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NONLINEAR MIXED SKEW BI-SKEW JORDAN n-DERIVATIONS ON $*$-ALGEBRAS

Asma Ali, Claus Haetinger, Tooba Naz and Mohd Tasleem

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ABSTRACT. Let $\mathcal{A}$ be a unital $*$-algebra containing nontrivial projections P_j $(j = 1, 2)$. In this article, we examine the nature of nonlinear mixed skew bi-skew Jordan n-derivations on $\mathcal{A}$. Further, we extend our main results to some special classes of $*$-algebras such as prime $*$-algebras, factor von Neumann algebras and standard operator algebras and prove that on these $*$-algebras every nonlinear mixed skew bi-skew Jordan n-derivation is an additive $*$-derivation.

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1. Introduction

Consider $\mathcal{A}$ to be a unital $*$- algebra with unity I over the ground field $\mathbb{C}$ (the field of complex numbers). An additive map $\varrho : \mathcal{A} \to \mathcal{A}$ is called an additive derivation if $\varrho(R_1R_2) = \varrho(R_1)R_2 + R_1\varrho(R_2)$ for all $R_1, R_2 \in \mathcal{A}$. Recall that a map $* : \mathcal{A} \to \mathcal{A}$ is said to be an involution if $(R + S)^* = R^* + S^*$; $(RS)^* = S^*R^*$; $(\alpha R)^* = \overline{\alpha}R^*$ and $(R^*)^* = R$ for all $R, S \in \mathcal{A}$, $\alpha \in \mathbb{C}$. An additive derivation $\varrho : \mathcal{A} \to \mathcal{A}$ is called an additive $*$-derivation if $\varrho(R^*) = \varrho(R)^*$ for all $R \in \mathcal{A}$.

Let us denote some different kinds of products based on involution operation, such as $R_1 \bullet R_2 = R_1R_2 + R_2R_1^*$ (skew Jordan product), $R_1 \diamond R_2 = R_1R_2^* + R_2R_1^*$ (bi-skew Jordan product), $R_1 \odot R_2 = R_1^*R_2 + R_2^*R_1$ (left bi-skew Jordan product) for any $R_1, R_2 \in \mathcal{A}$. These products have captured keen interest of many researchers who have worked to characterize certain mappings concerning these types of products on different kinds of rings and algebras (see [1,3,4,10–15,18,20,21,24–26]). A linear mapping $\varrho : \mathcal{A} \to \mathcal{A}$ is called a skew Jordan n-derivation if

$$\varrho(R_1 \bullet R_2 \bullet \cdots \bullet R_n) = \sum_{i=1}^{n} R_1 \bullet \cdots \bullet \varrho(R_i) \bullet \cdots \bullet R_n$$

holds for all $R_i \in \mathcal{A}$, $(1 \leq i \leq n)$. In particular, for $n = 2, 3$ the skew Jordan 2-derivation and skew Jordan 3-derivation are respectively called as skew Jordan derivation and skew Jordan triple derivation. If ϱ is not linear in the above definition, then it is called a nonlinear skew Jordan n-derivation. Exploring the nature of skew Jordan derivations and uncovering the relationship between $*$-derivations and skew Jordan derivations many mathematicians devote themselves and obtain several results. For instance, Taghavi et al. [22] investigated the behaviour of a nonlinear derivation on factor von Neumann algebras and proved that every nonlinear $*$-Jordan derivation on factor von Neumann algebras is an additive $*$-derivation. Darvish et al. [6] extended this result to nonlinear skew Jordan triple derivation on prime $*$-algebras. Li et al. [15] considered the more general case of nonlinear skew Jordan n-derivation and proved that every nonlinear $*$-Jordan n-derivation on $*$-algebras having certain conditions is an additive $*$-derivation. A map (not necessarily linear) $\varrho : \mathcal{A} \to \mathcal{A}$ is said to be a nonlinear bi-skew Jordan n-derivation if

$$\varrho(R_1 \odot R_2 \odot \cdots \odot R_n) = \sum_{i=1}^{n} R_1 \odot \cdots \odot \varrho(R_i) \odot \cdots \odot R_n$$

holds for all $R_i \in \mathcal{A}$, $(1 \leq i \leq n)$.

Darvish et al. [5, 7] considered the structure of prime $*$-algebras and proved that every nonlinear bi-skew Jordan derivation and nonlinear bi-skew Jordan triple derivation become additive $*$-derivations on them. Furthermore, Zhao et al. [26] extended these results to bi-skew Jordan n-derivations and characterized nonlinear bi-skew Jordan n-derivations on $*$-algebras.

In recent years, many researchers constructed new types of products by mixing different kinds of products as mentioned above (see [2, 9, 16, 17, 19, 23]). Li and Zhang [16, 17] studied the products $([R, S]_* \bullet T)$ and $([R \bullet T, S]_*)$, where $R \bullet T$ is the bi-skew Jordan product and $[R, S]_* = RS - SR^*$ is known as the skew Lie product of R and S. Very recently, Ali et al. [2] constructed a new mixed triple product known as the Jordan bi-skew Lie triple product and obtained the structure of nonlinear mixed Jordan bi-skew Lie triple derivations on $*$-algebras. They proved that a map on a unital $*$-algebra $\mathcal{A}$ containing a nontrivial projection and satisfying

$$\varrho([R \circ S, T]_\bullet) = [\varrho(R) \circ S, T]_\bullet + [R \circ \varrho(S), T]_\bullet + [R \circ S, \varrho(T)]_\bullet$$

for all $R, S, T \in \mathcal{A}$, is an additive $*$-derivation. Furthermore, a map (not necessarily linear) $\varrho : \mathcal{A} \to \mathcal{A}$, satisfying

$$\varrho(R_1 \circ R_2 \circ \cdots \circ R_{n-1} \odot R_n) = \sum_{i=1}^{n} R_1 \circ \cdots \circ \varrho(R_i) \circ \cdots \circ R_{n-1} \odot R_n$$

for all $R_i \in \mathcal{A}$, $(1 \leq i \leq n)$ is called a nonlinear mixed $*$-Jordan n-derivation. In [9], the authors proved that every nonlinear mixed $*$-Jordan-type derivation is an additive $*$-derivation.

In view of the above observations, we construct a mixed Jordan n-product by taking the following products: $R \star S = RS^* + SR$ and $R \odot S = R^*S + S^*R$. Considering these products the mixed Jordan n-product can be defined in four different ways, according as n is even or odd, which are as follows: $(R_1 \odot R_2 \star R_3 \odot \cdots \star R_n)$; $(R_1 \odot R_2 \star R_3 \odot R_4 \star \cdots \odot R_n)$; $(R_1 \star R_2 \odot R_3 \star \cdots \odot R_n)$ and $(R_1 \star R_2 \odot R_3 \star R_4 \odot \cdots \star R_n)$. These mixed Jordan n-products are evaluated from left to right, that is, $R_1 \odot R_2 \star R_3 \odot \cdots \star R_n = (\ldots((R_1 \odot R_2) \star R_3) \odot \cdots \star R_n)$. A map $\varrho : \mathcal{A} \to \mathcal{A}$ (not necessarily linear) termed as a nonlinear mixed skew bi-skew Jordan n-derivation if

$$\varrho(R_1 \odot R_2 \star R_3 \odot \cdots \star R_n) = \sum_{k=1}^{n} R_1 \odot R_2 \star R_3 \odot \cdots \star \varrho(R_k) \odot \cdots \star R_n$$

holds for all $R_k \in \mathcal{A}$, $1 \leq k \leq n$ and $n \geq 3$ is a fixed positive integer. The main objective of this article is to examine the nature of nonlinear mixed skew bi-skew n-Jordan derivations on $*$-algebras with some mild conditions and extend these results to special classes of $*$-algebras such as prime $*$-algebras, factor von Neumann algebras and standard operator algebras.

2. Main results

Theorem 2.1. *Let $\mathcal{A}$ be a unital $*$-algebra with identity I, containing nontrivial projections P_j $(j = 1, 2)$ such that $U\mathcal{A}P_j = (0)$ implies that $U = 0$ for any $U \in \mathcal{A}$. If a map $\varrho : \mathcal{A} \to \mathcal{A}$ satisfies*

$$\varrho(R_1 \odot R_2 \star \cdots \odot R_{n-1} \star R_n) = \sum_{k=1}^{n} R_1 \odot R_2 \star \cdots \odot \varrho(R_k) \star \cdots \odot R_{n-1} \star R_n$$

for all $R_1, R_2 \in \mathcal{A}$ and $R_i = \frac{1}{2}I$ for all $i \in \{3, 4, \ldots, n\}$ where $n \geq 3$ is any fixed odd integer, then ϱ is an additive $*$-derivation.

Theorem 2.2. *Let $\mathcal{A}$ be a unital $*$-algebra with identity I, containing nontrivial projections P_j $(j = 1, 2)$ such that $U\mathcal{A}P_j = (0)$ implies that $U = 0$ for any $U \in \mathcal{A}$. If a map $\varrho : \mathcal{A} \to \mathcal{A}$ satisfies*

$$\varrho(R_1 \odot R_2 \star R_3 \odot \cdots \star R_{n-1} \odot R_n) = \sum_{k=1}^{n} R_1 \odot R_2 \star R_3 \odot \cdots \star \varrho(R_k) \odot \cdots \star R_{n-1} \odot R_n$$

for all $R_1, R_2 \in \mathcal{A}$ and $R_i = \frac{1}{2}I$ for all $i \in \{3, 4, \ldots, n\}$ where $n \geq 4$ is any fixed even integer, then ϱ is an additive $*$-derivation.

Theorem 2.3. *Let $\mathcal{A}$ be a unital $*$-algebra with identity I, containing nontrivial projections P_j $(j = 1, 2)$ such that $U\mathcal{A}P_j = (0)$ implies that $U = 0$ for any $U \in \mathcal{A}$. If a map $\varrho : \mathcal{A} \to \mathcal{A}$ satisfies*

$$\varrho(R_1 \star R_2 \odot \cdots \star R_{n-1} \odot R_n) = \sum_{k=1}^{n} R_1 \star R_2 \odot \cdots \star \varrho(R_k) \odot \cdots \star R_{n-1} \odot R_n$$

for all $R_1, R_2 \in \mathcal{A}$ and $R_i = \frac{1}{2}I$ for all $i \in \{3, 4, \ldots, n\}$ where $n \geq 3$ is any fixed odd integer, then ϱ is an additive $*$-derivation.

Theorem 2.4. *Let $\mathcal{A}$ be a unital $*$-algebra with identity I, containing nontrivial projections P_j $(j = 1, 2)$ such that $U\mathcal{A}P_j = (0)$ implies that $U = 0$ for any $U \in \mathcal{A}$. If a map $\varrho : \mathcal{A} \to \mathcal{A}$ satisfies*

$$\varrho(R_1 \star R_2 \odot R_3 \star \cdots \odot R_{n-1} \star R_n) = \sum_{k=1}^{n} R_1 \star R_2 \odot R_3 \star \cdots \odot \varrho(R_k) \star \cdots \odot R_{n-1} \star R_n$$

for all $R_1, R_2 \in \mathcal{A}$ and $R_i = \frac{1}{2}I$ for all $i \in \{3, 4, \ldots, n\}$ where $n \geq 4$ is any fixed even integer, then ϱ is an additive $*$-derivation.

3. Proof of our main theorems

Here we prove only Theorem 2.1 and the remaining theorems can be proved in the similar manner.

We begin by introducing some notations that are essential for proving our results. Throughout the article, we assume that $P_j \in \mathcal{A}$ $(j = 1, 2)$ are two nontrivial projections such that $P_2 = I - P_1$ and $\mathcal{A}_{ij} = P_i\mathcal{A}P_j$, where $i, j \in \{1, 2\}$. Moreover, by Peirce decomposition of $\mathcal{A}$, we can write it as $\mathcal{A} = \mathcal{A}_{11} + \mathcal{A}_{12} + \mathcal{A}_{21} + \mathcal{A}_{22}$. Additionally, let $\mathfrak{K} = \{U \in \mathcal{A} : U^* = U\}$ and $\mathfrak{L} = \{V \in \mathcal{A} : V^* = -V\}$. Denote $\mathfrak{K}_{ii} = P_i\mathfrak{K}P_i$ and $\mathfrak{K}_{12} = \{P_1UP_2 + P_2UP_1 \mid U \in \mathfrak{K}\}$. For every $U \in \mathfrak{K}$, we can write $U = U_{11} + U_{12} + U_{22}$ for every $U_{ii} \in \mathfrak{K}_{ii}$ $(i = 1, 2)$ and $U_{12} \in \mathfrak{K}_{12}$.

Using the given notation in [8], we can denote a monomial q of degree n by the following expression

$$q(X_1, X_2, \ldots, X_n) = X_1 \odot X_2 \star X_3 \odot \cdots \star X_n.$$

Also, if the last $n - 2$ variables in the monomial take the same value, then it can be written as

$$q(X_1, X_2, R, R, \ldots, R) = \eta_R(X_1, X_2)$$

for any $R \in \mathcal{A}$.

The following series of claims are essential to prove Theorem 2.1.

Claim 3.1. $\varrho(0) = 0$.

By the hypothesis, we have

$$\begin{aligned}\varrho(0) &= \varrho(\eta_R(0,0)) \\ &= \eta_R(\varrho(0),0) + \eta_R(0,\varrho(0)) + \sum_{\varrho(R)} \eta_{\varrho(R)}(0,0) \\ &= 0.\end{aligned}$$

Claim 3.2. *$\varrho(U)^* = \varrho(U)$ for any $U \in \mathfrak{K}$, i.e., ϱ preserves self adjoint elements of $\mathcal{A}$.*

For any $U \in \mathfrak{K}$, we have $U = \eta_{\frac{1}{2}I}(U, \frac{1}{2}I)$. Therefore

$$\begin{aligned}\varrho(U) &= \varrho(\eta_{\frac{1}{2}I}(U, \frac{1}{2}I)) \\ &= \eta_{\frac{1}{2}I}(\varrho(U), \frac{1}{2}I) + \eta_{\frac{1}{2}I}(U, \varrho(\frac{1}{2}I)) \\ &+ \sum_{\varrho(\frac{1}{2}I)} \eta_{\varrho(\frac{1}{2}I)}(U, \frac{1}{2}I) \\ &= \big(\frac{1}{2}\varrho(U)^* + \frac{1}{2}\varrho(U)\big) + \big(U\varrho(\frac{1}{2}I) + \varrho(\frac{1}{2}I)^*U\big) \\ &+ \big(U\varrho(\frac{1}{2}I)^* + \varrho(\frac{1}{2}I)U\big) + \cdots + \big(U\varrho(\frac{1}{2}I) + \varrho(\frac{1}{2}I)^*U\big) \\ &= \frac{1}{2}\big(\varrho(U)^* + \varrho(U)\big) + \frac{n-1}{2}\big(U\varrho(\frac{1}{2}I) + \varrho(\frac{1}{2}I)^*U\big) \\ &+ \frac{n-1}{2}\big(U\varrho(\frac{1}{2}I)^* + \varrho(\frac{1}{2}I)U\big).\end{aligned}$$

Thus

$$\begin{aligned}\varrho(U) = \varrho(U)^* &+ (n-1)\big(U\varrho(\frac{1}{2}I) + \varrho(\frac{1}{2}I)^*U\big) \\ &+ (n-1)\big(U\varrho(\frac{1}{2}I)^* + \varrho(\frac{1}{2}I)U\big). \qquad (1)\end{aligned}$$

Also

$$\begin{aligned}\varrho(U)^* = \varrho(U) &+ (n-1)\big(\varrho(\frac{1}{2}I)^*U + U\varrho(\frac{1}{2}I)\big) \\ &+ (n-1)\big(\varrho(\frac{1}{2}I)U + U\varrho(\frac{1}{2}I)^*\big). \qquad (2)\end{aligned}$$

Using equations (1) and (2), we get

$$\varrho(U)^* = \varrho(U)$$

for any $U \in \mathfrak{K}$.

Claim 3.3. *For any $X_{11} \in \mathfrak{K}_{11}, Y_{12} \in \mathfrak{K}_{12}$ and $Z_{22} \in \mathfrak{K}_{22}$, we have*

(a) $\varrho(X_{11} + Y_{12}) = \varrho(X_{11}) + \varrho(Y_{12})$;

(b) $\varrho(Y_{12} + Z_{22}) = \varrho(Y_{12}) + \varrho(Z_{22})$.

(a) Let $G = \varrho(X_{11} + Y_{12}) - \varrho(X_{11}) - \varrho(Y_{12})$. Then, we can easily see by Claim 3.2 that $G^* = G$. We have to show that $G = 0$, where $G = G_{11} + G_{12} + G_{22}$. Since $\eta_{\frac{1}{2}I}(P_2, X_{11}) = 0$, using Claim 3.1, we have

$$\begin{aligned}
&\eta_{\frac{1}{2}I}\big(\varrho(P_2), X_{11} + Y_{12}\big) + \eta_{\frac{1}{2}I}\big(P_2, \varrho(X_{11} + Y_{12})\big) \\
&+ \sum_{\varrho(\frac{1}{2}I)} \eta_{\varrho(\frac{1}{2}I)}\big(P_2, X_{11} + Y_{12}\big) \\
&= \varrho\big(\eta_{\frac{1}{2}I}(P_2, X_{11} + Y_{12})\big) \\
&= \varrho\big(\eta_{\frac{1}{2}I}(P_2, X_{11})\big) + \varrho\big(\eta_{\frac{1}{2}I}(P_2, Y_{12})\big) \\
&= \eta_{\frac{1}{2}I}\big(\varrho(P_2), X_{11}\big) + \eta_{\frac{1}{2}I}\big(P_2, \varrho(X_{11})\big) \\
&+ \sum_{\varrho(\frac{1}{2}I)} \eta_{\varrho(\frac{1}{2}I)}\big(P_2, X_{11}\big) + \eta_{\frac{1}{2}I}\big(\varrho(P_2), Y_{12}\big) \\
&+ \eta_{\frac{1}{2}I}\big(P_2, \varrho(Y_{12})\big) + \sum_{\varrho(\frac{1}{2}I)} \eta_{\varrho(\frac{1}{2}I)}\big(P_2, Y_{12}\big) \\
&= \eta_{\frac{1}{2}I}\big(\varrho(P_2), X_{11} + Y_{12}\big) + \eta_{\frac{1}{2}I}\big(P_2, \varrho(X_{11}) + \varrho(Y_{12})\big) \\
&= \sum_{\varrho(\frac{1}{2}I)} \eta_{\varrho(\frac{1}{2}I)}\big(P_2, X_{11} + Y_{12}\big).
\end{aligned}$$

This implies that, $\eta_{\frac{1}{2}I}\big(P_2, G\big) = 0$, using $G^* = G$, we get $G_{12} = 0$. Similarly, $\eta_{\frac{1}{2}I}\big(Y_{12}, (P_2 - P_1)\big) = 0$, so we have

$$\begin{aligned}
&\varrho\big(\eta_{\frac{1}{2}I}(X_{11} + Y_{12}, (P_2 - P_1))\big) \\
&= \varrho\big(\eta_{\frac{1}{2}I}(X_{11}, (P_2 - P_1))\big) + \varrho\big(\eta_{\frac{1}{2}I}(Y_{12}, (P_2 - P_1))\big) \\
&= \eta_{\frac{1}{2}I}(\varrho(X_{11}), (P_2 - P_1)) + \eta_{\frac{1}{2}I}(X_{11}, \varrho(P_2 - P_1)) \\
&+ \sum_{\varrho(\frac{1}{2}I)} \eta_{\varrho(\frac{1}{2}I)}\big(X_{11}, (P_2 - P_1)\big) + \eta_{\frac{1}{2}I}(\varrho(Y_{12}), (P_2 - P_1)) \\
&+ \eta_{\frac{1}{2}I}(Y_{12}, \varrho(P_2 - P_1)) + \sum_{\varrho(\frac{1}{2}I)} \eta_{\varrho(\frac{1}{2}I)}\big(Y_{12}, (P_2 - P_1)\big).
\end{aligned}$$

Furthermore

$$\begin{aligned}
\varrho\big(\eta_{\frac{1}{2}I}(X_{11} + Y_{12}, (P_2 - P_1))\big) &= \eta_{\frac{1}{2}I}(\varrho(X_{11} + Y_{12}), (P_2 - P_1)) \\
&+ \eta_{\frac{1}{2}I}(X_{11} + Y_{12}, \varrho(P_2 - P_1)) \\
&+ \sum_{\varrho(\frac{1}{2}I)} \eta_{\varrho(\frac{1}{2}I)}\big(X_{11} + Y_{12}, P_2 - P_1\big).
\end{aligned}$$

From the last two equations, we get $\eta_{\frac{1}{2}I}\big(G,(P_2-P_1)\big)=0$. Thus, by using Claim 3.2, we have $G(P_2-P_1)+(P_2-P_1)G=0$ which implies that $G_{11}=G_{22}=0$. Hence $G=0$. In the similar manner one can prove (b).

Claim 3.4. *For any $X_{11}\in\mathfrak{K}_{11}, Z_{22}\in\mathfrak{K}_{22}$ and $Y_{12}\in\mathfrak{K}_{12}$, we have*

$$\varrho(X_{11}+Y_{12}+Z_{22})=\varrho(X_{11})+\varrho(Y_{12})+\varrho(Z_{22}).$$

Let $G=\varrho(X_{11}+Y_{12}+Z_{22})-\varrho(X_{11})-\varrho(Y_{12})-\varrho(Z_{22})$. Then, by Claims 3.1, 3.3 and $\eta_{\frac{1}{2}I}(Z_{22},P_1)=0$, we have

$$\begin{aligned}
&\eta_{\frac{1}{2}I}\big(\varrho(X_{11}+Y_{12}+Z_{22}),P_1\big)+\eta_{\frac{1}{2}I}\big(X_{11}+Y_{12}+Z_{22},\varrho(P_1)\big)\\
&+\sum_{\varrho(\frac{1}{2}I)}\eta_{\varrho(\frac{1}{2}I)}\big(X_{11}+Y_{12}+Z_{22},P_1\big)\\
&=\varrho\big(\eta_{\frac{1}{2}I}((X_{11}+Y_{12}+Z_{22}),P_1)\big)\\
&=\varrho\big(\eta_{\frac{1}{2}I}(X_{11}+Y_{12},P_1)\big)+\varrho\big(\eta_{\frac{1}{2}I}(Z_{22},P_1)\big)\\
&=\eta_{\frac{1}{2}I}\big(\varrho(X_{11}+Y_{12}),P_1\big)+\eta_{\frac{1}{2}I}\big(X_{11}+Y_{12},\varrho(P_1)\big)\\
&+\sum_{\varrho(\frac{1}{2}I)}\eta_{\varrho(\frac{1}{2}I)}\big(X_{11}+Y_{12},P_1\big)+\eta_{\frac{1}{2}I}\big(\varrho(Z_{22}),P_1\big)\\
&+\eta_{\frac{1}{2}I}\big(Z_{22},\varrho(P_1)\big)+\sum_{\varrho(\frac{1}{2}I)}\eta_{\varrho(\frac{1}{2}I)}\big(Z_{22},P_1\big)\\
&=\eta_{\frac{1}{2}I}\big(\varrho(X_{11})+\varrho(Y_{12}),P_1\big)+\eta_{\frac{1}{2}I}\big(X_{11}+Y_{12},\varrho(P_1)\big)\\
&+\sum_{\varrho(\frac{1}{2}I)}\eta_{\varrho(\frac{1}{2}I)}\big(X_{11}+Y_{12},P_1\big)+\eta_{\frac{1}{2}I}\big(\varrho(Z_{22}),P_1\big)\\
&+\eta_{\frac{1}{2}I}\big(Z_{22},\varrho(P_1)\big)+\sum_{\varrho(\frac{1}{2}I)}\eta_{\varrho(\frac{1}{2}I)}\big(Z_{22},P_1\big)\\
&=\eta_{\frac{1}{2}I}\big(\varrho(X_{11})+\varrho(Y_{12})+\varrho(Z_{22}),P_1\big)+\eta_{\frac{1}{2}I}\big(X_{11}+Y_{12}+Z_{22},\varrho(P_1)\big)\\
&+\sum_{\varrho(\frac{1}{2}I)}\eta_{\varrho(\frac{1}{2}I)}\big(X_{11}+Y_{12}+Z_{22},P_1\big).
\end{aligned}$$

This implies that $\eta_{\frac{1}{2}I}\big(G,P_1\big)=0$ and by using Claim 3.2, we get $G_{11}=G_{12}=0$. Similarly, if we replace P_1 by P_2 in the above calculation, we can easily get $G_{22}=0$. Hence $G=0$, which is our aim.

Claim 3.5. *For any $X_{12},Y_{12}\in\mathfrak{K}_{12}$, we have*

$$\varrho(X_{12}+Y_{12})=\varrho(X_{12})+\varrho(Y_{12}).$$

For any $A_{12}, B_{12} \in \mathcal{A}_{12}$, let $X_{12} = A_{12} + A_{12}^* \in \mathfrak{K}_{12}$ and $Y_{12} = B_{12} + B_{12}^* \in \mathfrak{K}_{12}$. Then

$$\begin{aligned}
&\eta_{\frac{1}{2}I}\big((P_2 + B_{12} + B_{12}^*), (P_1 + A_{12} + A_{12}^*)\big)\\
&= (A_{12} + A_{12}^*) + (B_{12} + B_{12}^*) + (A_{12}B_{12}^* + B_{12}A_{12}^* + A_{12}^*B_{12} + B_{12}^*A_{12})\\
&= X_{12} + Y_{12} + X_{12}Y_{12}^* + Y_{12}X_{12}^*,
\end{aligned}$$

where

$$\begin{aligned}
X_{12}Y_{12}^* + Y_{12}X_{12}^* &= A_{12}B_{12}^* + B_{12}A_{12}^* + A_{12}^*B_{12} + B_{12}^*A_{12}\\
&= U_{11} + V_{22}.
\end{aligned}$$

Note that $U_{11} = (A_{12}B_{12}^* + B_{12}A_{12}^*) \in \mathfrak{K}_{11}$ and $V_{22} = (A_{12}^*B_{12} + B_{12}^*A_{12}) \in \mathfrak{K}_{22}$. Since $A_{12} + A_{12}^*, B_{12} + B_{12}^* \in \mathfrak{K}_{12}$, using Claims 3.3 and 3.4, we have

$$\begin{aligned}
&\varrho(X_{12} + Y_{12}) + \varrho(U_{11}) + \varrho(V_{22})\\
&= \varrho(X_{12} + Y_{12} + U_{11} + V_{22})\\
&= \varrho(X_{12} + Y_{12} + X_{12}Y_{12}^* + Y_{12}X_{12}^*)\\
&= \varrho\big(\eta_{\frac{1}{2}I}(P_2 + B_{12} + B_{12}^*), (P_1 + A_{12} + A_{12}^*)\big)\\
&= \eta_{\frac{1}{2}I}\big(\varrho(P_2) + \varrho(B_{12} + B_{12}^*), (P_1 + A_{12} + A_{12}^*)\big)\\
&+ \eta_{\frac{1}{2}I}\big(P_2 + B_{12} + B_{12}^*, \varrho(P_1) + \varrho(A_{12} + A_{12}^*)\big)\\
&+ \sum_{\varrho(\frac{1}{2}I)} \eta_{\varrho(\frac{1}{2}I)}\big(P_2 + B_{12} + B_{12}^*, P_1 + A_{12} + A_{12}^*\big)\\
&= \eta_{\frac{1}{2}I}(\varrho(P_2), P_1) + \eta_{\frac{1}{2}I}(P_2, \varrho(P_1))\\
&+ \sum_{\varrho(\frac{1}{2}I)} \eta_{\varrho(\frac{1}{2}I)}(P_2, P_1) + \eta_{\frac{1}{2}I}\big(\varrho(P_2), (A_{12} + A_{12}^*)\big)\\
&+ \eta_{\frac{1}{2}I}\big(P_2, \varrho(A_{12} + A_{12}^*)\big) + \sum_{\varrho(\frac{1}{2}I)} \eta_{\varrho(\frac{1}{2}I)}(P_2, A_{12} + A_{12}^*)\\
&+ \eta_{\frac{1}{2}I}\big(\varrho(B_{12} + B_{12}^*, P_1)\big) + \eta_{\frac{1}{2}I}\big(B_{12} + B_{12}^*, \varrho(P_1)\big)\\
&+ \sum_{\varrho(\frac{1}{2}I)} \eta_{\varrho(\frac{1}{2}I)}\big(B_{12} + B_{12}^*, P_1\big) + \eta_{\frac{1}{2}I}\big(\varrho(B_{12} + B_{12}^*, A_{12} + A_{12}^*)\big)\\
&+ \eta_{\frac{1}{2}I}\big(B_{12} + B_{12}^*, \varrho(A_{12} + A_{12}^*)\big) + \sum_{\varrho(\frac{1}{2}I)} \eta_{\varrho(\frac{1}{2}I)}\big(B_{12} + B_{12}^*, A_{12} + A_{12}^*\big)\\
&= \varrho\big(\eta_{\frac{1}{2}I}(P_2, P_1)\big) + \varrho\big(\eta_{\frac{1}{2}I}(P_2, A_{12} + A_{12}^*)\big)\\
&+ \varrho\big(\eta_{\frac{1}{2}I}(B_{12} + B_{12}^*, P_1)\big) + \varrho\big(\eta_{\frac{1}{2}I}(B_{12} + B_{12}^*, A_{12} + A_{12}^*)\big)
\end{aligned}$$

$$= \varrho(X_{12}) + \varrho(Y_{12}) + \varrho(A_{12}B_{12}^* + B_{12}A_{12}^* + A_{12}^*B_{12} + B_{12}^*A_{12})$$
$$= \varrho(X_{12}) + \varrho(Y_{12}) + \varrho(U_{11}) + \varrho(V_{22}).$$

This gives

$$\varrho(X_{12} + Y_{12}) = \varrho(X_{12}) + \varrho(Y_{12}).$$

Claim 3.6. *For any* $X_{ii}, Y_{ii} \in \mathfrak{K}_{ii}$ $(i = 1, 2)$*, we have*

$$\varrho(X_{ii} + Y_{ii}) = \varrho(X_{ii}) + \varrho(Y_{ii}).$$

We assume that $G = \varrho(X_{11} + Y_{11}) - \varrho(X_{11}) - \varrho(Y_{11})$ and show that $G = 0$. For it, using Claim 3.1, and $\eta_{\frac{1}{2}I}(X_{11}, P_2) = \eta_{\frac{1}{2}I}(Y_{11}, P_2) = 0$, we have

$$\begin{aligned}
&\eta_{\frac{1}{2}I}\big(\varrho(X_{11} + Y_{11}), P_2\big) + \eta_{\frac{1}{2}I}\big(X_{11} + Y_{11}, \varrho(P_2)\big) \\
&+ \sum_{\varrho(\frac{1}{2}I)} \eta_{\varrho(\frac{1}{2}I)}\big(X_{11} + Y_{11}, P_2\big) \\
&= \varrho\big(\eta_{\frac{1}{2}I}(X_{11} + Y_{11}, P_2)\big) \\
&= \varrho\big(\eta_{\frac{1}{2}I}(X_{11}, P_2)\big) + \varrho\big(\eta_{\frac{1}{2}I}(Y_{11}, P_2)\big) \\
&= \eta_{\frac{1}{2}I}\big(\varrho(X_{11}), P_2\big) + \eta_{\frac{1}{2}I}\big(X_{11}, \varrho(P_2)\big) \\
&+ \sum_{\varrho(\frac{1}{2}I)} \eta_{\varrho(\frac{1}{2}I)}\big(X_{11}, P_2\big) + \eta_{\frac{1}{2}I}\big(\varrho(Y_{11}), P_2\big) \\
&+ \eta_{\frac{1}{2}I}\big(Y_{11}, \varrho(P_2)\big) + \sum_{\varrho(\frac{1}{2}I)} \eta_{\varrho(\frac{1}{2}I)}\big(Y_{11}, P_2\big) \\
&= \eta_{\frac{1}{2}I}\big(\varrho(X_{11}) + \varrho(Y_{11}), P_2\big) + \eta_{\frac{1}{2}I}\big(X_{11} + Y_{11}, \varrho(P_2)\big) \\
&+ \sum_{\varrho(\frac{1}{2}I)} \eta_{\varrho(\frac{1}{2}I)}\big(X_{11} + Y_{11}, P_2\big).
\end{aligned}$$

Therefore, $\eta_{\frac{1}{2}I}\big(G, P_2\big) = 0$. By using Claim 3.2, we get $G_{12} = G_{22} = 0$. Now it only remains to show that $G_{11} = 0$. For it, let $Z = U_{12} + U_{12}^* \in \mathfrak{K}_{12}$ for some $U_{12} \in \mathcal{A}_{12}$. Thus $\eta_{\frac{1}{2}I}\big(X_{11}, Z\big), \eta_{\frac{1}{2}I}\big(Y_{11}, Z\big) \in \mathfrak{K}_{12}$ also using Claim 3.5, we have

$$\begin{aligned}
&\varrho\big(\eta_{\frac{1}{2}I}(X_{11} + Y_{11}, Z)\big) \\
&= \varrho\big(\eta_{\frac{1}{2}I}(X_{11}, Z)\big) + \varrho\big(\eta_{\frac{1}{2}I}(Y_{11}, Z)\big) \\
&= \eta_{\frac{1}{2}I}\big(\varrho(X_{11}) + \varrho(Y_{11}), Z\big) + \varrho\big(\eta_{\frac{1}{2}I}(X_{11} + Y_{11}, \varrho(Z))\big) \\
&+ \sum_{\varrho(\frac{1}{2}I)} \eta_{\varrho(\frac{1}{2}I)}\big(X_{11} + Y_{11}, Z\big).
\end{aligned}$$

Alternatively,

$$\begin{aligned}&\varrho\big(\eta_{\frac{1}{2}I}(X_{11}+Y_{11},Z)\big)\\&=\eta_{\frac{1}{2}I}\big(\varrho(X_{11}+Y_{11}),Z\big)+\eta_{\frac{1}{2}I}\big(X_{11}+Y_{11},\varrho(Z)\big)\\&+\sum_{\varrho(\frac{1}{2}I)}\eta_{\varrho(\frac{1}{2}I)}\big(X_{11}+Y_{11},Z\big).\end{aligned}$$

Thus, we have $\eta_{\frac{1}{2}I}\big(G,Z\big)=0$ which implies that $G_{11}(U_{12}+U_{12}^*)+(U_{12}+U_{12}^*)G_{11}=0$. Multiplying P_1 from the left and P_2 from the right, we get $P_1GP_1\mathcal{A}P_2=0$ and using the hypothesis, we get $G_{11}=0$.

Remark 3.7. From Claims 3.3–3.6, we can say that ϱ is additive on $\mathfrak{K}$.

Claim 3.8. $\varrho(I)=0$.

Since $\eta_{\frac{1}{2}I}\big(I,P_1\big)=2P_1$, and using Claim 3.2, Remark 3.7, we get

$$\begin{aligned}2\varrho(P_1)&=\varrho\big(\eta_{\frac{1}{2}I}(I,P_1)\big)\\&=\eta_{\frac{1}{2}I}\big(\varrho(I),P_1\big)+\eta_{\frac{1}{2}I}\big(I,\varrho(P_1)\big)\\&+\sum_{\varrho(\frac{1}{2}I)}\eta_{\varrho(\frac{1}{2}I)}(I,P_1)\\&=2\varrho(P_1)+(n-1)\big(\varrho(I)P_1+P_1\varrho(I)\big).\end{aligned}$$

Thus, $\varrho(I)P_1+P_1\varrho(I)=0$. From this, we get $P_1\varrho(I)P_1=P_1\varrho(I)P_2=P_2\varrho(I)P_1=0$. Similarly, replacing P_1 by P_2 in the above calculation, we can easily see that $P_2\varrho(I)P_2=0$. Hence $\varrho(I)=0$.

Claim 3.9. $\varrho(V)^*=-\varrho(V)$ *for every* $V\in\mathfrak{L}$.

Since $\eta_{\frac{1}{2}I}\big(I,V\big)=0$, from Claims 3.1, 3.8 and Remark 3.7, we have

$$\begin{aligned}0&=\varrho\big(\eta_{\frac{1}{2}I}(I,V)\big)\\&=\eta_{\frac{1}{2}I}\big(I,\varrho(V)\big)\\&=\varrho(V)+\varrho(V)^*.\end{aligned}$$

Hence $\varrho(V)^*=-\varrho(V)$ for every $V\in\mathfrak{L}$.

Claim 3.10. $\varrho(iI)$ *is a central element of* $\mathcal{A}$.

For any $U \in \mathfrak{K}$, using Claim 3.9 and Remark 3.7, we get

$$\begin{aligned}0 &= \varrho\big(\eta_{\frac{1}{2}I}(iI, U)\big)\\ &= \eta_{\frac{1}{2}I}\big(\varrho(iI), U\big) + \eta_{\frac{1}{2}I}\big(iI, \varrho(U)\big)\\ &= \varrho(iI)^*U + U\varrho(iI)\\ &= -\varrho(iI)U + U\varrho(iI).\end{aligned}$$

Thus $\varrho(iI)U = U\varrho(iI)$ for any $U \in \mathfrak{K}$. Since any $W \in \mathcal{A}$ can be written as $W = W_1 + iW_2$ for some $W_1, W_2 \in \mathfrak{K}$, we have $\varrho(iI)W = W\varrho(iI)$ for all $W \in \mathcal{A}$, i.e., $\varrho(iI)$ is a central element of $\mathcal{A}$.

Claim 3.11. *For any $V \in \mathfrak{L}$, $\varrho(iV) = i\varrho(V) + \varrho(iI)V$.*

Since, $\eta_{\frac{1}{2}I}(iI, V) = -2iV$, we have

$$\begin{aligned}-2\varrho(iV) = \varrho(-2iV) &= \varrho\big(\eta_{\frac{1}{2}I}(iI, V)\big)\\ &= \eta_{\frac{1}{2}I}\big(\varrho(iI), V\big) + \eta_{\frac{1}{2}I}\big(iI, \varrho(V)\big)\\ &= -2V\varrho(iI) - 2i\varrho(V).\end{aligned}$$

Thus $\varrho(iV) = i\varrho(V) + \varrho(iI)V$.

Claim 3.12. *ϱ is additive on $\mathfrak{L}$.*

Let $V_1, V_2 \in \mathfrak{L}$. Then from Claim 3.11 and Remark 3.7, we obtain

$$\begin{aligned}&\varrho(iI)(V_1 + V_2) + i\varrho(V_1 + V_2)\\ &= \varrho\big(i(V_1 + V_2)\big)\\ &= \varrho(iV_1 + iV_2)\\ &= \varrho(iV_1) + \varrho(iV_2)\\ &= \varrho(iI)V_1 + i\varrho(V_1) + \varrho(iI)V_2 + i\varrho(V_2).\end{aligned}$$

Hence, we get

$$\varrho(V_1 + V_2) = \varrho(V_1) + \varrho(V_2).$$

Claim 3.13. *ϱ is additive on $\mathcal{A}$.*

Let $V, V' \in \mathfrak{L}$. Using Remark 3.7, Claims 3.8, 3.10 and 3.11, we have

$$\begin{aligned}
2\big(V'\varrho(iI) + i\varrho(V')\big) &= 2\varrho(iV') \\
&= \varrho(2iV') \\
&= \varrho\big(\eta_{\frac{1}{2}I}(I, V + iV')\big) \\
&= \eta_{\frac{1}{2}I}\big(I, \varrho(V + iV')\big) \\
&= \varrho(V + iV') + \varrho(V + iV')^*. \qquad (3)
\end{aligned}$$

Furthermore, we have

$$\begin{aligned}
-2i\varrho(V) - 2\varrho(iI)V &= -2\varrho(iV) \\
&= \varrho(-2iV) \\
&= \varrho\big(\eta_{\frac{1}{2}I}(iI, V + iV')\big) \\
&= \eta_{\frac{1}{2}I}\big(\varrho(iI), V + iV'\big) + \eta_{\frac{1}{2}I}\big(iI, \varrho(V + iV')\big) \\
&= -2V\varrho(iI) - i\varrho(V + iV') + i\varrho(V + iV')^*.
\end{aligned}$$

Thus

$$2\varrho(V) = \varrho(V + iV') - \varrho(V + iV')^*. \qquad (4)$$

From (3) and (4), we have

$$\varrho(iI)V' + i\varrho(V') + \varrho(V) = \varrho(V + iV') \qquad (5)$$

for all $V, V' \in \mathfrak{L}$. Now assume that $R, S \in \mathcal{A}$ such that $R = R_1 + iR_2$ and $S = S_1 + iS_2$ for all $R_1, R_2, S_1, S_2 \in \mathfrak{L}$. Using (5) and Claim 3.12, we get

$$\begin{aligned}
\varrho(R + S) &= \varrho\big((R_1 + iR_2) + (S_1 + iS_2)\big) \\
&= \varrho\big((R_1 + S_1) + i(R_2 + S_2)\big) \\
&= \varrho(R_1 + S_1) + i\varrho(R_2 + S_2) + \varrho(iI)(R_2 + S_2) \\
&= \big(\varrho(R_1) + i\varrho(R_2) + \varrho(iI)R_2\big) \\
&\quad + \big(\varrho(S_1) + i\varrho(S_2) + \varrho(iI)S_2\big) \\
&= \varrho(R_1 + iR_2) + \varrho(S_1 + iS_2) \\
&= \varrho(R) + \varrho(S).
\end{aligned}$$

Hence ϱ is additive on $\mathcal{A}$.

Claim 3.14. *$\varrho(T^*) = \varrho(T)^*$ for all $T \in \mathcal{A}$.*

Any element $T \in \mathcal{A}$ can be written as $T = T_1 + iT_2$ for $T_1, T_2 \in \mathfrak{L}$. Now using (5) and Claim 3.9, we obtain

$$\begin{aligned}\varrho(T)^* &= \varrho(T_1 + iT_2)^* \\ &= \big(\varrho(T_1) + i\varrho(T_2) + \varrho(iI)T_2\big)^* \\ &= \varrho(-T_1 + iT_2) \\ &= \varrho(T^*).\end{aligned}$$

Claim 3.15. *ϱ is a derivation on $\mathfrak{L}$.*

For any $L_1, L_2 \in \mathfrak{L}$, it follows from Claims 3.9, 3.11 and 3.13 that

$$\begin{aligned}-\varrho\big(L_1L_2 + L_2L_1\big) &= \varrho\big(\eta_{\frac{1}{2}I}(L_1, L_2)\big) \\ &= \eta_{\frac{1}{2}I}\big(\varrho(L_1), L_2\big) + \eta_{\frac{1}{2}I}\big(L_1, \varrho(L_2)\big) \\ &= -\big(\varrho(L_1)L_2 + L_1\varrho(L_2) + \varrho(L_2)L_1 + L_2\varrho(L_1)\big).\end{aligned}$$

Thus,

$$\varrho(L_1L_2) + \varrho(L_2L_1) = \varrho(L_1)L_2 + L_1\varrho(L_2) + \varrho(L_2)L_1 + L_2\varrho(L_1). \tag{6}$$

Furthermore,

$$\begin{aligned}\varrho\big(\eta_{\frac{1}{2}I}(iL_1, L_2)\big) &= \eta_{\frac{1}{2}I}\big(\varrho(iL_1), L_2\big) + \eta_{\frac{1}{2}I}\big(iL_1, \varrho(L_2)\big) \\ &= \varrho(iL_1)^*L_2 + L_2^*\varrho(iL_1) + (iL_1)^*\varrho(L_2) + \varrho(L_2)^*(iL_1) \\ &= i\varrho(L_1)L_2 + \varrho(iI)L_1L_2 - iL_2\varrho(L_1) \\ &\quad - \varrho(iI)L_2L_1 + iL_1\varrho(L_2) - i\varrho(L_2)L_1.\end{aligned} \tag{7}$$

On the other side, we get

$$\begin{aligned}\varrho\big(\eta_{\frac{1}{2}I}(iL_1, L_2)\big) &= \varrho\big(i(L_1L_2 - L_2L_1)\big) \\ &= i\varrho(L_1L_2 - L_2L_1) + \varrho(iI)(L_1L_2 - L_2L_1).\end{aligned} \tag{8}$$

Using (7) and (8), we have

$$\varrho(L_1L_2) - \varrho(L_2L_1) = \varrho(L_1)L_2 + L_1\varrho(L_2) - \varrho(L_2)L_1 - L_2\varrho(L_1). \tag{9}$$

Hence from (6) and (9), we get our desire result, i.e.,

$$\varrho(L_1L_2) = \varrho(L_1)L_2 + L_1\varrho(L_2).$$

Claim 3.16. $\varrho(iI) = 0$.

Since $\eta_{\frac{1}{2}I}\big(iI, \frac{1}{2}iI\big) = I$, using Claims 3.1, 3.8, 3.9 and Remark 3.7, we get

$$\begin{aligned} 0 = \varrho(I) &= \varrho\big(\eta_{\frac{1}{2}I}(iI, \frac{1}{2}iI)\big) \\ &= \eta_{\frac{1}{2}I}\big(\varrho(iI), \frac{1}{2}iI\big) + \eta_{\frac{1}{2}I}\big(iI, \varrho(\frac{1}{2}iI)\big) \\ &= -2i\varrho(iI). \end{aligned}$$

Thus $\varrho(iI) = 0$.

Claim 3.17. *$\varrho(iD) = i\varrho(D)$ for all $D \in \mathcal{A}$.*

Since $\eta_{\frac{1}{2}I}(iD, iI) = D^* + D$, using Claims 3.2, 3.8, 3.14, 3.16 and Remark 3.7, we have

$$\begin{aligned} \varrho(D)^* + \varrho(D) &= \varrho(D^*) + \varrho(D) \\ &= \varrho\big(\eta_{\frac{1}{2}I}(iD, iI)\big) \\ &= \eta_{\frac{1}{2}I}\big(\varrho(iD), iI\big) \\ &= i\varrho(iD)^* - i\varrho(iD). \end{aligned}$$

Thus we get

$$-\varrho(iD)^* + \varrho(iD) = i\varrho(D)^* + i\varrho(D). \tag{10}$$

Furthermore

$$\begin{aligned} \varrho\big(\eta_{\frac{1}{2}I}(iD, I)\big) &= \varrho\big(\eta_{\frac{1}{2}I}(D, -iI)\big) \\ \eta_{\frac{1}{2}I}\big(\varrho(iD), I\big) &= \eta_{\frac{1}{2}I}\big(\varrho(D), -iI\big) \\ \varrho(iD)^* + \varrho(iD) &= -i\varrho(D)^* + i\varrho(D). \end{aligned} \tag{11}$$

From (10) and (11), we obtain $\varrho(iD) = i\varrho(D)$.

Claim 3.18. *For any $C, D \in \mathcal{A}$, $\varrho(CD) = \varrho(C)D + C\varrho(D)$.*

From Claim 3.14, for any $C, D \in \mathcal{A}$, we have

$$\begin{aligned} \varrho(CD + D^*C^*) &= \varrho\big(\eta_{\frac{1}{2}I}(C^*, D)\big) \\ &= \eta_{\frac{1}{2}I}\big(\varrho(C^*), D\big) + \eta_{\frac{1}{2}I}\big(C^*, \varrho(D)\big) \\ &= \varrho(C^*)^*D + D^*\varrho(C^*) + C\varrho(D) + \varrho(D)^*C^* \\ &= \varrho(C)D + C\varrho(D) + \varrho(D^*)C^* + D^*\varrho(C^*). \end{aligned} \tag{12}$$

Using (12), we get

$$\begin{aligned} \varrho\big(i(CD - D^*C^*)\big) &= \varrho\big(C(iD) + (iD)^*C^*\big) \\ &= \varrho(C)iD + C\varrho(iD) + \varrho\big((iD)^*\big)C^* + (iD)^*\varrho(C^*). \end{aligned}$$

From Claim 3.17, we have

$$\varrho(CD - D^*C^*) = \varrho(C)D + C\varrho(D) - \varrho(D)^*C^* - D^*\varrho(C^*). \tag{13}$$

Thus (12) and (13) yield that

$$\varrho(CD) = \varrho(C)D + C\varrho(D).$$

Hence by Claims 3.13 and 3.14, ϱ is an additive $*$-derivation.

4. Corollaries

The following are the immediate consequences of Theorem 2.1. Since a prime $*$-algebra satisfies the hypothesis of Theorem 2.1, so we have

Corollary 4.1. *Let $\mathcal{A}$ be a unital prime $*$-algebra containing nontrivial projections P_j (j=1,2). If a map $\varrho : \mathcal{A} \to \mathcal{A}$ satisfies*

$$\varrho(R_1 \star R_2 \odot R_3 \star \cdots \odot R_n) = \sum_{k=1}^{n} R_1 \star R_2 \odot R_3 \star \cdots \odot \varrho(R_k) \star \cdots \odot R_n$$

for all $R_1, R_2 \in \mathcal{A}$ and $R_i = \frac{1}{2}I$ for all $i \in \{3, 4, \ldots n\}$ where $n \geq 3$, then ϱ is an additive $*$-derivation.

A von Neumann algebra $\mathcal{A}$ is a weakly closed, self-adjoint algebra of operators on a Hilbert space H. Moreover, a factor von Neumann algebra is a von Neumann algebra whose center is trivial, i.e., it contains only scalar operators. Also, a factor von Neumann algebra is a prime $*$-algebra, so we have the following corollary:

Corollary 4.2. *Let $\mathcal{A}$ be a factor von Neumann algebra with dim($\mathcal{A}$) ≥ 2. If a map $\varrho : \mathcal{A} \to \mathcal{A}$ satisfies*

$$\varrho(R_1 \star R_2 \odot R_3 \star \cdots \odot R_n) = \sum_{k=1}^{n} R_1 \star R_2 \odot R_3 \star \cdots \odot \varrho(R_k) \star \cdots \odot R_n$$

for all $R_1, R_2 \in \mathcal{A}$ and $R_i = \frac{1}{2}I$ for all $i \in \{3, 4, \ldots n\}$ where $n \geq 3$, then ϱ is an additive $*$-derivation.

Suppose that $F(H)$ ($\subseteq B(H)$) denotes the subalgebra of all bounded finite rank operators. If a subalgebra $\mathcal{A}$ of $B(H)$ contains $F(H)$, we call it a standard operator algebra. As we know, a standard operator algebra is a prime $*$-algebra, we have the following:

Corollary 4.3. *Suppose that $\mathcal{A}$ is a standard operator algebra on an infinite dimensional complex Hilbert space H with identity operator I and is closed under the adjoint operation. If a map $\varrho : \mathcal{A} \to \mathcal{A}$ satisfies*

$$\varrho(R_1 \star R_2 \odot R_3 \star \cdots \odot R_n) = \sum_{k=1}^{n} R_1 \star R_2 \odot R_3 \star \cdots \odot \varrho(R_k) \star \cdots \odot R_n$$

for all $R_1, R_2 \in \mathcal{A}$ and $R_i = \frac{1}{2}I$ for all $i \in \{3, 4, \ldots n\}$ where $n \geq 3$, then ϱ is an additive $*$-derivation.

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Asma Ali, **Tooba Naz** (Corresponding Author) and **Mohd Tasleem**
Department of Mathematics
Faculty of Science
Aligarh Muslim University
Aligarh, India
e-mails: asma_ali2@rediffmail.com (A. Ali)
naztooba06@gmail.com (T. Naz)
tasleemh59@gmail.com (M. Tasleem)

Claus Haetinger
Institute of Mathematics, Statistics and Physics-IMEF
Federal University of Rio Grande-FURG
Rio Grande 96201-900, Brazil
e-mail: claus.haetinger@gmail.com