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ARTINIAN RINGS AND MODULES EVERYWHERE

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Abstract. For any commutative ring A, the finiteness conditions are a useful tool for approximating its structure. These finiteness conditions are reflected in some way in its spectrum; for example, if A is a Noetherian ring, then $\operatorname{Spec}(A)$ is a Noetherian topological space; the converse is not necessarily true. Noetherianness of $\operatorname{Spec}(A)$ has an interesting consequence in the behaviour of hereditary torsion theories in **Mod**– A: they are of finite type; that is, for any hereditary torsion theory σ in **Mod**– A there exists a cofinal set of $\mathcal{L}(\sigma)$ consisting of finitely generated ideals. The aim of this work is to study rings and modules via finite type hereditary torsion theories. Therefore, we restrict ourselves to considering hereditary torsion theories defined by finitely generated ideals and finiteness conditions relative to these theories, extending some type of rings and modules as (totally) Noetherian, (totally) Artinian or Artinian*.

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1. Introduction

One of the techniques for studying commutative rings is to consider their spectra since the spectrum of a commutative ring contains enough information about the ring itself; this is the case for a Noetherian or Artinian ring. Furthermore, $\operatorname{Spec}(A)$ has information in other weaker cases; for example, if $\operatorname{Spec}(A)$ is a Noetherian topological space. For every Noetherian ring A, the spectrum $\operatorname{Spec}(A)$ is a Noetherian topological space, but the converse does not necessarily hold; another example of a ring with a Noetherian spectrum are Laskerian rings (see [3], [8]). Our goal is to delve deeper into the study of rings with Noetherian spectra.

Since in every Noetherian topological space, every open set is quasi–compact, every prime ideal is the radical of a finitely generated ideal, and the converse also

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holds. For more general sets as generically closed subsets $\mathcal{K} \subseteq \operatorname{Spec}(A)$, since they are an intersection of open subsets, they also are quasi–compact subsets whenever $\operatorname{Spec}(A)$ is Noetherian.

Generically closed subsets appear in the following example: for any prime ideal $\mathfrak{p} \in \operatorname{Spec}(A)$, the multiplicative subset $\Sigma = A \setminus \mathfrak{p}$ defines a hereditary torsion theory $\sigma_{A\setminus\mathfrak{p}}$ with Gabriel filter $\mathcal{L}(\sigma_{A\setminus\mathfrak{p}}) = \{\mathfrak{h} \subseteq A \mid \mathfrak{h} \nsubseteq \mathfrak{p}\}$. In general, a hereditary torsion theory σ is defined by its Gabriel filter $\mathcal{L}(\sigma)$, and produces a partition of $\operatorname{Spec}(A)$ in two subsets $\mathcal{K}(\sigma)$ and $\mathcal{Z}(\sigma) = \operatorname{Spec}(A) \cap \mathcal{L}(\sigma)$, being $\mathcal{K}(\sigma)$ closed under generalizations. Furthermore, for any prime ideal $\mathfrak{p} \in \mathcal{K}(\sigma)$, we have $\sigma \leq \sigma_{A\setminus\mathfrak{p}}$; that is, $\mathcal{L}(\sigma) \subseteq \mathcal{L}(\sigma_{A\setminus\mathfrak{p}})$, so $\sigma \leq \wedge\{\sigma_{A\setminus\mathfrak{p}} \mid \mathfrak{p} \in \mathcal{K}(\sigma)\}$. The equality holds for *half–centered* hereditary torsion theories. On the other hand, for any closed under generalization subset $\mathcal{K} \subseteq \operatorname{Spec}(A)$, if $\mathcal{K}$ is quasi–compact, then $\wedge_{\mathfrak{p}\in\mathcal{K}}\sigma_{A\setminus\mathfrak{p}}$ satisfies the following property: for every ideal $\mathfrak{k} \in \mathcal{L}(\wedge_{\mathfrak{p}\in\mathcal{K}(\sigma_\Sigma)}\sigma_{A\setminus\mathfrak{p}})$, there exists $\mathfrak{h} \in \mathcal{L}(\wedge_{\mathfrak{p}\in\mathcal{K}}\sigma_{A\setminus\mathfrak{p}})$ finitely generated such that $\mathfrak{h} \subseteq \mathfrak{k}$; that is, $\wedge_{\mathfrak{p}\in\mathcal{K}}\sigma_{A\setminus\mathfrak{p}}$ is a *finite type* hereditary torsion theory.

The most known example of a finite type hereditary torsion theory is provided by a multiplicative subset $\Sigma \subseteq A$; thus, for the hereditary torsion theory σ_Σ, the Gabriel filter has a basis constituted by principal ideals; $\mathcal{K}(\sigma_\Sigma) = \{\mathfrak{p} \in \operatorname{Spec}(A) \mid \mathfrak{p} \cap \Sigma = \varnothing\}$ is quasi–compact, and $\sigma_\Sigma = \wedge_{\mathfrak{p}\in\mathcal{K}}\sigma_{A\setminus\mathfrak{p}}$.

We shall use quasi–compact subsets of $\operatorname{Spec}(A)$, hence half–centered finite type hereditary torsion theories, for studying commutative rings and finiteness conditions.

If A is a Noetherian or Artinian commutative ring with respect to a hereditary torsion theory σ, then σ is of finite type; in consequence, the background on finite type hereditary torsion theories will be an excellent tool to study finiteness conditions, as is the Noetherian condition on $\operatorname{Spec}(A)$ or on the generically closed subset $\mathcal{K}(\sigma) \subseteq \operatorname{Spec}(A)$.

To do this, in Section 2, starting from a finitely generated ideal $\mathfrak{a} \subseteq A$ we construct a finite type hereditary torsion theory $\sigma_\mathfrak{a}$ with Gabriel filter

$$\mathcal{L}(\sigma_\mathfrak{a}) = \{\mathfrak{h} \subseteq A \mid \text{ there exists } n \in \mathbb{N} \text{ such that } \mathfrak{a}^n \subseteq \mathfrak{h}\},$$

and we extend this definition to any set $\mathcal{S}$ of finitely generated ideals. Given a finite type hereditary torsion theory σ, we have a plethora of finite type hereditary torsion theories $\{\sigma_\mathfrak{a} \mid \mathfrak{a} \in \mathcal{L}_f(A) \setminus \mathcal{L}(\sigma)\}$, properly containing σ, and a hereditary torsion theory τ_σ, defined as the intersection $\wedge_{\mathfrak{a}\in\mathcal{L}_f(A)\setminus\mathcal{L}(\sigma)}\sigma_\mathfrak{a}$.

After studying properties relative to hereditary torsion theories in the set $\{\sigma_\mathfrak{a} \mid \mathfrak{a} \in \mathcal{L}_f(A) \setminus \mathcal{L}(\sigma)\}$, we compare them with properties of τ_σ. This is the main aim of

Sections 3 in which we work some examples, and 4, where we study the Noetherian case. In Section 5 we extend some result of [1] on Artinian* rings and modules; the Artinian case, which has the particularity that its dimension is zero, hence elements in $\mathcal{K}(\sigma)$ are maximal and minimal. In these sections, we consider finite type hereditary torsion theories extensions of a torsion theory σ, and particularize to the case where $\sigma = o$.

In Section 6, we consider a more general context. Given a finite type hereditary torsion theory σ, and the partition $\mathrm{Spec}(A) = \mathcal{K}(\sigma) \cup \mathcal{Z}(\sigma)$, for any prime ideal $\mathfrak{p} \in \mathcal{K}(\sigma)$, we build a new hereditary torsion theory $\tau_{\mathfrak{p}} = \sigma \vee \eta_{\mathfrak{p}}$, satisfying $\mathcal{K}(\tau_{\mathfrak{p}}) = \mathcal{K}(\sigma) \setminus X(\mathfrak{p})$. In this way, for prime ideals $\mathfrak{p}_1, \mathfrak{p}_2 \in \mathcal{K}(\sigma)$ such that $\mathfrak{p}_2 \in \mathcal{K}(\tau_{\mathfrak{p}_1})$, we obtain a chain of hereditary torsion theories $\sigma \leq \tau_{\mathfrak{p}_1} \leq \tau_{\mathfrak{p}_2}$ whenever $\tau_{\mathfrak{p}_1}$ is of finite type: that is, whenever $\mathcal{K}(\tau_{\mathfrak{p}_1})$ is quasi–compact. A sufficient condition is that $\mathfrak{p}_1$ be the σ–radical of a finitely generated ideal, in order to obtain a filtration of finite type hereditary torsion theories $\sigma \leq \tau_{\mathfrak{p}_1} \leq \tau_{\mathfrak{p}_2} \leq \cdots$, and so on, in order to verify the properties of A by studying this filtration. It should be noted that this has applications if $\mathcal{K}(\sigma)$ is a Noetherian ring, since in the latter case every hereditary torsion theory $\tau \geq \sigma$ is of finite type.

2. A new hereditary torsion theory

We work on a commutative ring A, in the category of A–modules and with hereditary torsion theories in $\mathbf{Mod}\text{–}\,A$. Each hereditary torsion theory σ is determined by a class of modules $\mathcal{T}_\sigma$, which is closed under submodules, homomorphic images, group extensions and direct sums: the *torsion* class, or equivalently by a *torsionfree* class: $\mathcal{F}_\sigma$.

For any module M, there is a submodule $\sigma M \subseteq M$ maximal among those belonging to $\mathcal{T}_\sigma$: therefore, $M \in \mathcal{T}_\sigma$ if and only if $\sigma M = M$, and $M \in \mathcal{F}_\sigma$ if and only if $\sigma M = 0$. Furthermore, σ is determined by a filter of ideals $\mathcal{L}(\sigma) = \{\mathfrak{h} \subseteq A \mid A/\mathfrak{h} \in \mathcal{T}_\sigma\}$, characterized by the following property: for any ideal $\mathfrak{a} \subseteq A$, if there is an ideal $\mathfrak{h} \in \mathcal{L}(\sigma)$ such that $(\mathfrak{a} : h) \in \mathcal{L}(\sigma)$ for any $h \in \mathfrak{h}$, then $\mathfrak{a} \in \mathcal{L}(\sigma)$. In consequence, $\sigma M = \{m \in M \mid \ (0 : m) \in \mathcal{L}(\sigma)\}$ for any module M. More about hereditary torsion theories can be found in [4] and references therein.

Let $\sigma \leq \tau$ be hereditary torsion theories in $\mathbf{Mod}\text{–}\,A$, for any A–module M, if M is totally σ–Noetherian, then it is totally τ–Noetherian. The same holds if we consider the σ–Artinian property. Our interest lies in the inverse problem; that is, if M is totally τ–Noetherian, when is it totally σ–Noetherian?

With this generality, the problem only concerns the properties of σ and τ; in a more general context, we want to characterize, in terms of σ, when a module is totally σ–Noetherian for all hereditary torsion theories $\tau > \sigma$.

In [4], it is proved that if the ring A is totally σ–Noetherian, then the hereditary torsion theory σ is of finite type, and trivially holds if we are studying the problem with respect to a multiplicatively closed subset $S \subseteq A$: S–Noetherian modules. Therefore, it makes sense for the torsion theory σ to be of finite type. This restriction is enforced by the fact that every totally S–Artinian ring is also totally S–Noetherian.

Thus we shall consider a finite type hereditary torsion theory σ in **Mod**– A, and finite type hereditary torsion theories $\tau > \sigma$.

Recall that every finite type hereditary torsion theory σ is determined by its Gabriel filter $\mathcal{L}(\sigma)$; furthermore, for any ideal $\mathfrak{a} \subseteq A$ we have:

Lemma 2.1. *For any ideal $\mathfrak{a} \subseteq A$, we represent by $\sigma_{\mathfrak{a}}$ the smallest hereditary torsion theory σ such that $\mathfrak{a} \in \mathcal{L}(\sigma)$. The Gabriel filter of $\sigma_{\mathfrak{a}}$ is*

$$\mathcal{L} = \{\mathfrak{h} \subseteq A \mid \text{ there exists } n \in \mathbb{N} \text{ such that } \mathfrak{a}^n \subseteq \mathfrak{h}\}$$

whenever $\mathfrak{a}$ is finitely generated. If $\mathfrak{a}$ is not finitely generated, we only have the inclusion "$\mathcal{L} \subseteq \mathcal{L}(\sigma_{\mathfrak{a}})$".

Proof. In the case where $\mathfrak{a}$ is finitely generated, let $\mathfrak{h} \in \mathcal{L}$, and $\mathfrak{c} \subseteq A$ such that $(\mathfrak{c} : h) \in \mathcal{L}$ for any $h \in \mathfrak{h}$. We can assume that $\mathfrak{h} = \langle h_1, \ldots, h_t \rangle$ is finitely generated. For any $i = 1, \ldots, t$ there exists $n_i \in \mathbb{N}$ such that $\mathfrak{a}^{n_i} \subseteq (\mathfrak{c} : h_i)$, so that $\cap_i \mathfrak{a}^{n_i} \subseteq \cap_i (\mathfrak{c} : h_i) = (\mathfrak{c} : \mathfrak{h})$. Therefore, $\mathfrak{h}(\cap_i \mathfrak{a}^{n_i}) \subseteq \mathfrak{c}$, and $\mathfrak{c} \in \mathcal{L}$. □

We can extend this result to consider a set $\mathcal{S}$ of finitely generated ideals.

Given a set $\mathcal{S}$ of finitely generated ideals of A, we define a new hereditary torsion theory $\sigma_{\mathcal{S}}$ as follows:

$$\sigma_{\mathcal{S}} = \vee\{\sigma_{\mathfrak{a}} \mid \mathfrak{a} \in \mathcal{S}\}.$$

If $\langle \mathcal{S} \rangle$ is the family of all products of elements of $\mathcal{S}$, then we have:

$$\sigma_{\mathcal{S}} = \vee_{\mathfrak{a} \in \mathcal{S}} \sigma_{\mathfrak{a}} \leq \sigma_{\langle \mathcal{S} \rangle},$$

and since $\mathfrak{a}_1\mathfrak{a}_2 \in \sigma_{\mathcal{S}}$ for any $\mathfrak{a}_1, \mathfrak{a}_2 \in \mathcal{S}$, the equality holds; that is, $\sigma_{\mathcal{S}} = \sigma_{\langle \mathcal{S} \rangle}$. The Gabriel filter of $\sigma_{\mathcal{S}}$ is

$$\mathcal{L}(\sigma_{\mathcal{S}}) = \{\mathfrak{a} \subseteq A \mid \text{ there exist } \mathfrak{a}_1, \ldots, \mathfrak{a}_t \in \mathcal{S} \text{ such that } \mathfrak{a}_1 \cdots \mathfrak{a}_t \subseteq \mathfrak{a}\}.$$

In particular, given a finite type hereditary torsion theory σ, we can take $\mathcal{S} = \mathcal{L}_f(\sigma)$ (the set of all finitely generated ideals in $\mathcal{L}(\sigma)$), and relate properties of σ with properties of $\sigma_{\mathfrak{a}}$, for any $\mathfrak{a} \in \mathcal{L}_f(\sigma)$. In fact, we have: $\sigma = \vee\{\sigma_{\mathfrak{a}} \mid \mathfrak{a} \in \mathcal{L}_f(\sigma)\} = \sigma_{\mathcal{S}}$.

Lemma 2.2. *For any two finitely generated ideals $\mathfrak{a}_1, \mathfrak{a}_2 \subseteq A$, the following statements hold:*

(1) *If $\mathfrak{a}_1 \subseteq \mathfrak{a}_2$, then $\sigma_{\mathfrak{a}_2} \leq \sigma_{\mathfrak{a}_1}$.*

(2) $\sigma_{\mathfrak{a}_1} \vee \sigma_{\mathfrak{a}_2} = \sigma_{\mathfrak{a}_1\mathfrak{a}_2}$.

(3) $\sigma_{\mathfrak{a}_1} \wedge \sigma_{\mathfrak{a}_2} = \sigma_{\mathfrak{a}_1+\mathfrak{a}_2}$.

Proof. We call $\sigma_{\mathfrak{a}_i} = \sigma_i$, for $i = 1, 2$. Since σ_i is of finite type, $\sigma_i = \wedge\{\sigma_{A\setminus\mathfrak{p}} \mid \mathfrak{p} \in \mathcal{K}(\sigma_i)\}$, being

$$\mathcal{K}(\sigma_i) = \{\mathfrak{p} \in \operatorname{Spec}(A) \mid \mathfrak{a}_i \nsubseteq \mathfrak{p}\},$$
$$\mathcal{Z}(\sigma_i) = \{\mathfrak{p} \in \operatorname{Spec}(A) \mid \mathfrak{a}_i \subseteq \mathfrak{p}\}.$$

In addition, σ is of finite type, if and only if $\sigma = \wedge_{\mathcal{K}(\sigma)}\sigma_{A\setminus\mathfrak{p}}$ and $\mathcal{K}(\sigma) \subseteq \operatorname{Spec}(A)$ is quasi–compact.

(1) It is clear that if $\mathfrak{a}_1 \subseteq \mathfrak{a}_2$, then $\mathfrak{a}_2 \in \mathcal{L}(\sigma_1)$, hence $\sigma_2 \leq \sigma_1$.

(2) Since $\sigma_i \leq \sigma_1 \vee \sigma_2$, we have $\mathfrak{a}_i \in \mathcal{L}(\sigma_1 \vee \sigma_2)$, and $\mathfrak{a}_1\mathfrak{a}_2 \in \mathcal{L}(\sigma_1 \vee \sigma_2)$. On the other hand, $\mathcal{K}(\sigma_1 \vee \sigma_2) = \mathcal{K}(\sigma_1) \cap \mathcal{K}(\sigma_2) = \{\mathfrak{p} \in \operatorname{Spec}(A) \mid \mathfrak{a}_1, \mathfrak{a}_2 \nsubseteq \mathfrak{p}\} = \{\mathfrak{p} \in \operatorname{Spec}(A) \mid \mathfrak{a}_1\mathfrak{a}_2 \nsubseteq \mathfrak{p}\} = \mathcal{K}(\sigma_{\mathfrak{a}_1\mathfrak{a}_2})$. Thus we have

$$\sigma_{\mathfrak{a}_1\mathfrak{a}_2} \leq \sigma_1 \vee \sigma_2 \leq \wedge_{\mathcal{K}(\sigma_{\mathfrak{a}_1\mathfrak{a}_2})}\sigma_{A\setminus\mathfrak{p}}.$$

Since $\mathfrak{a}_1\mathfrak{a}_2$ is finitely generated, $\sigma_{\mathfrak{a}_1\mathfrak{a}_2}$ is of finite type and the equality holds.

(3) We have $\sigma_1 \wedge \sigma_2 = \wedge\{\sigma_{A\setminus\mathfrak{p}} \mid \mathfrak{p} \in \mathcal{K}(\sigma_1) \cup \mathcal{K}(\sigma_2)\}$, so it is half–centered. On the other hand, since the finite union of quasi–compact subsets is quasi–compact, $\sigma_1 \wedge \sigma_2$ is of finite type.

We have $\mathcal{Z}(\sigma_1 \wedge \sigma_2) = \mathcal{Z}(\sigma_1) \cap \mathcal{Z}(\sigma_2) = \{\mathfrak{p} \in \operatorname{Spec}(A) \mid \mathfrak{a}_1 + \mathfrak{a}_2 \subseteq \mathfrak{p}\}$, hence $\sigma_{\mathfrak{a}_1+\mathfrak{a}_2} \subseteq \sigma_1 \wedge \sigma_2$. If $\mathfrak{h} \in \mathcal{L}(\sigma_1 \wedge \sigma_2)$, there exists $n \in \mathbb{N}$ such that $\mathfrak{a}_i^n \subseteq \mathfrak{h}$, hence $(\mathfrak{a}_1 + \mathfrak{a}_2)^{2n} \subseteq \mathfrak{h}$, and we have the equality $\sigma_1 \wedge \sigma_2 = \sigma_{\mathfrak{a}_1+\mathfrak{a}_2}$. □

For any A–module M, we define:

- $C(M, \sigma) = \{N \subseteq M \mid M/N \in \mathcal{F}_\sigma\}$, and
- $\mathcal{L}(M, \sigma) = \{N \subseteq M \mid M/N \in \mathcal{T}_\sigma\}$.

Given a submodule $N \subseteq M$, the σ–**closure** of N in M is $\operatorname{Cl}_\sigma^M(N)$, which is defined by the equation

$$\sigma(M/N) = \operatorname{Cl}_\sigma^M(N)/N.$$

The set $\mathcal{L}(M, \sigma)$ is a filter in the lattice $\mathcal{L}(M)$ of all submodules of M, and $C(M, \sigma)$ is a lattice with operations:

- $N_1 \vee N_2 = \mathrm{Cl}_\sigma^M(N_1 + N_2)$ for any $N_1, N_2 \in C(M, \sigma)$, and
- $N_1 \wedge N_2 = N_1 \cap N_2$ for any $N_1, N_2 \in C(M, \sigma)$.

In the following, we assume σ is a finite type hereditary torsion theory in **Mod**– A.

As we have seen above, if σ is a finite type hereditary torsion theory, then σ is completely determined by the finitely generated ideals in the Gabriel filter; now we will study what happens with the hereditary torsion theories that extend σ.

Theorem 2.3. *Let σ be a finite type hereditary torsion theory. For any finitely generated ideal $\mathfrak{a} \subseteq A$, consider the hereditary torsion theory $\sigma_\mathfrak{a}$, and define $\tau_\mathfrak{a} = \sigma \vee \sigma_\mathfrak{a}$. The following assertions hold:*

- *If $\mathfrak{a} \in \mathcal{L}(\sigma)$, then $\tau_\mathfrak{a} = \sigma$.*
- *If $\mathfrak{a} \notin \mathcal{L}(\sigma)$, then $\tau_\mathfrak{a} > \sigma$.*

In any case, the description of $\tau_\mathfrak{a}$ is as follows:

$$\mathcal{L}(\tau_\mathfrak{a}) = \{\mathfrak{b} \subseteq A \mid \text{ there exist } \mathfrak{h} \in \mathcal{L}(\sigma), \text{ and } n \in \mathbb{N} \text{ such that } \mathfrak{h}\mathfrak{a}^n \subseteq \mathfrak{b}\}.$$

Furthermore, the following statements hold:

(1) *$\mathfrak{b} \in \mathcal{L}(\tau_\mathfrak{a})$ if and only if $\mathrm{Cl}_\sigma^A(\mathfrak{b}) \in \mathcal{L}(\tau_\mathfrak{a})$ for any ideal $\mathfrak{b} \subseteq A$.*

(2) *Given ideals $\mathfrak{a}_1, \mathfrak{a}_2 \subseteq A$, we have:*

$$\mathfrak{a}_1\mathfrak{a}_2 \subseteq \mathfrak{a}_1\mathrm{Cl}_\sigma^A(\mathfrak{a}_2) \subseteq \mathrm{Cl}_\sigma^A(\mathfrak{a}_1)\mathrm{Cl}_\sigma^A(\mathfrak{a}_2) \subseteq \mathrm{Cl}_\sigma^A(\mathfrak{a}_1\mathfrak{a}_2).$$

(3) *For any finitely generated ideal $\mathfrak{a} \subseteq A$, we have $\tau_\mathfrak{a} = \tau_{\mathrm{Cl}_\sigma^A(\mathfrak{a})}$.*

Proof. (1) We only need to prove the sufficient condition. If $\mathrm{Cl}_\sigma^A(\mathfrak{b}) \in \mathcal{L}(\tau_\mathfrak{a})$, hence, for every $x \in \mathrm{Cl}_\sigma^A(\mathfrak{b})$, we have $(\mathfrak{b} : x) \in \mathcal{L}(\sigma) \subseteq \mathcal{L}(\tau_\mathfrak{a})$; therefore, $\mathrm{Cl}_\sigma^A(\mathfrak{b}) \in \mathcal{L}(\tau_\mathfrak{a})$.

(2) Let $x_1 \in \mathrm{Cl}_\sigma^A(\mathfrak{a}_1)$, and $x_2 \in \mathrm{Cl}_\sigma^A(\mathfrak{a}_2)$; there is $\mathfrak{h} \in \mathcal{L}(\sigma)$ such that $x_i\mathfrak{h} \subseteq \mathrm{Cl}_\sigma^A(\mathfrak{a}_i)$, for $i = 1, 2$, so $x_1x_2\mathfrak{h}^2 \subseteq \mathfrak{a}_1\mathfrak{a}_2$, and $x_1x_2 \in \mathrm{Cl}_\sigma^A(\mathfrak{a}_1\mathfrak{a}_2)$.

(3) Given that $\mathfrak{a} \subseteq \mathrm{Cl}_\sigma^A(\mathfrak{a})$, we have the inclusion: $\tau_{\mathrm{Cl}_\sigma^A(\mathfrak{a})} \subseteq \tau_\mathfrak{a}$. On the other hand, if $\mathfrak{h} \in \tau_\mathfrak{a}$, there are $\mathfrak{b} \in \mathcal{L}(\sigma)$, and $n \in \mathbb{N}$ such that $\mathfrak{b}\mathfrak{a}^n \subseteq \mathfrak{h}$; so

$$\mathfrak{b}\mathrm{Cl}_\sigma^A(\mathfrak{a})^n \subseteq \mathrm{Cl}_\sigma^A(\mathfrak{b}\mathfrak{a}^n) \subseteq \mathrm{Cl}_\sigma^A(\mathfrak{h}).$$

Therefore, $\mathrm{Cl}_\sigma^A(\mathfrak{h}) \in \mathcal{L}(\tau_{\mathrm{Cl}_\sigma^A(\mathfrak{a})})$; in consequence, $\mathfrak{h} \in \mathcal{L}(\tau_{\mathrm{Cl}_\sigma^A(\mathfrak{a})})$. □

The last result in (3) is natural because $\tau_\mathfrak{a}$ subsumes the σ–closure.

Given a family of finitely generated ideals $\mathcal{S} = \{\mathfrak{a}_i \mid i \in I \text{ and } \mathfrak{a}_i \in \mathcal{L}_f(A) \setminus \mathcal{L}(\sigma)\}$, we define $\tau_\mathcal{S} = \wedge\{\tau_\mathfrak{a} \mid \mathfrak{a} \in \mathcal{S}\}$. Therefore, for any ideal $\mathfrak{a} \in \mathcal{S}$, we have $\sigma \leq \tau_\mathcal{S} \leq \tau_\mathfrak{a}$.

In the case where $\mathcal{S} = \mathcal{L}_f(A) \setminus \mathcal{L}(\sigma)$; that is, $\mathcal{S}$ is the set of all finitely generated ideals $\mathfrak{a} \subseteq A$ such that $\mathfrak{a} \notin \mathcal{L}(\sigma)$, the hereditary torsion theory $\tau_{\mathcal{S}}$ is represented as τ_σ. We ask whether τ_σ is of finite type.

Consider the set $\{a \in A \mid aA \notin \mathcal{L}(\sigma)\}$; our first objective will be to relate τ_σ and the hereditary torsion theory $\wedge\{\tau_a \mid aA \notin \mathcal{L}(\sigma)\}$.

Proposition 2.4. *With the above notation* $\tau_\sigma = \wedge\{\tau_a \mid aA \notin \mathcal{L}(\sigma)\}$.

Proof. For any $\mathfrak{a} \in \mathcal{L}_f(A) \setminus \mathcal{L}(\sigma)$, we have $\tau_{\mathfrak{a}} \leq \tau_a$, for any $a \in \mathfrak{a}$ such that $aA \notin \mathcal{L}(\sigma)$, therefore, $\tau_{\mathfrak{a}} \leq \wedge\{\tau_a \mid a \in \mathfrak{a}\}$. On the other hand, if $\mathfrak{b} \in \wedge\tau_a$, and $a_1, \ldots, a_t \in \mathfrak{a}$ is a system of generators of $\mathfrak{a}$, for any index $i \in I$, there are $\mathfrak{h}_i \in \mathcal{L}(\sigma)$, and $n_i \in \mathbb{N}$ such that $\mathfrak{h}_i a_i^{n_i} \subseteq \mathfrak{b}$. If $n > n_1 + \cdots + n_t$, then $\mathfrak{h}_1 \cdots \mathfrak{h}_t \mathfrak{a}^n \subseteq \mathfrak{b}$, so $\mathfrak{b} \in \mathcal{L}(\tau_{\mathfrak{a}})$. Consequently, $\tau_{\mathfrak{a}} = \wedge\{\tau_a \mid a \in \mathfrak{a}\}$, and we obtain $\tau_\sigma = \wedge\{\tau_a \mid aA \notin \mathcal{L}(\sigma)\}$. □

Let M be an A–module, and consider an *abstract property* of modules, we say M has the

- σ–**property everywhere-II** whenever M has the $\tau_{\mathfrak{a}}$–**property** for every finitely generated ideal $\mathfrak{a} \in \mathcal{L}_f(A) \setminus \mathcal{L}(\sigma)$.
- σ–**property everywhere-I** whenever M has the τ_a–**property** for every $a \in A$ such that $aA \notin \mathcal{L}(\sigma)$.

Our goal is to relate the σ–property, the τ_σ–property and the σ–properties everywhere.

In particular, we are interested in the following properties: Noetherian, totally Noetherian, Artinian, totally Artinian, and so on, and its relationships with the theoretical frameworks of hereditary torsion theory mentioned above.

Remark 2.5. As mentioned above, another problem that interests us is to determining when τ_σ is of finite type. Recall that σ is of finite type and that $\tau_\sigma = \wedge\{\tau_a \mid aA \notin \mathcal{L}(\sigma)\}$, where each τ_a is of finite type, but τ_σ is not necessarily so.

3. Examples

For every finitely generated ideal $\mathfrak{a} \subseteq A$ such that $\mathfrak{a} \notin \mathcal{L}(\sigma)$, we have the following hereditary torsion theories: $\sigma \leq \tau_\sigma \leq \tau_{\mathfrak{a}}$. By Theorem 2.3, we can consider $\mathfrak{a} = \mathrm{Cl}_\sigma^A(\mathfrak{a})$; therefore, an element of $C(A, \sigma)$ that is σ–finitely generated. We can represent by $C(A, \sigma)_f$ the set of all σ–finitely generated σ–closed ideals of A.

The simplest case is when $\sigma = o$; that is, $\mathcal{L}(o) = \{A\}$, or equivalently, 0 is the only torsion module. Let us study in this case τ_o.

Proposition 3.1. *In the above situation, we have one of the following possibilities:*

(1) $o \neq \tau_o$; *whence* A *is a local ring with maximal ideal* $\mathfrak{m}$, *and*

$$\mathcal{L}(\tau_o) = \{\mathfrak{h} \subseteq A \mid \operatorname{rad}(\mathfrak{h}) = \mathfrak{m}\}.$$

(2) $\mathcal{L}(\tau_o) = \{A\}$, *if* A *is not local.*

Proof. (1) If there exists $\mathfrak{h} \in \mathcal{L}(\tau_o) \setminus \{A\}$, let $\mathfrak{m} \subseteq A$ be a maximal ideal such that $\mathfrak{m} \supseteq \mathfrak{h}$. Let A^* be the set of all regular elements of A. For any $a \in A^* \setminus U(A)$, given that $\tau_o \leq \tau_a$, there exists $m \in \mathbb{N}$ such that $a^m \in \mathfrak{h} \subseteq \mathfrak{m}$, so $a \in \mathfrak{m}$; therefore, $\mathfrak{m}$ is the only maximal ideal of A; that is, A is a local ring with maximal ideal $\mathfrak{m}$.

Similarly, for any prime ideal $\mathfrak{p} \in \mathcal{Z}(\tau_o)$ and any $a \in A^* \setminus U(A)$, we also have $a \in \mathfrak{p}$; therefore, $\mathfrak{p} = \mathfrak{m}$, and $\mathcal{Z}(\tau_o) = \{\mathfrak{m}\}$.

Since $\mathcal{Z}(\tau_o) = \{\mathfrak{m}\}$, for any $\mathfrak{h} \in \mathcal{L}(\tau_o)$, we have $\operatorname{rad}(\mathfrak{h}) = \mathfrak{m}$. The converse is also true; in fact, if $\operatorname{rad}(\mathfrak{h}) = \mathfrak{m}$, for any $a \in \mathfrak{m} = A \setminus U(A)$, there exists $m \in \mathbb{N}$ such that $a^m \in \mathfrak{h}$, so $\mathfrak{h} \in \mathcal{L}(\tau_o)$,

$$\mathcal{L}(\tau_o) = \{\mathfrak{h} \subseteq A \mid \operatorname{rad}(\mathfrak{h}) = \mathfrak{m}\}.$$

Is τ_o necessarily of finite type? The answer is No, see Example 3.2 below.

(2) We have the following equivalent statements:

(a) A is not local.
(b) For any maximal ideal $\mathfrak{m} \subseteq A$, there exists $a \in A^* \setminus U(A)$ such that $a \notin \mathfrak{m}$.
(c) For any maximal ideal $\mathfrak{m} \subseteq A$, there exists $a \in A^* \setminus U(A)$ such that $a^m \notin \mathfrak{m}$, for any $m \in \mathbb{N}$.
(d) $\mathcal{Z}(\tau_o) = \varnothing$.
(e) $\tau_o = o$. □

Observe that in the local case, we also have:

$$\mathcal{Z}(\tau_o) = \{\mathfrak{m}\},$$
$$\mathcal{K}(\tau_o) = \operatorname{Spec}(A) \setminus \{\mathfrak{m}\}.$$

First, we aim to study when τ_o is of finite type. Recall that if τ_o is of finite type, then $\mathcal{K}(\tau_o)$ is quasi–compact. We will prove that, in general, this is not the case.

Example 3.2. Consider the lexicographical order in the group $G = \mathbb{Z}^{(\mathbb{N})}$, and the field $K = \mathbb{C}(X_n \mid n \in \mathbb{N})$. There is a valuation v on K over $\mathbb{C}$ such that $v(X_n) = e_n = (e_{in})_i \in G$. Let V be the valuation ring and $\mathfrak{m}$ its maximal ideal.

We have: $\mathfrak{m} = (X_n \mid n \in \mathbb{N})$. Since $e_0 > e_1 > e_2 > \cdots$, there is a strictly increasing chain of ideals:

$$(X_0) \subsetneqq (X_1) \subsetneqq (X_2) \subsetneqq \cdots .$$

If $\mathfrak{p}_n = \mathrm{rad}(X_n)$, there is a strictly increasing chain of prime ideals: $\mathfrak{p}_0 \subsetneqq \mathfrak{p}_1 \subsetneqq \mathfrak{p}_2 \subsetneqq \cdots$. Consequently, V has infinite Krull dimension, and $\mathfrak{m}$ is not finitely generated.

We assert that $Y = \mathrm{Spec}(V) \setminus \{\mathfrak{m}\}$ is not quasi–compact. In fact, if Y is quasi–compact, there are elements $y_1, \ldots, y_s \in V$ such that $Y = X(y_1) \cup \ldots \cup X(y_s)$, so there is $y \in V$ such that $Y = X(y) = \{\mathfrak{p} \in \mathrm{Spec}(V) \mid \ y \notin \mathfrak{p}\}$, which is a contradiction.

To complete this example we can show that in this case τ_o is not of finite type. For any ideal $\mathfrak{a} \in \mathcal{L}(\tau_o)$, we have $\mathrm{rad}(\mathfrak{a}) = \mathfrak{m}$; if there is a finitely generated ideal $\mathfrak{b} \subseteq \mathfrak{a}$ such that $\mathfrak{b} \in \mathcal{L}(\tau_o)$; that is, $\mathrm{rad}(\mathfrak{b}) = \mathfrak{m}$, then $X(\mathfrak{b}) = Y$ which is a contradiction because $\mathfrak{b}$ is finitely generated, hence principal.

In fact, we can characterize when τ_o is of finite type.

Proposition 3.3. *If we consider the hereditary torsion theory τ_o, the following statements are equivalent:*

(a) *τ_o is of finite type.*
(b) *$\mathfrak{m}$ is the radical of a finitely generated ideal.*

Proof. (a) $\Rightarrow$ (b) is clear.

(b) $\Rightarrow$ (a) Let $\mathfrak{h} \in \mathcal{L}(\tau_o)$, then $\mathrm{rad}(\mathfrak{h}) = \mathfrak{m}$. If $\mathfrak{k} \subseteq A$ is finitely generated and $\mathrm{rad}(\mathfrak{k}) = \mathfrak{m}$, there is $m \in \mathbb{N}$ such that $\mathfrak{k}^m \subseteq \mathfrak{h}$, hence τ_o is of finite type. □

This result partially answers the question raised in Remark 2.5. In particular, this is the case when Spec(A) is a Noetherian space in the Zariski topology. See [7, Proposition 3.2].

Remark 3.4. Consequently, there are three possibilities:

(1) $\mathfrak{m}$ is finitely generated. In this case $\mathcal{L}(\tau_0) = \{\mathfrak{h} \subseteq A \mid \ \mathfrak{m} \subseteq \mathfrak{h}\}$, and it is of finite type.
(2) $\mathfrak{m}$ is not finitely generated but there exists $\mathfrak{h} \subseteq A$ finitely generated such that $\mathrm{rad}(\mathfrak{h}) = \mathfrak{m}$. In this case $\mathcal{L}(\tau_o)$ is of finite type. See Example 3.5 below.
(3) There is no $\mathfrak{h} \subseteq A$ finitely generated such that $\mathrm{rad}(\mathfrak{h}) = \mathfrak{m}$. In this case τ_o is not of finite type. See Example 3.2 above.

Example 3.5. Consider a valuation domain V with value group $(\mathbb{Q}, +, \leq)$ and maximal ideal $\mathfrak{m}$. We have that $\mathfrak{m}$ is not finitely generated, but it is the radical of a finitely generated ideal.

Let us consider the following example related to (2) in Remark 3.4.

Example 3.6. Let K be a field and $A = K[x_n \mid n \in \mathbb{N}] = \dfrac{K[X_n \mid n \in \mathbb{N}]}{\langle X_n^{n+1} \mid n \in \mathbb{N}\rangle}$. The ring A has a unique prime ideal, therefore, maximum: $\mathfrak{m} = \langle x_n \mid n \in \mathbb{N}\rangle \subseteq A$. For any finitely generated ideal $\mathfrak{a} \in \mathcal{L}_f(A) \setminus \{A\}$, given that $\mathfrak{a}$ is nilpotent, then $\sigma_{\mathfrak{a}} = 1$; that is, every A–module is $\sigma_{\mathfrak{a}}$–torsion, so $\tau_o = 1$. In particular, $\mathfrak{m} \in \mathcal{L}(\tau_o)$, and $\mathcal{L}(\tau_o)$ is different from the set $\{\mathfrak{h} \subseteq A \mid \text{ there exists } n \in \mathbb{N} \text{ such that } \mathfrak{m}^n \subseteq \mathfrak{h}\}$.

Relative to a hereditary torsion theory. A similar result can be realized if we consider a finite type hereditary torsion theory σ instead of o, as we will show below.

An element $N \in C(M, \sigma)$ is **maximal** whenever $N \neq M$ and for any $X \in C(M, \sigma)$ if $N \subseteq X$, then either $X = M$ or $N = X$.

Lemma 3.7. *Let σ be a (non–necessarily of finite type) hereditary torsion theory, if $\mathfrak{a} \in C(A, \sigma)$ is maximal, then $\mathfrak{a} \subseteq A$ is a prime ideal and $\mathfrak{a} \in \mathcal{K}(\sigma)$.*

Proof. We just need to show that $\mathfrak{a} \subseteq A$ is a prime ideal. Let $\mathfrak{a}_1, \mathfrak{a}_2 \subseteq A$ such that $\mathfrak{a}_1\mathfrak{a}_2 \subseteq \mathfrak{a}$ and $\mathfrak{a} \subsetneqq \mathfrak{a}_1, \mathfrak{a}_2$. By the maximality of $\mathfrak{a}$, we have $\mathfrak{a}_1, \mathfrak{a}_2 \notin C(M, \sigma)$, but $\mathrm{Cl}_\sigma^A(\mathfrak{a}_1), \mathrm{Cl}_\sigma^A(\mathfrak{a}_1) \in C(M, \sigma)$, so $\mathrm{Cl}_\sigma^A(\mathfrak{a}_1) = A = \mathrm{Cl}_\sigma^A(\mathfrak{a}_2)$, and $\mathfrak{a}_1, \mathfrak{a}_2 \in \mathcal{L}(\sigma)$; therefore, $\mathfrak{a}_1\mathfrak{a}_2 \in \mathcal{L}(\sigma)$, which is a contradiction. □

The set of all maximal elements in $C(A, \sigma)$, or in $\mathcal{K}(\sigma)$, is denoted by $\mathcal{C}(\sigma)$.

In the case of finite type hereditary torsion theories, we can say more about the maximal elements in $C(A, \sigma)$.

Proposition 3.8. *Let σ be a finite type hereditary torsion theory, for any $\mathfrak{a} \notin \mathcal{L}(\sigma)$, there is $\mathfrak{c} \in C(A, \sigma)$, maximal, such that $\mathfrak{c} \supseteq \mathfrak{a}$. In particular, $\mathfrak{c} \in \mathcal{K}(\sigma)$ is maximal.*

Proof. For any $\mathfrak{a} \notin \mathcal{L}(\sigma)$, we have $\mathrm{Cl}_\sigma^A(\mathfrak{a}) \neq A$ belongs to $C(A, \sigma)$. If we consider the family $\Gamma = \{\mathfrak{c} \in C(A, \sigma) \mid \mathfrak{c} \supseteq \mathfrak{a}\}$, we assert that Γ is inductive. In fact, it is non–empty, and for any ascending chain $\{\mathfrak{c}_i \mid i \in I\}$ in Γ, the union $\cup_i \mathfrak{c}_i$ belongs to Γ. Otherwise, there is an element $a \in A \setminus \cup_i \mathfrak{c}_i$, and an ideal $\mathfrak{h} \in \mathcal{L}(\sigma)$ such that $a\mathfrak{h} \subseteq \cup_i \mathfrak{c}_i$; since we can take $\mathfrak{h}$ finitely generated, there exists an index $j \in I$ such that $a\mathfrak{h} \subseteq \mathfrak{c}_j$, so $a \in \mathfrak{c}_j \subseteq \cup_i \mathfrak{c}_i$, which is a contradiction by Zorn's lemma. □

If σ is a hereditary torsion theory in **Mod**– A, the ring A is called σ–**local** whenever the set $\mathcal{K}(\sigma)$ has a unique maximal element; that is, $\mathcal{C}(\sigma)$ is a unitary set.

Proposition 3.9. *Let σ be a finite type hereditary torsion theory, then either*

(1) *$\sigma \neq \tau_\sigma$; hence A is σ–local with $\mathcal{C}(\sigma) = \{\mathfrak{m}\}$, and $\mathcal{L}(\tau_\sigma) = \{\mathfrak{h} \subseteq A \mid \mathrm{rad}(\mathfrak{h}) = \mathfrak{m}\} \cup \mathcal{L}(\sigma)$.*

(2) $\sigma = \tau_\sigma$, *if* A *is not* σ*–local.*

Proof. (1) If $\sigma \neq \tau_\sigma$, there exists $\mathfrak{h} \in \mathcal{L}(\tau_\sigma)\setminus\mathcal{L}(\sigma)$; let $\mathfrak{m} \in \mathcal{C}(\sigma)$ be such that $\mathfrak{h} \subseteq \mathfrak{m}$. For any $a \in A$ such that $aA \notin \mathcal{L}(\sigma)$, there exists $m \in \mathbb{N}$ such that $a^m \in \mathfrak{h} \subseteq \mathfrak{m}$; therefore, $\mathfrak{m}$ is the only element of $\mathcal{C}(\sigma)$. Therefore, A is σ–local.

For any prime ideal $\mathfrak{p} \in \mathcal{Z}(\tau_\sigma)$, we also have $\mathfrak{p} = \mathfrak{m}$; whence $\mathcal{Z}(\tau_\sigma) = \{\mathfrak{m}\}$, and for any $\mathfrak{h} \in \mathcal{L}(\tau_\sigma)$, we have $\mathrm{rad}(\mathfrak{h}) = \mathfrak{m}$.

On the other hand, if $\mathfrak{h} \subseteq A$ and $\mathrm{rad}(\mathfrak{h}) = \mathfrak{m}$, for any $a \in A$ such that $aA \notin \mathcal{L}(\sigma)$, we have $a \in \mathfrak{m}$, hence there exists $m_a \in \mathbb{N}$ such that $a^{m_a} \in \mathfrak{h}$. Therefore, $\mathfrak{h} \in \mathcal{L}(\tau_\sigma)$. In this case, we have

$$\mathcal{L}(\tau_\sigma) = \{\mathfrak{h} \subseteq A \mid \ \mathrm{rad}(\mathfrak{h}) = \mathfrak{m}\} \cup \mathcal{L}(\sigma).$$

(2) We have the following equivalent statements:

(a) $\mathcal{Z}(\tau_\sigma) = \mathcal{Z}(\sigma)$.
(b) For any $\mathfrak{p} \in \mathcal{C}(\sigma)$, there exists $a \in A$ such that $aA \notin \mathcal{L}(\sigma)$ and $a \notin \mathfrak{p}$.
(c) For any $\mathfrak{p} \in \mathcal{C}(\sigma)$, there exists $a \in A$ such that $aA \notin \mathcal{L}(\sigma)$ and $a^m \notin \mathfrak{p}$ for any $m \in \mathbb{N}$.
(d) $\mathcal{C}(\sigma)$ has more than one element.
(e) A is not σ–local. □

In the non σ–local case we have $\sigma = \tau_\sigma$, in the σ–local case we have:

$$\mathcal{Z}(\tau_\sigma) = \mathcal{Z}(\sigma) \cup \{\mathfrak{m}\},$$
$$\mathcal{K}(\tau_\sigma) = \mathcal{K}(\sigma) \setminus \{\mathfrak{m}\}.$$

Example 3.10. Let $A = \mathbb{F}_2^{(\mathbb{N})} + \mathbb{F}_2$ whose maximal ideals are:

- $\mathfrak{p} = \mathbb{F}_2^{(\mathbb{N})}$,
- $\mathfrak{p}_m = (1 - e_m)A$, being $e_m = (\delta_{m,i})_i$ for any $m \in \mathbb{N}$.

If $\sigma = \sigma_{A\setminus\mathfrak{p}_m}$, then A is σ–Noetherian, hence $\mathfrak{p}_m$ is σ–finitely generated, and σ is of finite type.

Since $\mathcal{K}(\sigma) = \{\mathfrak{p}_m\}$, we have $\mathcal{K}(\tau_\sigma) = \varnothing$, and $\tau_\sigma = 1$.

It is also possible to determine when τ_σ is of finite type.

Proposition 3.11. *If we consider the hereditary torsion theory* τ_σ*, the following statements are equivalent:*

(a) τ_σ *is of finite type.*
(b) $\mathfrak{m}$ *is the radical of a finitely generated ideal.*
(c) $\mathfrak{m}$ *is the radical of a* σ*–finitely generated ideal.*

Proof. (a) $\Rightarrow$ (b) If τ_σ is of finite type, given that $\mathfrak{m} \in \mathcal{L}(\tau_\sigma)$, there exists $\mathfrak{h} \in \mathcal{L}(\tau_\sigma)$, finitely generated, such that $\mathfrak{h} \subseteq \mathfrak{m}$, whence $\operatorname{rad}(\mathfrak{h}) = \mathfrak{m}$.

(b) $\Rightarrow$ (a) Let $\mathfrak{h} \in \mathcal{L}(\tau_\sigma)$, then $\operatorname{rad}(\mathfrak{h}) = \mathfrak{m} = \operatorname{rad}(\mathfrak{k})$ for some finitely generated ideal $\mathfrak{k} \subseteq A$. Since $\mathfrak{k} \subseteq \operatorname{rad}(\mathfrak{h})$, there exists $n \in \mathbb{N}$ such that $\mathfrak{k}^n \subseteq \mathfrak{h}$, and τ_σ is of finite type.

(b) $\Rightarrow$ (c) is obvious.

(c) $\Rightarrow$ (b) By the hypothesis, there are $\mathfrak{k} \subseteq \mathfrak{h} \subseteq \mathfrak{m}$ such that: $\mathfrak{k} \subseteq A$ is finitely generated, $\mathfrak{h}/\mathfrak{k}$ is σ–torsion, $\operatorname{rad}(\mathfrak{h}) = \mathfrak{m}$. From the short exact sequence $0 \to \mathfrak{k} \to \mathfrak{h} \to \mathfrak{h}/\mathfrak{k} \to 0$, since $\mathfrak{h} \in \mathcal{L}(\tau_\sigma)$, and $\mathfrak{h}/\mathfrak{k}$ is τ_σ–torsion, we have that $\mathfrak{k} \in \mathcal{L}(\tau_\sigma)$, whence $\operatorname{rad}(\mathfrak{k}) = \mathfrak{m}$. $\square$

4. Noetherian modules everywhere

Recall that an A–module M is **totally σ–Noetherian everywhere II** whenever is totally $\tau_{\mathfrak{a}}$–Noetherian for every ideal $\mathfrak{a} \in \mathcal{L}_f(A) \setminus \mathcal{L}(\sigma)$, and similarly for totally σ–Noetherian everywhere I. Hereinafter, we shall refer them simply as totally σ–Noetherian everywhere.

We have that totally σ–Noetherian modules everywhere can be characterized, similarly to [1], as follows:

Let $\mathcal{X}$ be a family of submodules of $\mathcal{L}(M)$.

- The family $\mathcal{X}$ is **totaly σ–saturated everywhere** if it is totally $\tau_{\mathfrak{a}}$–saturated for every ideal $\mathfrak{a} \in \mathcal{L}_f(A) \setminus \mathcal{L}(\sigma)$.
- An element $N \in \mathcal{X}$ is **totally σ–maximal everywhere** if it is totally $\tau_{\mathfrak{a}}$–maximal for every ideal $\mathfrak{a} \in \mathcal{L}_f(A) \setminus \mathcal{L}(\sigma)$.
- The module M satisfies the **totally σ–maximal condition** if every non–empty family of submodules has a totally σ–maximal element, and satisfies the **totally σ–maximal condition everywhere** whenever every non–empty family of submodules has a totally σ–maximal everywhere element.
- The module M is **totally σ–finitely generated** if for every non–empty family of submodules $\mathcal{X}$ if $\sum \mathcal{X} = M$, the sum of all element of $\mathcal{F}$ is M, there exists $\mathfrak{h} \in \mathcal{L}(\sigma)$, and a finite subfamily $\mathcal{F} \subseteq \mathcal{X}$ such that $(\sum \mathcal{F})\mathfrak{h} = M$; and it is **totally σ–finitely generated everywhere** whenever if it is totally $\tau_{\mathfrak{a}}$–finitely generated for every ideal $\mathfrak{a} \in \mathcal{L}_f(A) \setminus \mathcal{L}(\sigma)$.

With these definitions, we have the following characterization theorem for totally σ–Noetherian modules everywhere.

Theorem 4.1. *Let M be an A–module, the following statements are equivalent:*

(a) *M is totally σ–Noetherian everywhere.*

(b) *Every non–empty totally σ–saturated family everywhere $\mathcal{X}$ of submodules of M has a maximal element.*

(c) *Every non–empty family of submodules $\mathcal{X}$ of M has a totally σ–maximal element everywhere.*

(d) *Every submodule of M is totally σ–finitely generated everywhere.*

Proof. (a) $\Rightarrow$ (b) Let Γ be a non–empty totally σ–saturated family everywhere of submodules of M; for any ascending chain $\{N_i \mid i \in I\}$, and any $\mathfrak{a} \in \mathcal{L}_f(A) \setminus \mathcal{L}(\sigma)$, there exist $j_\mathfrak{a} \in I$ and $\mathfrak{h}_\mathfrak{a} \in \mathcal{L}(\tau_\mathfrak{a})$ such that $\sum_i N_i\mathfrak{h}_\mathfrak{a} \subseteq N_{j_\mathfrak{a}}$. Since $N_{j_\mathfrak{a}} \in \Gamma$, we have $\sum_i N_i \in \Gamma$. Finally, by Zorn's lemma, Γ has maximal elements.

(b) $\Rightarrow$ (c) Given Γ, a non–empty family of submodules of M, define

$\Omega = \{N \subseteq M \mid \exists H \in \Gamma \text{ such that } \forall \mathfrak{a} \in \mathcal{L}_f(A) \setminus \mathcal{L}(\sigma), \exists \mathfrak{h}_\mathfrak{a} \in \mathcal{L}(\tau_\mathfrak{a}) \text{ such that } N\mathfrak{h}_\mathfrak{a} \subseteq H\}$.

We can assume that $H \subseteq N$. We have $\Gamma \subseteq \Omega$, and Ω is totally σ–saturated everywhere. In fact, if $L, N \subseteq M$, $N \in \Omega$, and for any $\mathfrak{a} \in \mathcal{L}_f(A) \setminus \mathcal{L}(\sigma)$, there exists $\mathfrak{h}_\mathfrak{a} \in \mathcal{L}(\tau_\mathfrak{a})$ such that $L\mathfrak{h}_\mathfrak{a} \subseteq N$, there exist $H \in \Gamma$ and $\mathfrak{k}_\mathfrak{a} \in \mathcal{L}(\tau_\mathfrak{a})$ such that $N\mathfrak{k}_\mathfrak{a} \subseteq H$, whence $L\mathfrak{h}_\mathfrak{a}\mathfrak{k}_\mathfrak{a} \subseteq N\mathfrak{k}_\mathfrak{a} \subseteq H$, and $L \in \Omega$.

By the hypothesis, there exists $N \in \Omega$, maximal, and there exists $H \in \Gamma$, such that for any $\mathfrak{a} \in \mathcal{L}_f(A) \setminus \mathcal{L}(\sigma)$, there exists an ideal $\mathfrak{h}_\mathfrak{a} \in \mathcal{L}(\tau_\mathfrak{a})$ such that $N\mathfrak{h}_\mathfrak{a} \subseteq H$; we can assume $N \supseteq H$, as $N + H \in \Omega$. Now we check that $H \in \Gamma$ is totally σ–maximal. If there is $L \in \Gamma$ such that $H \subseteq L$, then $N + L \in \Omega$ because $(N + L)\mathfrak{h}_\mathfrak{a} \subseteq N\mathfrak{h}_\mathfrak{a} + L \subseteq H + L = L$; by the maximality of $N \in \Omega$, we have $L = N$, hence $L\mathfrak{h}_\mathfrak{a} = N\mathfrak{h}_\mathfrak{a} \subseteq H$.

(c) $\Rightarrow$ (d) Let $\{N_i \mid i \in I\}$ be the family of finitely generated submodules of M, and consider $\Gamma = \left\{\sum_{j\in F} N_j \mid F \subseteq I \text{ finite}\right\}$. By assumption, there exists $N = \sum_{j\in F} N_j \in \Gamma$ that is totally σ–maximal everywhere. For any index $i \in I \setminus F$, we have $N + N_j \in \Gamma$, hence, for any $\mathfrak{a} \in \mathcal{L}_f(A) \setminus \mathcal{L}(\sigma)$, there is $\mathfrak{h}_\mathfrak{a} \in \mathcal{L}(\tau_\mathfrak{a})$ such that $(N + N_j)\mathfrak{h}_\mathfrak{a} \subseteq N$; that is, $N_i\mathfrak{h}_\mathfrak{a} \subseteq N$, and we have $M\mathfrak{h}_\mathfrak{a} = (\sum_i N_i)\,\mathfrak{h}_\mathfrak{a} \subseteq N$. Therefore, M is totally σ–finitely generated everywhere.

(d) $\Rightarrow$ (a) For any ascending chain of submodules $\{N_i \mid i \in I\}$ of M, define $N = \sum_i N_i \subseteq M$, which is totally σ–finitely generated everywhere. There exists $H \subseteq N$ finitely generated such that for any $\mathfrak{a} \in \mathcal{L}_f(A) \setminus \mathcal{L}(\sigma)$, there exists $\mathfrak{h}_\mathfrak{a} \in \mathcal{L}(\tau_\mathfrak{a})$ such that $N\mathfrak{h}_\mathfrak{a} \subseteq H \subseteq N$. Therefore, there exists $j \in I$ such that $H \subseteq N_j$, and we have $\sum_i N_i\mathfrak{h}_\mathfrak{a} \subseteq N_j$. □

If σ is a finite type hereditary torsion theory, for every ideal $\mathfrak{a} \notin \mathcal{L}(\sigma)$, there is an ideal $\mathfrak{m} \in \mathcal{C}(\sigma)$ such that $\mathfrak{a} \subseteq \mathfrak{m}$.

We know the following implications with σ–Noetherian modules:

M is Noetherian $\Rightarrow$ M is totally σ–Noetherian $\Rightarrow$ M is totally σ–Noetherian everywhere.

The converse not necessarily holds. In a particular case we have:

Proposition 4.2. *Let A be a non σ–local ring, then every totally σ–Noetherian module everywhere is totally σ–Noetherian.*

Proof. Let $\{N_i \mid i \in I\}$ be an ascending chain of submodules of M. For every ideal $\mathfrak{a} \in \mathcal{L}_f(A) \setminus \mathcal{L}(\sigma)$, there exist $j_\mathfrak{a} \in I$ and $\mathfrak{h}_\mathfrak{a} \in \mathcal{L}(\tau_\mathfrak{a})$ such that $(\sum_i N_i)\,\mathfrak{h}_\mathfrak{a} \subseteq N_{j_\mathfrak{a}}$.

We define $\mathfrak{h} = \sum\{\mathfrak{h}_\mathfrak{a} \mid \mathfrak{a} \in \mathcal{L}_f(A) \setminus \mathcal{L}(\sigma)\}$. If $\mathfrak{h} \notin \mathcal{L}(\sigma)$, there is $\mathfrak{m} \in \mathcal{C}(\sigma)$ such that $\mathfrak{h} \subseteq \mathfrak{m}$; therefore, for every $\mathfrak{a} \in \mathcal{L}_f(A) \setminus \mathcal{L}(\sigma)$, there exist $\mathfrak{k}_\mathfrak{a} \in \mathcal{L}(\sigma)$, and $n_\mathfrak{a} \in \mathbb{N}$ such that $\mathfrak{k}_\mathfrak{a}\mathfrak{a}^{n_\mathfrak{a}} \subseteq \mathfrak{h}_\mathfrak{a} \subseteq \mathfrak{h} \subseteq \mathfrak{m}$, hence $\mathfrak{a} \subseteq \mathfrak{m}$, which is a contradiction. Consequently, $\mathfrak{h} \in \mathcal{L}(\sigma)$, and there exist finitely many $\mathfrak{a}_1, \ldots, \mathfrak{a}_t \in \mathcal{L}_f(A)\setminus\mathcal{L}(\sigma)$ such that $\mathfrak{h}' = \sum_{j=1}^{t} \mathfrak{h}_{\mathfrak{a}_j} \in \mathcal{L}(\sigma)$. If we take $j = \max\{j_{\mathfrak{a}_1}, \ldots, j_{\mathfrak{a}_t}\}$, then $(\sum_i N_i)\,\mathfrak{h}' \subseteq N_j$, and M is totally σ–Noetherian. □

Let A be a ring and σ a hereditary torsion theory in **Mod**– A, for any A–module M, we define the σ–**dimension** of M as the dimension of $\mathrm{Supp}(M) \cap \mathcal{K}(\sigma)$. In a similar way, the σ–**dimension** of A is the dimension of $\mathcal{K}(\sigma)$. The σ–dimension of M is represented by $\dim_\sigma(M)$.

Corollary 4.3. *Given a totally σ–Noetherian ring A, we have that τ_σ is of finite type whenever either A is not σ–local or $\mathcal{K}(A)$ has dimension zero.*

Proof. If A is not σ–local, $\tau_\sigma = \sigma$ is of finite type. If A is σ–local and $\mathcal{C}(\sigma) = \{\mathfrak{m}\}$, then $\mathcal{L}(\tau_\sigma) = \{\mathfrak{h} \subseteq A \mid \mathrm{rad}(\mathfrak{h}) = \mathfrak{m}\}$; given that $\mathfrak{m} \in C(A, \sigma)$, there exists a finitely generated ideal $\mathfrak{k} \subseteq A$ such that $\mathrm{Cl}_\sigma^A(\mathfrak{k}) = \mathfrak{m}$. For any $\mathfrak{h} \in \mathcal{L}(\tau_\sigma)$, we have $\mathfrak{k} \subseteq \mathrm{rad}(\mathfrak{h})$, whence there is $n \in \mathbb{N}$ such that $\mathfrak{k}^n \subseteq \mathfrak{h}$; since $\mathcal{K}(\sigma)$ is zero–dimensional, then $\mathrm{rad}(\mathfrak{k}^n) = \mathfrak{m}$, and $\mathfrak{k}^n \in \mathcal{L}(\tau_\sigma)$. □

We discuss now the relationship between totally τ_σ–Noetherian, and totally σ–Noetherian everywhere in the case of σ–local rings.

Proposition 4.4. *Let A be a σ–local ring such that τ_σ is of finite type, for any A–module M, the following statements are equivalent:*

(a) *M is totally σ–Noetherian everywhere.*

(b) *M is totally τ_σ–Noetherian.*

Proof. Since $\tau_\sigma \leq \tau_\mathfrak{a}$ for any ideal $\mathfrak{a} \in \mathcal{L}_f(A) \setminus \mathcal{L}(\sigma)$, we have (b) $\Rightarrow$ (a).

(a) $\Rightarrow$ (b) Let $\{H_i \mid i \in I\}$ be an ascending chain of submodules of M; for any $\mathfrak{a} \in \mathcal{L}_f(A) \setminus \mathcal{L}(\sigma)$, there are $\mathfrak{h}_\mathfrak{a} \in \mathcal{L}(\tau_\mathfrak{a})$, and $j_\mathfrak{a} \in I$ such that $(\sum_i H_i)\,\mathfrak{h}_\mathfrak{a} \subseteq H_{j_\mathfrak{a}}$. If we define $\mathfrak{h} = \sum_\mathfrak{a} \mathfrak{h}_\mathfrak{a}$, then $\mathfrak{h} \in \mathcal{L}(\tau_\sigma)$.

Since τ_σ is of finite type, there is $\mathfrak{k} \subseteq \mathfrak{h}$ such that $\mathfrak{k} \in \mathcal{L}(\tau_\sigma)$, and $\mathfrak{k} \subseteq A$ is finitely generated. Therefore, there are $j_1, \ldots, j_t \in I$ such that $\mathfrak{k} \subseteq \sum_{n=1}^{t} \mathfrak{h}_{\mathfrak{a}_{j_n}}$; therefore, $\mathfrak{f} = \sum_{n=1}^{t} \mathfrak{h}_{\mathfrak{a}_{j_n}} \in \mathcal{L}(\tau_\sigma)$. If we take $j = \max\{j_{\mathfrak{a}_1}, \ldots, j_{\mathfrak{a}_t}\}$, then $(\sum_i H_i)\mathfrak{f} \subseteq H_j$, and M is totally τ_σ–Noetherian. □

As a consequence of Proposition 3.11, we have that if $\mathcal{C}(\sigma) = \{\mathfrak{m}\}$, then τ_σ is of finite type, provided that $\mathfrak{m}$ is the radical of a σ–finitely generated ideal; therefore, the previous proposition holds in the following cases: Spec(A) is Noetherian or, with more generality, whenever $\mathcal{K}(\sigma)$ is Noetherian. See [5] Jara et al. In particular, if A is totally σ–Noetherian.

5. Artinian modules everywhere

An A–module M is

- σ–**Artinian everywhere II** whenever M is $\tau_\mathfrak{a}$–Artinian for every finitely generated ideal $\mathfrak{a} \notin \mathcal{L}(\sigma)$.
- **totally** σ–**Artinian everywhere II** whenever M is totally $\tau_\mathfrak{a}$–Artinian for every finitely generated ideal $\mathfrak{a} \notin \mathcal{L}(\sigma)$.

Similarly, we define σ–**Artinian everywhere I** and **totally** σ–**Artinian everywhere I**. Obviously, every σ–Artinian module everywhere II is σ–Artinian everywhere I.

We have that totally σ–Artinian modules everywhere can be characterized, following [1], as follows:

Let $\mathcal{X}$ be a family of submodules of $\mathcal{L}(M)$.

- The family $\mathcal{X}$ is **totaly** σ–**cosaturated everywhere** if it is totally $\tau_\mathfrak{a}$–cosaturated for every ideal $\mathfrak{a} \in \mathcal{L}_f(A) \setminus \mathcal{L}(\sigma)$.
- An element $N \in \mathcal{X}$ is **totally** σ–**minimal everywhere** if it is totally $\tau_\mathfrak{a}$–minimal for every ideal $\mathfrak{a} \in \mathcal{L}_f(A) \setminus \mathcal{L}(\sigma)$.
- The module M satisfies the **totally** σ–**minimal condition** if every non–empty family of submodules has a totally σ–minimal element, and satisfies the **totally** σ–**minimal condition everywhere** whenever every non–empty family of submodules has a totally σ–minimal everywhere element.
- The module M is **totally** σ–**finitely cogenerated** if for every non–empty family of submodules $\mathcal{X}$, if $\cap\mathcal{X} = 0$, the intersection of all elements in $\mathcal{X}$ is zero, there exists $\mathfrak{h} \in \mathcal{L}(\sigma)$, and a finite subfamily $\mathcal{F} \subseteq \mathcal{X}$ such that $(\cap\mathcal{F})\mathfrak{h} = 0$; and it is **totally** σ–**finitely cogenerated everywhere** whenever if it is totally $\tau_\mathfrak{a}$–finitely cogenerated for every ideal $\mathfrak{a} \in \mathcal{L}_f(A) \setminus \mathcal{L}(\sigma)$.

With these definitions we have the following theorem of characterization of totally σ–Artinian modules everywhere. See [1] for totally σ–Artinian everywhere I or Theorem 4.1.

Theorem 5.1. *Let M be an A–module, the following statements are equivalent:*

(a) *M is totally σ–Artinian everywhere.*

(b) *Every non–empty totally σ–cosaturated family everywhere $\mathcal{X}$ of submodules of M has a minimal element.*

(c) *Every non–empty family of submodules $\mathcal{X}$ of M has a totally σ–minimal element everywhere.*

(d) *Every factor module of M is totally σ–finitely cogenerated everywhere.*

The behaviour of totally σ–Artinian modules everywhere is reflected in the following results.

Lemma 5.2. *Let M be an A–module, and $N \subseteq M$ a submodule, the following statements are equivalent:*

(a) *M is totally σ–Artinian everywhere II.*

(b) *N and M/N are totally σ–Artinian everywhere II.*

Similarly for totally Artinian everywhere I.

Proof. (a) $\Rightarrow$ (b) Let $\{H_i \mid i \in I\}$ and $\{K_i/N \mid i \in I\}$ be a decreasing chain of submodules of N and M/N, respectively. Since $\{H_i \mid i \in I\}$ is a chain of submodules of M, for any $\mathfrak{a} \in C(A,\sigma)_f$, there exist an index $j \in I$, and $n \in \mathbb{N}$ such that $H_j\mathfrak{a}^n \subseteq \cap_i H_i$, so the chain of submodules of N stabilizes. Given that $\{K_i \mid i \in I\}$ is a chain of submodules of M, there exist an index $j \in I$, and $n \in \mathbb{N}$ such that $K_j\mathfrak{a}^n \subseteq \cap_i K_i$, whence $\left(\frac{K_j}{N}\right)\mathfrak{a}^n \subseteq \cap_i \frac{K_i}{N}$, and the chain of submodules of M/N stabilizes.

(b) $\Rightarrow$ (a) Let $\{H_i \mid i \in I\}$ be a decreasing chain of submodules of M, consider $\{H_i \cap N \mid i \in I\}$ and $\{(H_i + N)/N \mid i \in I\}$ chains of submodules in N and M/N, respectively. For any ideal $\mathfrak{a} \in C(A,\sigma)_f$, there exist indices $j_1, j_2 \in I$, and $n_1, n_2 \in \mathbb{N}$, such that $(H_{j_1} \cap N)\mathfrak{a}^{n_1} \subseteq \cap_i (H_i \cap N)$, and $\frac{(H_{j_2}+N)}{N}\mathfrak{a}^{n_2} \subseteq \cap_i \frac{(H_i+N)}{N}$. Therefore, $\frac{H_{j_2}}{H_{j_2}\cap N}\mathfrak{a}^{n_2} \subseteq \cap_i \frac{H_i}{H_i \cap N}$. For $j \geq j_1, j_2$, $n \geq n_1, n_2$, and $i \geq j$, we have short exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H_i \cap N & \longrightarrow & H_i & \longrightarrow & H_j/(H_i \cap N) & \longrightarrow & 0 \\
 & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \\
0 & \longrightarrow & H_j \cap N & \longrightarrow & H_j & \longrightarrow & H_j/(H_j \cap N) & \longrightarrow & 0.
\end{array}$$

Since $\frac{H_j}{H_j \cap N}\mathfrak{a}^n = \frac{H_i}{H_i \cap N}$, and $(H_j \cap N)\mathfrak{a}^n = H_i \cap N$, we have $H_j\mathfrak{a}^n = H_i$. □

Lemma 5.3. *Let A be a totally σ–Artinian ring everywhere II, for any ideal $\mathfrak{c} \subseteq A$, the ring $A/\mathfrak{c}$ is totally σ'–Artinian everywhere II, being σ' the induced hereditary torsion theory in **Mod**– $A/\mathfrak{c}$.*

Similarly for totally Artinian everywhere I.

Proof. We have $\mathcal{L}(\sigma') = \{\mathfrak{b}/\mathfrak{c} \subseteq A/\mathfrak{c} \mid \mathfrak{b} \in \mathcal{L}(\sigma)\}$. In this case, we also have that $\mathfrak{b}/\mathfrak{c} \in C(A/\mathfrak{c}, \sigma')$ if and only if $\mathfrak{b} \in C(A, \sigma)$. In fact, they are equivalent to the following property: for any $x \in A$, and any $\mathfrak{h} \in \mathcal{L}(\sigma)$, if $x\mathfrak{h} \subseteq \mathfrak{b}$, then $x \in \mathfrak{b}$. On the other hand, for any finitely generated ideal $\mathfrak{b}/\mathfrak{c} \in C(A/\mathfrak{c}, \sigma')$, we have $\mathfrak{b} \in C(A, \sigma)$, and there is a finitely generated ideal $\mathfrak{b}' \subseteq \mathfrak{b}$ such that $(\mathfrak{b}' + \mathfrak{c})/\mathfrak{c} = \mathfrak{b}/\mathfrak{c}$.

Let $\{\mathfrak{b}_i/\mathfrak{c} \mid i \in I\}$ be a decreasing chain of ideals of $A/\mathfrak{c}$; for any $\mathfrak{a} \in C(A, \sigma)_f$, there are an index $j \in J$, and $n \in \mathbb{N}$, such that $\mathfrak{b}_j\mathfrak{a}^n \subseteq \cap_i \mathfrak{b}_i$. So, $(\mathfrak{b}_j/\mathfrak{c})\ ((\mathfrak{a}^n + \mathfrak{c})/\mathfrak{c}) \subseteq \cap_i(\mathfrak{b}_i/\mathfrak{c})$. Consequently, $A/\mathfrak{c}$ is totally σ'–Artinian everywhere II. □

Corollary 5.4. *Let $\mathfrak{c} \subseteq A$ be an ideal such that the A–module $A/\mathfrak{c}$ is totally σ'–Artinian everywhere II, then $A/\mathfrak{c}$ is a totally σ'–Artinian ring everywhere II.*

Similarly for totally Artinian everywhere I.

Example 5.5. If $\sigma = o$; that is, every A–module is σ–torsion–free, or equivalently, $\mathcal{L}(o) = \{A\}$, a module M is totally o–Artinian everywhere I if and only if M is Artinian* in the sense of Ansari and others, [1].

From the definition, it is obvious the following result:

Proposition 5.6. *The different classes of Artinian modules are related as follows.*

(1) *Every σ–torsion module is σ–Artinian.*

(2) *Every σ–Artinian module is τ_σ–Artinian.*

(3) *Every τ_σ–Artinian module is σ–Artinian everywhere II.*

In the case of totally torsion, we also have:

(1') *Every totally σ–torsion module is totally σ–Artinian.*

(2') *Every totally σ–Artinian module is totally τ_σ–Artinian.*

(3') *Every totally τ_σ–Artinian module is totally σ–Artinian everywhere II.*

Similarly for totally Artinian everywhere I.

Proof. (1) If M is σ–torsion, then $C(M, \sigma) = \{M\}$, hence σ–Artinian.

(2) If M is σ–Artinian, since $\sigma \leq \tau_\sigma$, we have $C(M, \tau_\sigma) \subseteq C(M, \sigma)$, and M is τ_σ–Artinian.

We have $\sigma \leq \tau_{\mathfrak{a}}$, whence $\mathrm{Cl}_{\sigma}^{M}(N) \subseteq \mathrm{Cl}_{\tau_{\mathfrak{a}}}^{M}(N)$ for every submodule $N \subseteq M$, and $\mathrm{Cl}_{\tau_{\mathfrak{a}}}^{M}(N) = \mathrm{Cl}_{\tau_{\mathfrak{a}}}^{M}(\mathrm{Cl}_{\sigma}^{M}(N))$, so for any decreasing chain $\{N_i \mid i \in I\}$, if there exists $j \in I$ such that $\mathrm{Cl}_{\sigma}^{M}(N_j) = \mathrm{Cl}_{\sigma}^{M}(N_i)$, for any $i \geq j$, then $\mathrm{Cl}_{\tau_{\mathfrak{a}}}^{M}(N_j) = \mathrm{Cl}_{\tau_{\mathfrak{a}}}^{M}(N_i)$. In conclusion, if M is σ–Artinian, then M is τ_{σ}–Artinian.

(3) Since $\tau_{\sigma} \leq \tau_{\mathfrak{a}}$ for every $\mathfrak{a} \notin \mathcal{L}(\sigma)$, the same argument as above can be applied.

(1') and (2') are obvious.

(3') If M is totally τ_{σ}–Artinian, for any decreasing chain of submodules $\{N_i \mid i \in I\}$, there exist $j \in I$ and $\mathfrak{h} \in \mathcal{L}(\tau_{\sigma})$ such that $N_j\mathfrak{h} \subseteq \cap_i N_i$; since $\mathfrak{h} \in \mathcal{L}(\tau_{\mathfrak{a}})$, for any $\mathfrak{a}$, M is totally σ–Artinian everywhere II. □

$$\begin{array}{ccccccccc}
\text{t. } \sigma\text{–torsion} & \Longrightarrow & \text{t. } \sigma\text{–art.} & \Longrightarrow & \text{t. } \tau_{\sigma}\text{–art.} & \Longrightarrow & \text{t. } \sigma\text{–art. every. II} & \Longrightarrow & \text{t. } \sigma\text{–art. every. I} \\
\Downarrow & & \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow \\
\sigma\text{–torsion} & \Longrightarrow & \sigma\text{–art.} & \Longrightarrow & \tau_{\sigma}\text{–art.} & \Longrightarrow & \sigma\text{–art. every. II} & \Longrightarrow & \sigma\text{–art. every. I}
\end{array}$$

Corollary 5.7. *For any A–module M, the following statements are equivalent:*

(a) *M is σ–Artinian everywhere II.*

(b) *$M/\sigma M$ is σ–Artinian everywhere II.*

Proof. Since σM is always a σ–Artinian module everywhere II, the result is a direct consequence of Lemma 5.2. □

Proposition 5.8. *If A is a non–local ring, and o the hereditary torsion theory with $\mathcal{L}(o) = \{A\}$, then the following statements are equivalent:*

(a) *M is Artinian.*

(b) *M is totally o–Artinian everywhere II.*

(c) *M is totally o–Artinian everywhere I.*

See also [2, Theorem 3.21].

Proof. (a) $\Rightarrow$ (b) $\Rightarrow$ (c) are clear. To prove that (c) $\Rightarrow$ (a), consider a decreasing chain of submodules, $\{N_i \mid i \in I\}$ of M; for any finitely generated ideal $\mathfrak{a} \subseteq A$ such that $\mathfrak{a} \neq A$, there exist $j_{\mathfrak{a}} \in I$ and $n_{\mathfrak{a}} \in \mathbb{N}$ such that $N_{j_{\mathfrak{a}}}\mathfrak{a}^{n_{\mathfrak{a}}} \subseteq \cap_i N_i$. Define $\mathfrak{b} = \sum_{\mathfrak{a}} \mathfrak{a}^{n_{\mathfrak{a}}}$. If $\mathfrak{b} \neq A$, there exists a maximal ideal $\mathfrak{m} \supseteq \mathfrak{b}$, whence $\mathfrak{a} \subseteq \mathfrak{m}$, for all $\mathfrak{a} \neq A$. Since A is non–local, we have a contradiction. On the other hand, if $\mathfrak{b} = A$, there are finitely many ideals $\mathfrak{a}_1, \ldots, \mathfrak{a}_t$ such that $\sum_{h=1}^{t} \mathfrak{a}_h^{n_{\mathfrak{a}_h}} = A$. Consequently, if $j = \max\{j_{\mathfrak{a}_1}, \ldots, j_{\mathfrak{a}_t}\}$ and $n = \max\{n_{\mathfrak{a}_1}, \ldots, n_{\mathfrak{a}_t}\}$, then $N_j \subseteq N_j\mathfrak{a}_h^n \subseteq \cap_i N_i$; that is $N_j = \cap_i N_i$, and the decreasing chain stabilizes. □

Proposition 5.9. *Let σ be a finite type hereditary torsion theory in **Mod**– A such that A is not σ–local, then the following statements are equivalent:*

(a) *M is totally σ–Artinian.*

(b) *M is totally σ–Artinian everywhere II.*

(c) *M is totally σ–Artinian everywhere I.*

Proof. (a) $\Rightarrow$ (b) $\Rightarrow$ (c) are obvious. On the other hand, to prove that (c) $\Rightarrow$ (a), consider a decreasing chain of submodules $\{N_i \mid i \in I\}$. For any finitely generated ideal $\mathfrak{a} \in A$ such that $\mathfrak{a} \notin \mathcal{L}(\sigma)$, there exist $\mathfrak{b}_\mathfrak{a} \in \mathcal{L}(\sigma)$, $n_\mathfrak{a} \in \mathbb{N}$ and $j_\mathfrak{a} \in I$ such that $N_{j_\mathfrak{a}}(\mathfrak{b}_\mathfrak{a}\mathfrak{a})^{n_\mathfrak{a}} \subseteq \cap_i N_i$. Define $\mathfrak{b} = \sum\{(\mathfrak{b}_\mathfrak{a}\mathfrak{a})^{n_a} \mid \mathfrak{a} \notin \mathcal{L}(\sigma)\}$. If $\mathfrak{b} \notin \mathcal{L}(\sigma)$, there is $\mathfrak{n} \in \mathrm{Max}(C(A,\sigma))$ such that $\mathfrak{b} \subseteq \mathfrak{n}$; therefore, for any $\mathfrak{a} \notin \mathcal{L}(\sigma)$, we have $\mathfrak{a} \subseteq \mathfrak{n}$, hence A is σ–local, which is a contradiction. Otherwise, if $\mathfrak{b} \in \mathcal{L}(\sigma)$, there is $\mathfrak{h} \in \mathcal{L}(\sigma)$, finitely generated, such that $\mathfrak{h} \subseteq \mathfrak{b}$, so there are ideals $\mathfrak{a}_{i_j}$ such that $\mathfrak{a}_{i_j}A \notin \mathcal{L}(\sigma)$, $j = 1, \ldots, t$, such that $\mathfrak{h} \subseteq \sum_{j=1}^{t}(\mathfrak{b}_{\mathfrak{a}_{i_j}}\mathfrak{a}_{i_j})^{n_{\mathfrak{a}_{i_j}}}$, whence for any $k \in I$ such that $i_j < k$, we have $N_k\mathfrak{h} \subseteq \cap_i N_i$. In conclusion, the decreasing chain $\{N_i \mid i \in I\}$ is totally σ–stable; therefore, M is totally σ–Artinian. $\square$

This result can be extended to consider σ–local rings, in the following sense:

Proposition 5.10. *Let A be a σ–local ring such that τ_σ is of finite type, for any A–module M, the following statements are equivalent:*

(a) *M is totally σ–Artinian everywhere.*

(b) *M is totally τ_σ–Artinian.*

Similar to Proposition 4.4.

Proposition 5.11. *If A is a totally σ–Artinian ring everywhere I, then* $\dim_\sigma(A) = 0$*, and every element of $\mathcal{K}(\sigma)$ is a maximal ideal of A; that is $\mathcal{K}(\sigma) \subseteq \mathrm{Max}(A)$.*

Proof. Let $\mathfrak{p}_0 \subsetneq \mathfrak{p}_1$ ideals in $\mathcal{K}(\sigma)$. Consider the ring $A/\mathfrak{p}_0$ and σ' the induced hereditary torsion theory in **Mod**– $A/\mathfrak{p}_0$. Since $A/\mathfrak{p}_0$ is a σ'–Artinian ring everywhere and $0 \neq \mathfrak{p}_1/\mathfrak{p}_0 \subseteq A/\mathfrak{p}_0$ is a prime ideal in $\mathcal{K}(\sigma')$, we can rename $A/\mathfrak{p}_0$ as A, and consider an integral domain σ–Artinian everywhere with prime ideals $0, \mathfrak{p} \in \mathcal{K}(\sigma)$, and $\mathfrak{p} \neq 0$.

For any $0 \neq x \in \mathfrak{p}$, let us consider the decreasing chain $xA \supseteq x^2A \supseteq \cdots$. By the hypothesis, since $xA \notin \mathcal{L}(\sigma)$, there are an index $j \in \mathbb{N}$, and $m \in \mathbb{N}$, such that $(x^jA)x^m \subseteq \cap_n x^nA$. Then we have $x^{j+m}A = x^{j+m+1}A$, and there is $x \in A$ such that $x^{j+n} = x^{j+n+1}y$; that is, $1 = xy$, and $x \in A$ is invertible. In consequence, $\mathfrak{p} = A$, which is a contradiction.

In conclusion, the integral domain A is a field and each ideal $\mathfrak{p}_0 \in \mathcal{K}(\sigma)$ is a maximal ideal; that is, $\mathcal{K}(\sigma) \subseteq \mathrm{Max}(A)$, and the σ–dimension of A is 0. $\square$

Now we can extend this result to modules as follows.

Lemma 5.12. *For any A–module M, we have* $\operatorname{Supp}(\sigma M) \subseteq \mathcal{Z}(\sigma)$.

Corollary 5.13. *If M is a non σ–torsion totally σ–Artinian module everywhere I, then the σ–dimension of M is zero, and* $\operatorname{Supp}(M) \cap \mathcal{K}(\sigma) \subseteq \operatorname{Max}(A)$.

Proof. Let M be a non σ–torsion A–module, and $\mathfrak{p} \in \operatorname{Supp}(M) \cap \mathcal{K}(\sigma)$; there exists $0 \neq m \in M$ such that $\operatorname{Ann}(m) \subseteq \mathfrak{p}$. Then $mA \cong A/\operatorname{Ann}(m)$ is a totally σ–Artinian A–module everywhere I, whence $A/\operatorname{Ann}(m)$ is a totally σ'–Artinian ring everywhere I, so $\mathcal{K}(\sigma') \subseteq \operatorname{Max}(A/\operatorname{Ann}(m))$. Consequently, $\mathfrak{p} \in \operatorname{Max}(A)$. □

Proposition 5.14. *If A is a totally σ–Artinian ring everywhere I, then $\mathcal{K}(\sigma)$ is finite, and if A is σ–local, then $\mathcal{K}(\sigma)$ is a singleton.*

Proof. If $\mathcal{K}(\sigma)$ is not local, then A is σ–Artinian, and $\mathcal{K}(\sigma)$ is finite.

If $\mathcal{K}(\sigma)$ is local, since $\dim(\mathcal{K}(\sigma)) = 0$, $\mathcal{K}(\sigma) = \{\mathfrak{m}\}$ is a singleton. □

Proposition 5.15. [Nakayama–like lemma] *Let M be a σ–finitely generated module, and $\mathfrak{a} \subseteq \cap\mathcal{C}(\sigma)$; if $M\mathfrak{a} = M$, then M is σ–torsion.*

Proof. Suppose that M is not σ–torsion, and consider

$$\Gamma = \{N \subsetneqq M \mid M/N \text{ is } \sigma\text{–torsionfree}\};$$

given that $\sigma M \in \Gamma$, then $\Gamma \neq \varnothing$. For any ascending chain $\{N_i \mid i \in I\}$ in Γ, we have $\cup_i N_i \in \Gamma$ because σ is of finite type. By Zorn's lemma, there are maximal elements in Γ. For any $N \in \Gamma$ maximal, and any submodule $N \subsetneqq H \subseteq M$, we have M/H is σ–torsion.

On the other hand, for any submodule $N \subsetneqq H \subseteq M$, we have that H/N is σ–torsionfree, and any proper quotient is σ–torsion. In particular, for any $m \in M \setminus N$, we have that $(N : m) \in C(A, \sigma)$ is maximal, hence $\mathfrak{p} := (N : m) \subseteq A$ is a prime ideal. For any $0 \neq x + N \in M/N$, we also have $(x + N)\mathfrak{p} = 0$, so $\mathfrak{p}$ annihilates M/N.

Consequently, for any ideal $\mathfrak{a} \subseteq \cap\mathcal{C}(\sigma)$, we have $\mathfrak{a} \subseteq \mathfrak{p}$, hence $M\mathfrak{a} \subseteq M\mathfrak{p} \subseteq N \neq M$, which is a contradiction. Therefore, M must be σ–torsion. □

The property totally σ–Artinian everywhere is inherited by localization.

Proposition 5.16. *Let A be a totally σ–Artinian ring and $\mathfrak{m} \in \mathcal{K}(\sigma)$, then $A_{\mathfrak{m}}$ is totally o–Artinian everywhere.*

Proof. Let $\{\mathfrak{b}_i \mid i \in I\}$ be a chain in $A_\mathfrak{m}$, and $\mathfrak{a}_i = \lambda^{-1}(\mathfrak{b}_i)$, for any $i \in I$, being $\lambda : A \longrightarrow A_\mathfrak{m}$ the canonical map. For any $0 \neq a/s \in \mathfrak{m}A_\mathfrak{m}$, given that $aA \notin \mathcal{L}(\sigma)$, there are $j \in I$ and $m \in \mathbb{N}$ such that $\mathfrak{a}_j a^m \subseteq \cap_i \mathfrak{a}_i$. Consequently, $\mathfrak{b}_j(a/s) \subseteq \cap_i \mathfrak{b}_i$. □

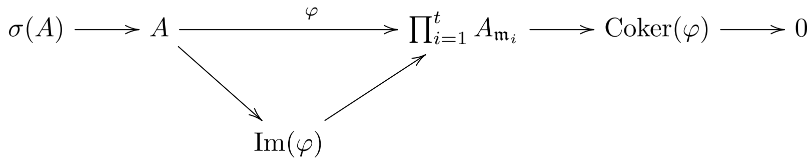
Theorem 5.17. *If A is a totally σ–Artinian ring everywhere which is not σ–local, then the following statements hold:*

(1) $\mathcal{K}(\sigma) = \mathcal{C}(\sigma) \subseteq \mathrm{Max}(A)$ *is finite.*

(2) $\sigma = \tau_\sigma$.

(3) *Every totally σ–Artinian module everywhere is totally σ–Artinian.*

(4) *If $\mathcal{K}(\sigma) = \{\mathfrak{m}_1, \ldots, \mathfrak{m}_t\}$, there is a commutative diagram*



and $\mathfrak{h} \in \mathcal{L}(\sigma)$ such that $(\prod_{i=1}^t A_{\mathfrak{m}_i})\mathfrak{h} \subseteq \mathrm{Im}(\varphi)$.

Proof. (1) is a direct consequence of Propositions 5.11 and 5.14 and Corollary 5.13.

(2) is a consequence of Proposition 3.9.

(3) is a consequence of Proposition 5.9.

(4) We can apply [6, Proposition 2.8], where we have that every local ring $A_{\mathfrak{m}_i}$ is totally o–Artinian, hence local Artinian, and A is totally $\sigma_{A\setminus\mathfrak{m}_i}$–Artinian.

In particular, A is totally σ–Noetherian and σA is totally σ–torsion. □

Following this approach to the totally σ–Artinian everywhere structure, an in-depth study of local totally o–Artinian rings everywhere remains to be carried out.

Theorem 5.18. *If A is σ–local, the following statements are equivalent:*

(a) *A is totally σ–Artinian everywhere.*

(b) $\tau_\mathfrak{a} = 1$, *for every $\mathfrak{a} \in \mathcal{L}_f(A) \setminus \mathcal{L}(\sigma)$.*

(c) *For every $\mathfrak{a} \in \mathcal{L}_f \setminus \mathcal{L}(\sigma)$, there is $n \in \mathbb{N}$ such that $\mathfrak{a}^n \subseteq \sigma A$.*

In particular, $\tau_\sigma = 1$.

Proof. (a) ⇒ (b) We have $\mathcal{K}(\sigma) = \mathcal{C}(\sigma) = \{\mathfrak{m}\}$ is unitary. For any $\mathfrak{a} \in \mathcal{L}_f \setminus \mathcal{L}(\sigma)$, since $\mathfrak{a} \subseteq \mathfrak{m}$, we have $\mathcal{K}(\tau_\mathfrak{a}) = \varnothing$. Since $\tau_\mathfrak{a}$ is of finite type, $\tau_\mathfrak{a} = \wedge\{\sigma_{A\setminus\mathfrak{p}} \mid \mathfrak{p} \in \mathcal{K}(\tau_\mathfrak{a})\} = 1$.

(b) ⇔ (c) ⇒ (a) are straightforward. □

Corollary 5.19. *If A is a local ring and $\sigma = o$, then the ring A is totally o–Artinian everywhere if and only if $A \setminus U(A) \subseteq \mathrm{Nil}(A)$.*

6. Extensions of finite type hereditary torsion theories

We can extend the construction of τ_σ, in the σ–local case, to a more general framework. Thus, starting from a finite type hereditary torsion theory σ and any prime ideal $\mathfrak{p} \in \mathcal{K}(\sigma)$, we clone the above construction of τ_σ to obtain a new hereditary torsion theory $\tau_\mathfrak{p}$. In particular, we are interested in determining when this new torsion theory $\tau_\mathfrak{p}$ is of finite type.

Given a prime ideal $\mathfrak{p} \in \mathrm{Spec}(A)$, define $\eta_\mathfrak{p}$ as

$$\mathcal{L}(\eta_\mathfrak{p}) = \{\mathfrak{h} \subseteq A \mid \mathrm{rad}(\mathfrak{h}) \supseteq \mathfrak{p}\}.$$

Lemma 6.1. *$\mathcal{L}(\eta_\mathfrak{p})$ is a Gabriel filter, therefore, $\eta_\mathfrak{p}$ is a hereditary torsion theory.*

Proof. Let $\mathfrak{a} \in \mathcal{L}(\eta_\mathfrak{p})$, and $\mathfrak{b} \subseteq A$ such that for all $a \in \mathfrak{a}$, we have $(\mathfrak{b} : a) \in \mathcal{L}(\eta_\mathfrak{p})$. For any $x \in \mathfrak{p}$, there exists $n \in \mathbb{N}$ such that $x^n \in \mathfrak{a}$, whence $(\mathfrak{b} : x^n) \in \mathcal{L}(\eta_\mathfrak{p})$, and $\mathfrak{p} \subseteq \mathrm{rad}(\mathfrak{b} : x^n)$; therefore, there exists $m \in \mathbb{N}$ such that $x^m \in (\mathfrak{b} : x^n)$. Thus we have $x^n x^m \in \mathfrak{b}$. This means that $\mathfrak{p} \subseteq \mathrm{rad}(\mathfrak{b})$, and $\mathfrak{b} \in \mathcal{L}(\eta_\mathfrak{p})$. □

Let us consider

$$\mathcal{Z}(\eta_\mathfrak{p}) = \{\mathfrak{q} \in \mathrm{Spec}(A) \mid \mathfrak{q} = \mathrm{rad}(\mathfrak{q}) \supseteq \mathfrak{p}\} = V(\mathfrak{p}), \text{ and}$$
$$\mathcal{K}(\eta_\mathfrak{p}) = \{\mathfrak{q} \in \mathrm{Spec}(A) \mid \mathfrak{q} = \mathrm{rad}(\mathfrak{q}) \not\supseteq \mathfrak{p}\} = X(\mathfrak{p}).$$

Therefore, we have $\eta_\mathfrak{p} \leq \wedge\{\sigma_{A\setminus\mathfrak{q}} \mid \mathfrak{q} \in \mathcal{K}(\eta_\mathfrak{p})\}$; furthermore, we have the equality.

Lemma 6.2. *$\eta_\mathfrak{p}$ is half–centered, i.e., $\eta_\mathfrak{p} = \wedge\{\sigma_{A\setminus\mathfrak{q}} \mid \mathfrak{q} \in \mathcal{K}(\eta_\mathfrak{p})\}$.*

Proof. Given $\mathfrak{h} \in \mathcal{L}(\wedge\sigma_{A\setminus\mathfrak{q}})$ for every $\mathfrak{q} \in \mathcal{K}(\eta_\mathfrak{p})$, we have $\mathfrak{h} \not\subseteq \mathfrak{q}$, whence $\mathfrak{h}$ is an intersection of prime ideals in $\mathcal{Z}(\eta_\mathfrak{p})$; since each of them contains $\mathfrak{p}$, $\mathfrak{p} \subseteq \mathrm{rad}(\mathfrak{h})$. □

A sufficient condition to be $\eta_\mathfrak{p}$ of finite type is that $\mathfrak{p}$ is a finitely generated ideal. We can even characterize when $\eta_\mathfrak{p}$ is of finite type as follows.

Proposition 6.3. *Let $\mathfrak{p} \subseteq A$ be a prime ideal, the following statements are equivalent:*

(a) *$\eta_\mathfrak{p}$ is of finite type.*
(b) *$\mathfrak{p}$ is the radical of a finitely generated ideal.*

Proof. (a) ⇒ (b) Since $\mathfrak{p} \in \mathcal{L}(\eta_\mathfrak{p})$, there is $\mathfrak{h} \subseteq A$ finitely generated such that $\mathfrak{h} \subseteq \mathfrak{p}$. Since $\mathrm{rad}(\mathfrak{h}) \supseteq \mathfrak{p}$, $\mathrm{rad}(\mathfrak{h}) = \mathfrak{p}$.

(b) $\Rightarrow$ (a) Let $\mathfrak{h} \in \mathcal{L}(\eta_\mathfrak{p})$, and $\mathfrak{k} \in A$ finitely generated such that $\operatorname{rad}(\mathfrak{k}) = \mathfrak{p}$, whence $\mathfrak{k} \in \mathcal{L}(\eta_\mathfrak{p})$ and $\mathfrak{k} \subseteq \operatorname{rad}(\mathfrak{h})$, then there exists $n \in \mathbb{N}$ such that $\mathfrak{k}^n \subseteq \mathfrak{h}$. In conclusion, $\eta_\mathfrak{p}$ is of finite type. □

Note that we have $\eta_\mathfrak{p}$ is of finite type if and only if $\mathfrak{p}$ is the radical of a finitely generated ideal if and only if $\mathcal{K}(\eta_\mathfrak{p})$ is quasi–compact. Sufficient conditions for $\eta_\mathfrak{p}$ to be of finite type are that $\operatorname{Spec}(A)$ is a Noetherian topological space, or A is a Noetherian ring.

Example 6.4. Let $A = \mathbb{F}_2^{(\mathbb{N})} + \mathbb{F}_2$,

- $\mathfrak{p} = \mathbb{F}_2^{(\mathbb{N})}$, and
- $\mathfrak{p}_m = (1 - e_m)A$, $e_m = (\delta_{mi})_i$, for any $m \in \mathbb{N}$,

are the prime ideals of A.

In this case, we have:

- $\mathcal{K}(\eta_\mathfrak{p}) = \{\mathfrak{q} \subseteq A \mid \mathfrak{q} \neq \mathfrak{p}\} = \{\mathfrak{p}_n \mid n \in \mathbb{N}\}$, and
- $\mathcal{K}(\eta_{\mathfrak{p}_m}) = \{\mathfrak{q} \subseteq A \mid \mathfrak{q} \neq \mathfrak{p}_m\} = \{\mathfrak{p}\} \cup \{\mathfrak{p}_n \mid n \in \mathbb{N} \setminus \{m\}\}$.

The basic open subsets of $\operatorname{Spec}(A)$ are:

- $X(a) = \{\mathfrak{p}_n \in \operatorname{Spec}(A) \mid a_n \neq 0\}$, for any $a \in \mathfrak{p}$; it has finitely many elements.
- If $a \notin \mathfrak{p}$, then a is finally constant equal to 1; let $k \in \mathbb{N}$ such that $a_h = 1$ for any $h \geq k$, and $a_h = 0$ if $h < k$. Since $a \in \mathfrak{p}_n$ whenever $a_n = 0$, $X(a) \subseteq \{\mathfrak{p}\} \cup \{\mathfrak{p}_h \mid h \geq k\}$.

In consequence, $\mathcal{K}(\eta_\mathfrak{p})$ is not quasi–compact, and $\mathcal{K}(\eta_{\mathfrak{p}_m})$ is quasi–compact for every $m \in \mathbb{N}$.

We have that

- $\eta_\mathfrak{p}$ is not of finite type. On the contrary, let $\mathfrak{p}$ be the radical of a finitely generated ideal; hence there are $e_0, \ldots, e_t \in \mathfrak{p}$ such that $\operatorname{rad}(e_1, \ldots, e_t) = \mathfrak{p}$, then $\operatorname{rad}(\sum_{i=0}^t e_i) = \mathfrak{p}$, which is a contradiction because $\operatorname{rad}(\sum_{i=0}^t e_i) = \mathfrak{p} \cap (\cap_{i>t} \mathfrak{p}_i)$,
- since $\mathfrak{p}_m$ is finitely generated, $\eta_{\mathfrak{p}_m}$ is of finite type for any $m \in \mathbb{N}$.

More in general, let σ be a finite type hereditary torsion theory; for any prime ideal $\mathfrak{p} \in \mathcal{K}(\sigma)$, consider $\mathcal{P} = \{\mathfrak{a} \subseteq A \mid \mathfrak{a} \in \mathcal{L}_f(A),\ \mathfrak{a} \subseteq \mathfrak{p}\}$. In this situation, if $\mathcal{L}(\sigma_\mathfrak{a}) = \{\mathfrak{h} \subseteq A \mid \mathfrak{a}^n \subseteq \mathfrak{h},\ \text{for any } n \in \mathbb{N}\}$, then we have

Lemma 6.5. $\eta_\mathfrak{p} = \wedge\{\sigma_\mathfrak{a} \mid \mathfrak{a} \in \mathcal{L}_f(A),\ \mathfrak{a} \subseteq \mathfrak{p}\}$.

Proof. For any $\mathfrak{h} \in \mathcal{L}(\eta_{\mathfrak{p}})$, given that $\mathrm{rad}(\mathfrak{h}) \supseteq \mathfrak{p}$, and for any $\mathfrak{a} \in \mathcal{P}$, we have $\mathfrak{a} \subseteq \mathfrak{p}$, then $\mathfrak{a} \subseteq \mathrm{rad}(\mathfrak{h})$, so there exists $n \in \mathbb{N}$ such that $\mathfrak{a}^n \subseteq \mathfrak{h}$. Therefore, $\mathfrak{h} \in \sigma_{\mathfrak{a}}$. In conclusion, $\eta_{\mathfrak{p}} \leq \wedge_{\mathcal{P}}\sigma_{\mathfrak{a}}$.

Given $\mathfrak{h} \in \wedge_{\mathcal{P}}\sigma_{\mathfrak{a}}$, for any $\mathfrak{a} \in \mathcal{P}$, there is $n \in \mathbb{N}$ such that $\mathfrak{a}^n \subseteq \mathfrak{h}$. Let us define $\mathfrak{c} = \sum_{\mathcal{P}} \mathfrak{a}^n$. By the hypothesis, $\mathfrak{c} \subseteq \mathfrak{p}$. For any $x \in \mathfrak{p}$, there is $n \in \mathbb{N}$ such that $x^n \in \mathfrak{c}$, whence $\mathrm{rad}(\mathfrak{c}) = \mathfrak{p}$, and $\mathfrak{c} \in \mathcal{L}(\eta_{\mathfrak{p}})$. In conclusion, $\wedge_{\mathcal{P}}\sigma_{\mathfrak{a}} = \eta_{\mathfrak{p}}$. Therefore, $\eta_{\mathfrak{p}}$ is the wedge of finite type hereditary torsion theories, defined by multiplicative subsets. □

Consequently, if $\eta_{\mathfrak{p}}$ is of finite type, there exists $\mathfrak{h} \in \mathcal{L}(\eta_{\mathfrak{p}}) \cap \mathcal{L}_f(A)$ such that $\mathfrak{h} \subseteq \mathfrak{p}$; given that $\mathrm{rad}(\mathfrak{h}) \supseteq \mathfrak{p}$, then $\mathfrak{p} = \mathrm{rad}(\mathfrak{h})$. In particular, $\eta_{\mathfrak{p}} = \sigma_{\mathfrak{h}}$. See Proposition 6.3.

Once we have the hereditary torsion theory $\eta_{\mathfrak{p}}$, we define a new one $\tau_{\mathfrak{p}}$ as follows: $\tau_{\mathfrak{p}} = \sigma \vee \eta_{\mathfrak{p}}$. This new hereditary torsion theory satisfies:

$$\mathcal{K}(\tau_{\mathfrak{p}}) = \mathcal{K}(\sigma) \cap X(\mathfrak{p}) \quad \text{and} \quad \mathcal{Z}(\tau_{\mathfrak{p}}) = \mathcal{Z}(\sigma) \cup V(\mathfrak{p}).$$

In addition, we have the following description:

$$\tau_{\mathfrak{p}} = \sigma \vee \eta_{\mathfrak{p}} = \sigma \vee (\wedge_{\mathcal{P}}\sigma_{\mathfrak{a}}) = \wedge_{\mathcal{P}}(\sigma \vee \sigma_{\mathfrak{a}}).$$

Consequently, $\tau_{\mathfrak{p}}$ is a wedge of finite type hereditary torsion theories.

Since σ and $\sigma_{\mathfrak{a}}$ are finite type hereditary torsion theories, the Gabriel filter of $\tau_{\mathfrak{a}} = \sigma \wedge \sigma_{\mathfrak{a}}$ is easily described:

$$\mathcal{L}(\tau_{\mathfrak{a}}) = \{\mathfrak{h} \subseteq A \mid \text{ there exist } \mathfrak{h}_1 \in \mathcal{L}(\sigma) \text{ and } n \in \mathbb{N} \text{ such that } \mathfrak{h}_1\mathfrak{a}^n \subseteq \mathfrak{h}\}.$$

A similar description for $\tau_{\mathfrak{p}}$ is possible whenever $\eta_{\mathfrak{p}}$ is of finite type.

Since $\tau_{\mathfrak{p}}$ is an intersection of finite type hereditary torsion theories, it is half–centered (an intersection of hereditary torsion theories $\sigma_{A\setminus\mathfrak{q}}$ for a family of prime ideals $\mathfrak{q}$). Consequently, $\tau_{\mathfrak{p}}$ is of finite type if and only if $\mathcal{K}(\tau_{\mathfrak{p}})$ is quasi–compact.

Our aim is to give sufficient conditions on $\mathfrak{p}$ so that $\tau_{\mathfrak{p}}$ is a finite type hereditary torsion theory.

Note that $\mathcal{K}(\tau_{\mathfrak{p}}) = \mathcal{K}(\sigma) \cap X(\mathfrak{p})$, since $\mathcal{K}(\sigma)$ is quasi–compact, it is enough to check that $X(\mathfrak{p})$ is quasi–compact; this is the case if $\mathfrak{p}$ is the radical of a finitely generated ideal, as we saw before. A stronger condition $\mathfrak{p} = \mathrm{rad}_{\sigma}(\mathfrak{k})$, for some finitely generated ideal $\mathfrak{k} \subseteq A$, gives the same result.

Proposition 6.6. *If $\mathfrak{p} = \mathrm{rad}_{\sigma}(\mathfrak{k})$, for some finitely generated ideal $\mathfrak{k} \subseteq A$, then $\tau_{\mathfrak{p}}$ is of finite type.*

Proof. If $\mathfrak{a} \subseteq A$, then $\mathcal{K}(\sigma) \cap V(\mathfrak{a}) = \mathcal{K}(\sigma) \cap V(\mathrm{Cl}_\sigma^A(\mathfrak{a}))$. In fact, given that $\mathfrak{a} \subseteq \mathrm{Cl}_\sigma^A(\mathfrak{a})$ we have an inclusion. On the other hand, for any $\mathfrak{q} \in \mathcal{K}(\sigma) \cap V(\mathfrak{a})$, since $\mathfrak{a} \subseteq \mathfrak{q}$, we have $\mathrm{Cl}_\sigma^A(\mathfrak{a}) \subseteq \mathfrak{q}$, and $\mathfrak{q} \in \mathcal{K}(\sigma) \cap V(\mathrm{Cl}_\sigma^A(\mathfrak{a}))$. In particular we have:

$$\mathcal{K}(\sigma) \cap X(\mathfrak{a}) = \mathcal{K}(\sigma) \cap X(\mathrm{Cl}_\sigma^A(\mathfrak{a})).$$

Since σ is of finite type, $\mathrm{rad}_\sigma(\mathfrak{a}) = \mathrm{rad}(\mathrm{Cl}_\sigma^A(\mathfrak{a}))$, so if $\mathfrak{p} = \mathrm{rad}_\sigma(\mathfrak{k})$, then $\mathfrak{p} = \mathrm{rad}(\mathrm{Cl}_\sigma^A(\mathfrak{k}))$, so $X(\mathfrak{p}) = X(\mathrm{Cl}_\sigma^A(\mathfrak{k}))$, and we have:

$$\mathcal{K}(\tau_\mathfrak{p}) = \mathcal{K}(\sigma) \cap X(\mathfrak{p}) = \mathcal{K}(\sigma) \cap X(\mathrm{Cl}_\sigma^A(\mathfrak{k})) = \mathcal{K}(\sigma) \cap X(\mathfrak{k}).$$

Since $X(\mathfrak{k})$ is quasi–compact, $\tau_\mathfrak{p}$ is of finite type. □

Example 6.7. Let $A = \mathbb{F}_2^{(\mathbb{N})} + \mathbb{F}_2$, and $\sigma = o$.

- $\mathcal{L}(\eta_\mathfrak{p}) = \{\mathfrak{p}, A\}$ is not of finite type; $\mathcal{K}(\eta_\mathfrak{p}) = \{\mathfrak{p}_n \mid n \in \mathbb{N}\}$ is a Noetherian topological space; $\mathfrak{p}$ is neither finitely generated, nor the radical of a finitely generated ideal; the localization $A_{\eta_\mathfrak{p}} \cong \mathrm{Hom}(\mathfrak{p}, A)$.
- For any $m \in \mathbb{N}$: $\mathcal{L}(\eta_{\mathfrak{p}_m}) = \{(1 - e_m)A, A\}$ is of finite type; $\mathcal{K}(\eta_{\mathfrak{p}_m}) = \{\mathfrak{p}\} \cup \{\mathfrak{p}_n \mid n \in \mathbb{N} \setminus \{m\}\}$ is not a Noetherian space; $\mathfrak{p}_m$ is finitely generated; the localization $A_{\eta_{\mathfrak{p}_m}} \cong \mathrm{Hom}(\mathfrak{p}_m, A)$ is isomorphic to A.

In the case in which $\mathcal{K}(\sigma)$ is a Noetherian space, see [5], we have that any $\mathfrak{p} \in \mathcal{K}(\sigma)$ is the σ–radical of a finitely generated ideal.

In this case, $\mathcal{K}(\sigma)$ is Noetherian, the process initiated in Section 3, see Propositions 3.9 and 3.11, can be carried out more generally. For any sequence of prime ideals $\{\mathfrak{p}_i \mid i = 1, \ldots, n\}$ in $\mathcal{K}(\sigma)$ such that $\mathfrak{p}_{i+1} \in \mathcal{K}(\tau_{\mathfrak{p}_i})$, for any index $i < n$, we obtain an ascending chain of finite type hereditary torsion theories σ, $\tau_{\mathfrak{p}_1}, \ldots, \tau_{\mathfrak{p}_n}$.

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