_L)$ is superfluous in $L$. Then, the corestriction of $f$ to its image lies in $Hom_{\mathcal{L}_\mathcal{M}}(L, f(1_L)/0_L) = \{0\}$. Therefore, $f = 0$, and consequently, $Z_\ell(S) = \{0\}$.

($\Leftarrow$) Let $a \in L$ be superfluous in $L$, and let $f \in Hom_{\mathcal{L}_\mathcal{M}}(L, a/0_L)$. If $\iota$ denotes the appropriate inclusion mapping, then the composite $\iota \circ f \in S$. Furthermore, $(\iota \circ f)(1_L) \leq a$, so that $(i \circ f)(1_L)$ is a superfluous element in $L$. Thus, $i \circ f \in Z_\ell(S) = \{0\}$ which implies that $f = 0$. Therefore, $Hom_{\mathcal{L}_\mathcal{M}}(L, a/0_L) = \{0\}$.

(c) Let us assume that $L$ is weakly co-compressible and that $S$ is not semiprime. Then, there exists a non-trivial left ideal $I$ of $S$ such that $I^k = \{0\}$ for some $k \geq 2$. Take the least such $k$, so that $I^{k-1} \neq \{0\}$. Now, given $0 \neq f \in I^{k-1}$, and in view of Theorem 3.19, $S^{k\geq}_{k_f} = S \circ f \subseteq I^{k-1}$. Furthermore, since $k_f \neq 1_L$ and $L$ is weakly co-compressible, there exists $h \in S^{k\geq}_{k_f}$ such that $h^2 \neq 0$. However, $h^2 \in I^{2k-2} = \{0\}$ because $2k - 2 \geq k$ for $k \geq 2$. Therefore, $S$ is semiprime.

(d) ($\Rightarrow$) Let $a \in L$ be a superfluous coatom. Since $L$ is semi-injective, Theorem 3.23, parts (a) and (f), gives that $S^{k\geq}_a$ is a simple and essential left ideal of $S$. It follows that any nonzero element of $S^{k\geq}_a$ lies in every nonzero left ideal of $S$.

($\Leftarrow$) Assume that $0 \neq g \in S$ belongs to every non-trivial left ideal of $S$. For each $x \in L$ with $x \neq 1_L$, $L$ being coretractable implies that $Hom_{\mathcal{L}_\mathcal{M}}(1_L/x, L) \neq \{0\}$. Further, by Lemma 3.18, $S^{k\geq}_x \neq \{0\}$, and so, $g \in S^{k\geq}_x$. It follows that $x \leq k_g$ for all $1_L > x \in L$. Moreover, as $g \neq 0$, $k_g \neq 1_L$, so that $k_g$ is a superfluous coatom.

(e) Let $\{x_a\}_{a\in A} \subseteq L$ be such that $\bigvee_{a\in A} x_a = 1_L$. Then,

$$\{0\} = S^{k\geq}_{\bigvee_{a\in A} x_a} = \bigcap_{a\in A} S^{k\geq}_{x_a}.$$

Hence, as $S$ is finitely cogenerated, there exists a finite subset $F \subseteq A$ such that

$$\bigcap_{a\in F} S^{k\geq}_{x_a} = S^{k\geq}_{\bigvee_{a\in F} x_a} = \{0\}.$$

By Lemma 3.18 and the fact that $L$ is coretractable, $\bigvee_{a\in F} x_a = 1_L$. Therefore, $L$ is finitely generated. $\square$

**Definition 3.52.** A lattice $L \in \mathcal{L}_\mathcal{M}$ is *co-Hopfian* if every linear monomorphism $L \xrightarrow{f} L$ is a linear epimorphism.

**Proposition 3.53.** *If $L \in \mathcal{L}_\mathcal{M}$ is semi-injective and Hopfian, then $L$ is co-Hopfian.*

**Proof.** Let $f : L \longrightarrow L$ be a linear monomorphism. Since $L$ is semi-injective, there exists a linear morphism $g : L \longrightarrow L$ that makes the diagram

$$\begin{array}{ccc} L & \xrightarrow{f} & L \\ {\scriptstyle Id}\downarrow & \swarrow {\scriptstyle g} & \\ L & & \end{array}$$

commutative, that is, $g \circ f = Id_L$. Hence, as a function, $g$ has a right inverse, so it is surjective. Then, $g$ is a linear epimorphism. Since $L$ is Hopfian, $g$ is also a linear monomorphism, and hence an isomorphism. Consequently,

$$1_L = g^{-1}(1_L) = g^{-1}((g \circ f)(1_L)) = f(1_L).$$

Thus, $f$ is an epimorphism, and so $L$ is co-Hopfian. □

**Definition 3.54.** We say that a lattice $L \in \mathcal{L}_{\mathcal{M}}$ is *directly finite* if it is not isomorphic to any initial interval $a/0_L$, where $a < 1_L$ and $a$ has a complement in $L$.

**Proposition 3.55.** *If $L \in \mathcal{L}_{\mathcal{M}}$ is co-Hopfian, then $L$ is directly finite.*

**Proof.** If $L$ is not directly finite, then, in particular, there exists an isomorphism $\alpha : L \longrightarrow a/0_L$ for some proper $a \in L$. Now, if $a/0_L \xrightarrow{\iota} L$ denotes the inclusion mapping, then the composite $\iota \circ \alpha : L \longrightarrow L$ is a monomorphism that is not an epimorphism. Thus, $L$ is not co-Hopfian. □

**Proposition 3.56.** *Let $L \in \mathcal{L}_{\mathcal{M}}$ be semi-injective, and let $a \in L$ be essential and strongly invariant in $L$. Then, $L$ is co-Hopfian if and only if $a/0_L$ is co-Hopfian.*

**Proof.** By Theorem 3.15, $a$ is strongly invariant in $L^{op}$. Further, the fact that $a$ is essential in $L$ implies that $a$ is superfluous in $L^{op}$. Moreover, since $L$ is semi-injective, Theorem 3.11 gives that $L^{op}$ is semi-projective. Hence, by [8, Proposition 3.58], $L^{op}$ is Hopfian if and only if $(a/0_L)^{op}$ is Hopfian. It follows that $L$ is co-Hopfian if and only if $L^{op}$ is Hopfian, which holds if and only if $(a/0_L)^{op}$ is Hopfian, which is equivalent to $a/0_L$ being co-Hopfian. □

We say that a lattice $L \in \mathcal{L}_{\mathcal{M}}$ is *atomic* if for every $a \in L$ with $a \neq 0_L$, the initial interval $a/0_L$ has at least one atom.

**Definition 3.57.** Let $L \in \mathcal{L}_{\mathcal{M}}$, and let $\mathcal{A}_L$ be the set of all atoms of $L$. Then, the *socle* of $L$ is

$$Soc(L) = \left(\bigvee \mathcal{A}_L\right)/0_L.$$

**Corollary 3.58.** *Let $L \in \mathcal{L}_{\mathcal{M}}$ be semi-injective and atomic. Then, $L$ is co-Hopfian if and only if $Soc(L)$ is co-Hopfian.*

**Proof.** According to [2, Proposition 2.2], $\bigvee \mathcal{A}_L$ is strongly invariant in $L$. Also, it is readily verified (and noted, for example, in [6, p. 120]) that, as $L$ is atomic, $\bigvee \mathcal{A}_L$ is the smallest essential element in $L$. The result follows from Proposition 3.56. □

**Definition 3.59.** For a lattice $L \in \mathcal{L}_{\mathcal{M}}$, we introduce the following conditions:

- C1: For each $a \in L$ with $0_L < a$, there exists a complement $c \in L$ such that $c \geq a$ and $a$ is essential in $c/0_L$.
- C2: If $a \in L$ is such that $a/0_L \cong c/0_L$ and $c$ a complement in $L$, then $a$ is a complement.
- C3: Given two complements $k$ and $c$ in $L$, with $k \wedge c = 0_L$, the element $k \vee c$ is a complement.
- D1: For each $a \in L$, there exists a complement $c \in L$ such that $c \leq a$ and $a$ is superfluous in $1_L/c$.
- D2: If $a \in L$ is such that $1_L/a \cong c/0_L$, with $c$ a complement in $L$, then, $a$ is a complement.
- D3: Given two complements $k$ and $c$ in $L$ with $k \vee c = 1_L$, the element $k \wedge c$ is a complement.

**Definition 3.60.** Let $L \in \mathcal{L}_{\mathcal{M}}$. We say that $L$ is *extending* if it satisfies condition C1. Moreover, we say that $L$ is *continuous* if it is extending and satisfies condition C2, and that it is *quasi-continuous* if it is extending and satisfies condition C3.

**Lemma 3.61.** *Any semi-injective lattice $L \in \mathcal{L}_{\mathcal{M}}$ is a C2 lattice.*

**Proof.** Let $L \in \mathcal{L}_{\mathcal{M}}$ be semi-injective, and let $a, c \in L$ such that $a/0_L \cong c/0_L$, with $c$ a complement in $L$. Note that, by Theorem 3.11, $L^{op}$ is semi-projective, so it satisfies condition D2 by [8, Lemma 3.63]. Now, as $a/0_L \cong c/0_L$ in $L$, $(a/0_L)^{op} \cong (c/0_L)^{op}$ in $L^{op}$. Furthermore, if $d$ denotes the complement of $c$ in $L$, by modularity, $(c/0_L) \cong (1_L/d)$ in $L$, so that $(c/0_L)^{op} \cong (1_L/d)^{op}$ in $L^{op}$. Therefore, $(a/0_L)^{op} \cong (1_L/d)^{op}$ in $L^{op}$. As $L^{op}$ is a D2 lattice, $a$ is a complement in $L^{op}$, and consequently, a complement in $L$. □

**Proposition 3.62.** *Let $L \in \mathcal{L}_{\mathcal{M}}$. If $L$ is a C2 lattice, then $L$ is a C3 lattice.*

**Proof.** See [4, Proposition 1.10(5)]. □

**Theorem 3.63.** *Let $L \in \mathcal{L}_{\mathcal{M}}$ be semi-injective. Then, the following statements are equivalent:*

(1) *$L$ is extending.*
(2) *$L$ is continuous.*

(3) *$L$ is quasi-continuous.*

**Proof.** (1) $\Rightarrow$ (2) By Lemma 3.61, seeing as $L$ is semi-injective.

(2) $\Rightarrow$ (3) By Proposition 3.62.

(3) $\Rightarrow$ (1) By definition. □

**Definition 3.64.** We say that a lattice $L \in \mathcal{L}_{\mathcal{M}}$ is *indecomposable* if the only complements in $L$ are $0_L$ and $1_L$.

**Proposition 3.65.** *Let $L \in \mathcal{L}_{\mathcal{M}}$ be indecomposable and semi-injective. Then, $L$ is continuous if and only if $L$ is uniform.*

**Proof.** ($\Rightarrow$) Since $L$ is continuous, it is extending. Then, for any nonzero $x \in L$, there exists a complement $c \geq x$ such that $x$ is essential in $c/0_L$. Since $L$ is indecomposable, necessarily $c = 1_L$. Thus, $x$ is essential in $L$. Therefore, $L$ is uniform.

($\Leftarrow$) Let $x \in L$ with $x \neq 0$. Since $L$ is uniform, we have that $x$ is essential in $L = 1_L/0_L$. As $1_L$ is always a complement in $L$, $L$ satisfies condition C1, that is, $L$ is extending. Lemma 3.61 ends the proof. □

**Definition 3.66.** We say that a lattice $L \in \mathcal{L}_{\mathcal{M}}$ is *pseudo semi-injective* if for any two linear morphisms $f, g \in End_{\mathcal{L}_{\mathcal{M}}}(L)$ with $k_f = k_g$, there exists $h \in End_{\mathcal{L}_{\mathcal{M}}}(L)$ such that $h \circ f = g$.

**Remark 3.67.** Any semi-injective lattice is pseudo semi-injective.

Next, we show a lattice that is not pseudo semi-injective, and a lattice that is pseudo semi-injective but not semi-injective.

**Example 3.68.** Consider the lattice $L = \{\frac{1}{n}\}_{n \in \mathbb{N} \setminus \{0\}} \cup \{0\}$ with the order induced by $\mathbb{R}$.

Set $g : L \longrightarrow L$ such that $g(0) = 0$, $g(1) = \frac{1}{2}$, $g(\frac{1}{2}) = \frac{1}{3}$, and $g(\frac{1}{n}) = \frac{1}{n+1}$ for all $n \geq 3$. Clearly, $g$ is a linear monomorphism. Then, for the diagram

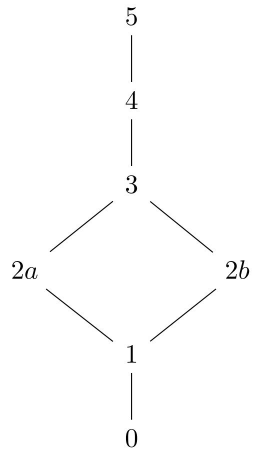
$$\begin{array}{ccc} L & \xrightarrow{g} & L \\ \downarrow{\scriptstyle Id_L} & & \\ L & & \end{array}$$

there is no $h \in End_{\mathcal{L}_{\mathcal{M}}}(L)$ such that $h \circ g = Id_L$. Indeed, if such endomorphism existed, it would happen that

$$1 = Id_L(1) = h(g(1)) = h(\tfrac{1}{2}) < h(1) \leq 1,$$

a contradiction. Therefore, $L$ is not a pseudo semi-injective lattice.

**Example 3.69.** Let us denote by $L$ the following lattice:




We first claim that $L$ is not semi-injective. Indeed, by [8, Example 3.72], $L^{op}$ is not semi-projective. Hence, by Theorem 3.11 $L$, is not semi-injective.

Let us now show that $L$ is pseudo semi-injective. Observe that only the quotient intervals $5/5$, $5/4$, $5/3$, and $5/0$ are isomorphic to an initial interval of $L$. Thus, the kernel of any linear endomorphism of $L$ lies in the set $\{5, 4, 3, 0\}$. Let $f, g \in End_{\mathcal{L_M}}(L)$ such that $k_f = k_g$. If $k_f = k_g = 5$, then $f = g$ and the map $h = Id_L$ satisfies $h \circ f = g$. Similarly, one shows that $f = g$ when $k_f = k_g = 4$. Suppose now that $k_f = k_g = 3$. Then,

$$f(5)/0 \cong 5/k_f = 5/k_g \cong g(5)/0.$$

Here, we find two possible cases: $f(5) = g(5)$ or $f(5) \neq g(5)$. If $f(5) = g(5)$, then $f(5) = g(5) \in \{2a, 2b\}$, $f(4) = 1 = g(4)$ and $f(x) = 0 = g(x)$ for all $x \leq 3$, so that $f = g$. If $f(5) \neq g(5)$, there are two following subcases: $f(5) = 2a$ and $g(5) = 2b$, or $f(5) = 2b$ and $g(5) = 2a$. Set $h : L \longrightarrow L$ by $h(2a) = 2b$, $h(2b) = 2a$, and $h(x) = x$ for $2a \neq x \neq 2b$. Clearly, $h$ is a linear morphism such that $h \circ f = g$.

Lastly, if $k_f = k_g = 0$, then $f$ and $g$ are linear monomorphisms, and thus, $L$ being finite, lattice isomorphisms. Hence, $h = g \circ f^{-1}$ satisfies $h \circ f = g$.

Therefore, $L$ is pseudo semi-injective.

**Lemma 3.70.** *A lattice $L \in \mathcal{L_M}$ is pseudo semi-injective if and only if for any $f, g \in End_{\mathcal{L_M}}(L)$ with $k_f = k_g$, one has that $End_{\mathcal{L_M}}(L) \circ f = End_{\mathcal{L_M}}(L) \circ g$.*

**Proof.** ($\Rightarrow$) Let $f, g \in End_{\mathcal{L_M}}(L)$ such that $k_f = k_g$. Since $L$ is pseudo semi-injective, there exists $h \in End_{\mathcal{L_M}}(L)$ such that $h \circ f = g$. Then,

$$End_{\mathcal{L_M}}(L) \circ g \subseteq End_{\mathcal{L_M}}(L) \circ f.$$

Similarly, $h' \circ g = f$ for some $h' \in End_{\mathcal{L_M}}(L)$, and so,

$$End_{\mathcal{L_M}}(L) \circ f \subseteq End_{\mathcal{L_M}}(L) \circ g.$$

Therefore, equality holds.

($\Leftarrow$) Let $f, g \in End_{\mathcal{L}_{\mathcal{M}}}(L)$ such that $k_f = k_g$. By hypothesis, we have that

$$End_{\mathcal{L}_{\mathcal{M}}}(L) \circ f = End_{\mathcal{L}_{\mathcal{M}}}(L) \circ g.$$

Thus, there exists $h \in End_{\mathcal{L}_{\mathcal{M}}}(L)$ such that $h \circ f = Id_L \circ g = g$. Hence, $L$ is pseudo semi-injective. □

**Remark 3.71.** For a pseudo semi-injective lattice $L \in \mathcal{L}_{\mathcal{M}}$, if $f, g \in End_{\mathcal{L}_{\mathcal{M}}}(L)$ are such that $k_f = k_g = 0_L$ and $f(1_L)$ is essential in $L$, then the linear morphism $h \in End_{\mathcal{L}_{\mathcal{M}}}(L)$ that makes the diagram

$$\begin{array}{ccc} L & \xrightarrow{f} & L \\ \downarrow g & \swarrow h & \\ L & & \end{array}$$

commutative is a monomorphism. Indeed, by the proof of [1, Lemma 2.1],

$$0_L = k_g = k_{h \circ f} = \overline{f}^{-1}(f(1_L) \wedge k_h).$$

Then, $f(1_L) \wedge k_h = 0_L$, so that, $f(1_L)$ being essential in $L$, $k_h = 0_L$. Hence, $h$ is a monomorphism.

**Proposition 3.72.** *If $L \in \mathcal{L}_{\mathcal{M}}$ is pseudo semi-injective and uniform, then $L$ is co-Hopfian.*

**Proof.** Note that the zero lattice is co-Hopfian. Assume then that $L$ is non-trivial. Let $f \in End_{\mathcal{L}_{\mathcal{M}}}(L)$ be a linear monomorphism, and let us consider the following commutative diagram.

$$\begin{array}{ccc} L & \xrightarrow{f} & L \\ \downarrow Id_L & \swarrow h & \\ L & & \end{array}$$

Since $L$ is uniform, the element $f(1_L)$ is essential in $L$, so, by Remark 3.71, $h$ is a linear monomorphism. Then, as

$$1_L = Id_L(1_L) = h(f(1_L)),$$

it follows that $f(1_L) = 1_L$. Thus, $f$ is a linear epimorphism. Therefore, $L$ is co-Hopfian. □

**Corollary 3.73.** *If $L \in \mathcal{L}_{\mathcal{M}}$ is pseudo semi-injective and uniform, then $L$ is directly finite.*

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