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HOMOLOGICAL INVARIANTS OF GENERALIZED BOUND PATH ALGEBRAS

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Abstract. We study homological invariants of a given generalized bound path algebra in terms of those of the algebras used in its construction. We discuss the particular case where the algebra is a generalized path algebra and give conditions for those algebras to be shod or quasitilted.

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1. Introduction

An important result in the representation theory of algebras states that every finite dimensional basic algebra A over an algebraically closed field k is isomorphic to a quotient of a path algebra kQ_A/I_A, where Q_A is a finite quiver and I_A is an admissible ideal (see below for details). This allows us to describe the finitely generated A-modules in terms of the representations of its corresponding quiver Q_A, a connection which proves to be essential in this theory.

In order to generalize this construction, Coelho and Liu introduced in [8] the notion of *generalized path algebras* (or *gp-algebras* for short). Instead of assigning the base field k to each vertex of a quiver Q as in the classical construction of the path algebra kQ, a finite dimensional k-algebra is assigned. This was further generalized by us in the article [5], where we considered certain quotients of the gp-algebras. Specifically, let Γ denote a quiver and $\mathcal{A} = \{A_i : i \in \Gamma_0\}$ denote a family of basic finite dimensional k-algebras indexed by the set Γ_0 of the vertices of Γ. Consider also a set of relations I on the paths of Γ which generates an admissible ideal of $k\Gamma$. To such data we have considered ([5]) the *generalized bound path algebra* $\Lambda = k(\Gamma, \mathcal{A}, I)$ (*gbp-algebra* for short) with a natural multiplication given not only by the concatenation of paths of the quiver but also by the multiplication of the algebras associated with the vertices of Γ, modulo the relations in I (see below for details).

Our idea behind such a construction is to obtain properties of a gbp-algebra Λ from those of the algebras in $\mathcal{A}$. In the seminal work [8], the focus was more ring theoretical, and, as mentioned, the authors only considered the case where $I = 0$. We mention, for instance, [9,12,13,14,15] for further works which are connected with this construction.

In [5,6], we have studied the case where I is not necessarily zero, thus extending the description of the representations of the algebra Λ given in [12]. Clearly, a path algebra A can be realized as a generalized one in two trivial ways: by its usual description as a quotient of the path algebra over the ordinary quiver Q_A, and also by considering a quiver with a single vertex and no arrows and assigning to it the whole algebra A. In [5], we discuss when there are other possibilities, apart from the two above, of realizing a path algebra as a generalized one. This is important because then we can relate properties of a gbp-algebra with those of the *smaller algebras* used in its definition. In [6], we studied the correspondence between modules over a gbp-algebra and representations of the corresponding quiver.

Also in [6], we gave a description of the projective and injective modules over a gbp-algebra. Using this, here we introduce a special case of gbp-algebras, which we call *terraced gbp-algebras*, and we show that we can study homological invariants of these algebras in terms of those of the "smaller" algebras used in their construction.

This is done in Sections 3 and 4 after devoting Section 2 to preliminary concepts needed along the paper. The particular case of gp-algebras is discussed in Section 4 where we prove, for instance, that the global dimension of a gp-algebra is the maximum between one and the global dimension of the algebras assigned to each vertex (Theorem 4.1).

Also, we provide a sufficient condition for a gp-algebra to belong to classes of algebras which can be defined using some homological invariants, such as *shod* or *quasitilted* algebras (see [7,10]). These classes of algebras were introduced with the idea of generalizing the class of *tilted algebras* through their homological properties. Also as a motivation, we mention that a very strong connection between shod/quasitilted algebras and the so-called silted algebras was recently found by Buan and Zhou [4], leading to a new line of investigation.

2. Preliminaries

Along this paper, k will denote an algebraically closed field. (We make this assumption in order to use results derived from [6,12]). For an algebra, we mean an associative and unitary basic finite dimensional k-algebra. Also, given an algebra

A, an A-module (or just a module) will be a finitely generated right module over A. We refer to [1,2,3] for unexplained details on modules and representation theory.

2.1. Path algebras. A *quiver* Q is given by a tuple (Q_0, Q_1, s, e), where Q_0 is the set of *vertices*, Q_1 is the set of *arrows* and $s, e\colon Q_1 \longrightarrow Q_0$ are maps which indicate, for each arrow $\alpha \in Q_1$, the *starting vertex* $s(\alpha) \in Q_0$ of α and the *ending vertex* $e(\alpha) \in Q_0$ of α. A vertex $i \in Q_0$ is called a *source* (respectively, a *sink*) provided there are no arrows ending (or starting, respectively) at i. A *path in Q of length* $n \geq 1$ is given by $\alpha_1 \cdots \alpha_n$, where for each $i = 1, \cdots, n-1$, $e(\alpha_i) = s(\alpha_{i+1})$. There are also *paths of length zero* which are in a one-to-one correspondence to the vertices of Q.

We shall assume that all quivers are finite, that is, both sets Q_0 and Q_1 are finite.

Given a quiver Q, one can assign a *path algebra* kQ with a k-basis given by all paths over Q and multiplication on that basis defined by concatenation. Even when Q is finite, the corresponding algebra does not need to be finite dimensional. However, a well-known result by P. Gabriel states that given an algebra A, there exists a finite quiver Q and a set of relations on the paths of Q which generates an admissible ideal I of kQ such that $A \cong kQ/I$ (see [1] for details).

2.2. Generalized bound path algebras (gbp-algebras). Let $\Gamma = (\Gamma_0, \Gamma_1, s, e)$ be a quiver and $\mathcal{A} = (A_i)_{i \in \Gamma_0}$ be a family of algebras indexed by Γ_0. An *$\mathcal{A}$-path of length n* over Γ is defined as follows: for $n = 0$, such a path is an element of $\bigcup_{i \in \Gamma_0} A_i$, and for $n > 0$, it is a sequence of the form

$$a_1 \beta_1 a_2 \ldots a_n \beta_n a_{n+1}$$

where $\beta_1 \ldots \beta_n$ is an ordinary path in the quiver Γ, $a_i \in A_{s(\beta_i)}$ if $i \leq n$, and $a_{n+1} \in A_{e(\beta_n)}$. Denote by $k[\Gamma, \mathcal{A}]$ the k-vector space spanned by all $\mathcal{A}$-paths over Γ.

Then we consider the quotient vector space $k(\Gamma, \mathcal{A}) = k[\Gamma, \mathcal{A}]/V$, where V is the subspace generated by all elements of the form

$$(a_1 \beta_1 \ldots \beta_{j-1} (a_{j,1} + \ldots + a_{j,m}) \beta_j a_{j+1} \ldots \beta_n a_{n+1}) - \sum_{l=1}^{m} (a_1 \beta_1 \ldots \beta_{j-1} a_{j,l} \beta_j \ldots \beta_n a_{n+1})$$

or, for $\lambda \in k$,

$$(a_1 \beta_1 \ldots \beta_{j-1} (\lambda a_j) \beta_j a_{j+1} \ldots \beta_n a_{n+1}) - \lambda \cdot (a_1 \beta_1 \ldots \beta_{j-1} a_j \beta_j a_{j+1} \ldots \beta_n a_{n+1}).$$

The space $k(\Gamma, \mathcal{A})$ has a naturally defined multiplication, induced by the multiplications of the algebras A_i's and the composition of the $\mathcal{A}$-paths. More explicitly, it is defined by linearity and the following rule:

$$(a_1\beta_1 \dots \beta_n a_{n+1})(b_1\gamma_1 \dots \gamma_m b_{m+1}) = a_1\beta_1 \dots \beta_n(a_{n+1}b_1)\gamma_1 \dots \gamma_m b_{m+1}$$

if $e(\beta_n) = s(\gamma_1)$, and

$$(a_1\beta_1 \dots \beta_n a_{n+1})(b_1\gamma_1 \dots \gamma_m b_{m+1}) = 0$$

otherwise.

With this multiplication, $k(\Gamma, \mathcal{A})$ is an associative algebra, and since we are assuming the quivers to be finite, it has also an identity element, which is equal to $\sum_{i\in\Gamma_0} 1_{A_i}$. Finally, it is easy to observe that $k(\Gamma, \mathcal{A})$ is finite dimensional over k if and only if so are the algebras A_i and Γ is acyclic. We call $k(\Gamma, \mathcal{A})$ the *generalized path algebra* (*gp-algebra*) over Γ and $\mathcal{A}$ (see [8]). In case $A_i = k$ for every $i \in \Gamma_0$, this construction gives the usual path algebra $k\Gamma$.

It was already observed in [8] that generalized path algebras can alternatively be constructed as tensor algebras, as follows: let Γ be a quiver, let $\mathcal{A} = \{A_i : i \in \Gamma_0\}$ be a set of finite-dimensional k-algebras, one for each vertex of Γ, and let $\mathcal{M} = \{M_{ij} : i, j \in \Gamma_0\}$ be a set of modules, such that, for each $i, j \in \Gamma_0$, M_{ij} is an $(A_i - A_j)$-bimodule, finitely generated from both sides, with M_{ij} free as an $A_i^{op} \otimes A_j$-module, having rank equal to the number of arrows $i \to j$ in Γ_0. (By the way, this structure is similar to those of *modulations* introduced by F. Li in [15], and of *pro-species of algebras* introduced by J. Külshammer in [13]. However, in their context, it is not necessary to assume, for example, that k is algebraically closed or that the M_{ij}'s are free as bimodules.)

Let now $A_{\mathcal{A}} = \prod_{i\in\Gamma_0} A_i$ be the product algebra and $M_{\mathcal{A}} = \bigoplus_{i,j\in\Gamma_0} M_{ij}$. Then, by restriction of scalars through the canonical projections $A_{\mathcal{A}} \to A_i$, we can make $M_{\mathcal{A}}$ into an $(A_{\mathcal{A}} - A_{\mathcal{A}})$-bimodule. Finally, the tensor algebra $T(A_{\mathcal{A}}, M_{\mathcal{A}})$ is isomorphic to the generalized path algebra $k(\Gamma, \mathcal{A})$ defined above.

2.3. Generalized bound path algebras. Let $k(\Gamma, \mathcal{A})$ be a generalized path algebra. Using the result mentioned above in 2.1, for each $i \in \Gamma_0$, we may fix a quiver Σ_i such that $A_i \cong k\Sigma_i/\Omega_i$ with Ω_i an admissible ideal of $k\Sigma_i$.

Following [5], we consider quotients of generalized path algebras by an ideal generated by relations. Namely, let I be a finite set of relations over Γ which generates an admissible ideal in $k\Gamma$. Consider the ideal $(\mathcal{A}(I))$ generated by the following subset of $k(\Gamma, \mathcal{A})$:

$$\mathcal{A}(I) = \left\{ \sum_{i=1}^{t} \lambda_i \beta_{i1} \overline{\gamma_{i1}} \beta_{i2} \ldots \overline{\gamma_{i(m_i-1)}} \beta_{im_i} : \right.$$
$$\left. \sum_{i=1}^{t} \lambda_i \beta_{i1} \ldots \beta_{im_i} \text{ is a relation in } I \text{ and } \gamma_{ij} \text{ is a path in } \Sigma_{e(\beta_{ij})} \right\}.$$

The quotient $\frac{k(\Gamma,\mathcal{A})}{(\mathcal{A}(I))}$ is said to be a *generalized bound path algebra* (*gbp-algebra*). We may also write $\frac{k(\Gamma,\mathcal{A})}{(\mathcal{A}(I))} = k(\Gamma, \mathcal{A}, I)$. When the context is clear, we simply denote the set $\mathcal{A}(I)$ by I.

We use the following notation in the sequel: Γ is an acyclic quiver, $\mathcal{A} = \{A_i : i \in \Gamma_0\}$ denotes a family of basic finite dimensional algebras over an algebraically closed field k, and I is a set of relations in Γ generating an admissible ideal in the path algebra $k\Gamma$. By $\Lambda = k(\Gamma, \mathcal{A}, I)$, we denote the gbp-algebra obtained from these data. Also, $A_{\mathcal{A}}$ will denote the product algebra $\prod_{i\in\Gamma_0} A_i$. We denote the identity element of the algebras A_i by 1_i instead of 1_{A_i}. Also, for an algebra A, we shall denote by mod$A$ the category of finitely generated right A-modules.

2.4. Representations. In [6], we have described the representations of a gbp-algebra, including those associated to projective and injective modules. We shall now recall the results needed in the sequel.

Definition 2.1. Let $\Lambda = k(\Gamma, \mathcal{A}, I)$ be a gbp-algebra.

(a) A *representation* of Λ is given by $((M_i)_{i\in\Gamma_0}, (M_\alpha)_{\alpha\in\Gamma_1})$ where
  (i) M_i is an A_i-module, for each $i \in \Gamma_0$;
  (ii) $M_\alpha : M_{s(\alpha)} \to M_{e(\alpha)}$ is a k-linear transformation, for each arrow $\alpha \in \Gamma_1$; and
  (iii) whenever $\gamma = \sum_{i=1}^{t} \lambda_t \alpha_{i1}\alpha_{i2} \ldots \alpha_{in_1}$ is a relation in I, with $\lambda_i \in k$ and $\alpha_{ij} \in \Gamma_1$, we have

$$\sum_{i=1}^{t} \lambda_t M_{\alpha_{in_i}} \circ \overline{\gamma_{in_i}} \circ \ldots \circ M_{\alpha_{i2}} \circ \overline{\gamma_{i2}} \circ M_{\alpha_{i1}} = 0$$

  for every choice of paths γ_{ij} over $\Sigma_{s(\alpha_{ij})}$, with $1 \le i \le t$, $2 \le j \le n_i$.

(b) We say that a representation $((M_i)_{i\in\Gamma_0}, (M_\alpha)_{\alpha\in\Gamma_1})$ of Λ is *finitely generated* if each of the A_i-modules M_i is finitely generated.

(c) Let $M = ((M_i)_{i\in\Gamma_0}, (M_\alpha)_{\alpha\in\Gamma_1})$ and $N = ((N_i)_{i\in\Gamma_0}, (N_\alpha)_{\alpha\in\Gamma_1})$ be representations of Λ. A *morphism of representations* $f : M \to N$ is given by a tuple $f = (f_i)_{i\in\Gamma_0}$, such that, for every $i \in \Gamma_0$, $f_i : M_i \to N_i$ is a morphism of

A_i-modules; and such that, for every arrow $\alpha : i \to j \in \Gamma_1$, it holds that $f_j M_\alpha = N_\alpha f_i$, that is, the following diagram commutes.

$$\begin{array}{ccc} M_i & \xrightarrow{M_\alpha} & M_j \\ {\scriptstyle f_i}\downarrow & & \downarrow{\scriptstyle f_j} \\ N_i & \xrightarrow[N_\alpha]{} & N_j \end{array}$$

We shall denote by $\operatorname{Rep}_k(\Gamma, \mathcal{A}, I)$ (or by $\operatorname{rep}_k(\Gamma, \mathcal{A}, I)$) the category of the representations (or finitely generated representations, respectively) of the algebra $k(\Gamma, \mathcal{A}, I)$.

Theorem 2.2. ([6], see also [12]) *There is a k-linear equivalence*

$$F : \operatorname{Rep}_k(\Gamma, \mathcal{A}, I) \to \operatorname{Mod} k(\Gamma, \mathcal{A}, I)$$

which restricts to an equivalence

$$F : \operatorname{rep}_k(\Gamma, \mathcal{A}, I) \to \operatorname{mod} k(\Gamma, \mathcal{A}, I).$$

2.5. Realizing an A_i-module as a Λ-module. Let $i \in \Gamma_0$ and let M be an A_i-module. We consider three ways of realizing M as a Λ-module (see [6] for further details of these constructions).

A) Natural inclusion. Define $\mathcal{I}(M) = ((M_j)_{j\in\Gamma_0}, (\phi_\alpha)_{\alpha\in\Gamma_1})$ to be the representation given by

$$M_j = \begin{cases} M & \text{if } j = i \\ 0 & \text{if } j \neq i \end{cases} \quad \text{and} \quad \phi_\alpha = 0 \quad \text{for all } \alpha \in \Gamma_1.$$

By abuse of notation, we shall identify $\mathcal{I}(M) = M$, since these two have the same underlying space.

B) Cones. As recalled from [8] above, consider the gp-algebra $k(\Gamma, \mathcal{A})$ as a tensor algebra $k(\Gamma, \mathcal{A}) \cong T(A_\mathcal{A}, M_\mathcal{A})$. Since M is naturally an $A_\mathcal{A}$-module and there is a canonical map $A_\mathcal{A} \to \Lambda = k(\Gamma, \mathcal{A})/I$, by extension of scalars, M originates a Λ-module $\mathcal{C}_i(M)$, which is called the *cone* over M.

We now recall the following results from [6].

Proposition 2.3. *Given $i \in \Gamma_0$, we have:*

(1) *The cone functor $\mathcal{C}_i : \operatorname{mod} A_i \to \operatorname{mod} \Lambda$ is exact.*
(2) *If P is a projective A_i-module, then $\mathcal{C}_i(P)$ is a projective Λ-module.*

C) Dual cones. The *dual cone* over M is given by $\mathcal{C}_i^*(M) \doteq D\mathcal{C}_i D(M)$, where $D = \mathrm{Hom}_k(-, k)$ is the usual duality functor. A dual result of Proposition 2.3 for injective modules holds true (see [6]).

3. Homological dimensions of gbp-algebras

We shall prove in this section general results involving gbp-algebras, leaving the particular case of gp-algebras for the next section. Using the notations established above, we shall compare some homological dimensions of a gbp-algebra Λ with those of the algebras A_i, $i \in \Gamma_0$, which are used in its construction. Given an algebra A and an A-module M, we denote by $\mathrm{pd}_A M$ and by $\mathrm{id}_A M$ the projective and the injective dimensions of M, respectively. Also, the global dimension of A is denoted by $\mathrm{gl.dim} A$.

3.1. First case. We analyse the natural inclusion of A_i-modules in mod Λ.

Lemma 3.1. *Let $i \in \Gamma_0$ and let M be an A_i-module. Then*

(a) $\mathrm{pd}_\Lambda M \geq \mathrm{pd}_{A_i} M$.
(b) *if i is a sink, then* $\mathrm{pd}_\Lambda M = \mathrm{pd}_{A_i} M$.
(c) $\mathrm{id}_\Lambda M \geq \mathrm{id}_{A_i} M$.
(d) *if i is a source, then* $\mathrm{id}_\Lambda M = \mathrm{id}_{A_i} M$.

Proof. We shall prove only (a) and (b) since the proofs of (c) and (d) are dual.

(a) There is nothing to show if $\mathrm{pd}_\Lambda M = \infty$. So, assume M has finite projective dimension m over Λ. It follows from the description of the projective modules over Λ (see [6], Subsection 5.1) that every component of a projective representation is projective (indeed, the i-th component is either a direct sum of indecomposable projective modules over A_i, copies of A_i, or zero modules). Therefore the i-th component of a minimal projective resolution of M as a Λ-module is a projective resolution of M_i as an A_i-module, which proves this item.

(b) Because i is a sink, every projective resolution of M over A_i is easily seen to yield a projective resolution of M over Λ with the same length. □

The next result follows easily.

Corollary 3.2. *$gl.dim\,\Lambda \geq \max\{gl.dim\, A_1, \ldots, gl.dim\, A_n\}$.*

We shall see below examples of when equality in the above statement holds and when it does not.

3.2. Cones and duals. The next result, which relates the projective and the injective dimensions of a module over A_i with the corresponding dimension of its cone or its dual cone, is a direct consequence of Proposition 2.3 and its dual.

Lemma 3.3. *Given $i \in \Gamma_0$ and M an A_i-module, we have*

(a) $\operatorname{pd}_{A_i} M = \operatorname{pd}_\Lambda \mathcal{C}_i(M)$.
(b) $\operatorname{id}_{A_i} M = \operatorname{id}_\Lambda \mathcal{C}_i^*(M)$.

Proof. We shall prove only (a) since the proof of (b) is dual. Let

$$0 \longrightarrow P_m \longrightarrow \dots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

be a minimal projective resolution of M in $\operatorname{mod} A_i$. Thus $m = \operatorname{pd}_{A_i} M$. Applying the functor $\mathcal{C}_i$, we have

$$0 \longrightarrow \mathcal{C}_i(P_m) \longrightarrow \dots \longrightarrow \mathcal{C}_i(P_1) \longrightarrow \mathcal{C}_i(P_0) \longrightarrow \mathcal{C}_i(M) \longrightarrow 0.$$

Because of Proposition 2.3, this sequence is exact. Moreover, also by Proposition 2.3, every term except possibly for $\mathcal{C}_i(M)$ is known to be projective. So this is a projective resolution in $\operatorname{mod} \Lambda$, proving that $\operatorname{pd}_\Lambda \mathcal{C}_i(M) \leq \operatorname{pd}_{A_i} M$. Since the i-th component of $\mathcal{C}_i(M)$ is M, we know from Proposition 3.1 that the inverse inequality also holds. □

3.3. General case. Having studied the projective and injective dimensions of modules which are inclusion or cones of A_i-modules, we turn our attention to general representations over Λ.

Definition 3.4. Let $M = ((M_i)_{i\in\Gamma_0}, (\phi_\alpha)_{\alpha\in\Gamma_1})$ be a representation over $k(\Gamma, \mathcal{A}, I)$. The *support* of M is defined as the set of vertices $\operatorname{supp} M \doteq \{i \in \Gamma_0 : M_i \neq 0\}$.

Proposition 3.5. *For a Λ-module M,*

(a) $\operatorname{pd}_\Lambda M \leq \max_{j\in\operatorname{supp} M}\{\operatorname{pd}_\Lambda M_j\}$,
(b) $\operatorname{id}_\Lambda M \leq \max_{j\in\operatorname{supp} M}\{\operatorname{id}_\Lambda M_j\}$.

Proof. For both items, it is sufficient to observe that since Γ is acyclic, M is an iterated extension of its components M_i, and so it must have its projective or injective dimensions over Λ limited by those of the M_i's. □

Remark 3.6. Although we have not explicitly stated, the results we gave so far hold in more general settings, for example in the context of tensor algebras over modulations by F. Li recalled above, provided that the defining bimodules M_{ij} are projective from both sides. However, the proofs of our upcoming results rely specifically on the structure of gbp-algebras.

3.4. The main lemma. We will adopt the following notation from here on: if i is a source vertex of Γ, then $\Gamma \setminus \{i\}$ shall denote the quiver obtained from Γ by deleting the vertex i and the arrows starting at i. Moreover, if Γ is equipped with a set of relations I, $I \setminus \{i\}$ will be the set obtained from I by excluding the relations starting at i. Also, since Γ is acyclic, we can iterate this process and enumerate $\Gamma_0 = \{1, \ldots, n\}$ in such a way that i is a source vertex of $\Gamma \setminus \{1, \ldots, i-1\}$ for every i. (Some authors call this a *topological ordering* of the vertices.)

Lemma 3.7. *Let $i \in \Gamma_0$, M be an A_i-module, and let (P, g) be its projective cover in $\operatorname{mod} A_i$. Then there is an exact sequence of Λ-modules:*

$$0 \longrightarrow \mathcal{C}_i(Ker\, g) \oplus L \longrightarrow \mathcal{C}_i(P) \longrightarrow M \longrightarrow 0$$

where L is a Λ-module with $\operatorname{supp} L \subseteq \{j \in \Gamma_0 : j \neq i$ and there is a path $i \rightsquigarrow j\}$. Moreover,

(a) *L_j is free for every vertex j, and*
(b) *If $i \in \Gamma_0$ is such that $I \setminus \{1, \ldots, i\} = I \setminus \{1, \ldots, i-1\}$, then L is projective over Λ.*

Proof. (a) It follows from [6, Proposition 5 and Remark 5] that $(\mathcal{C}_i(P))_i = P$. So, we can define a morphism of representations $g' : \mathcal{C}_i(P) \to M$ by establishing that $g'_i = g$ and that $g'_j = 0$ for $j \neq i$. We want to show that $\operatorname{Ker} g' = \mathcal{C}_i(\operatorname{Ker} g) \oplus L$, where L satisfies the conditions in the statement.

Let $\{p_1, \ldots, p_r\}$ be a k-basis of $\operatorname{Ker} g$ and complete it to a k-basis $\{p_1, \ldots, p_r, \ldots, p_s\}$ of P. Also let, for every $j \in \Gamma_0$, $\{a_1^j, \ldots, a_{n_j}^j\}$ be a k-basis of A_j. For a path $\gamma : i = l_0 \to l_1 \to \ldots \to l_t = j$ from i to j in Γ denote

$$\theta_{\gamma,h,i_1,\ldots,i_t} = p_h \otimes \overline{\gamma_1 a_{i_1}^{e(\gamma_1)} \gamma_2 a_{i_2}^{e(\gamma_2)} \ldots \gamma_t a_{i_t}^j} \in \operatorname{Ker} g'.$$

Remember that since g' was defined as a morphism of representations, it corresponds to a morphism of Λ-modules, because of Theorem 2.2. Therefore,

$$g'(\theta_{\gamma,h,i_1,\ldots,i_t}) = g'(p_h \otimes \overline{\gamma_1 a_{i_1}^{e(\gamma_1)} \gamma_2 a_{i_2}^{e(\gamma_2)} \ldots \gamma_t a_{i_t}^j}) = g(p_h).$$

So $\theta_{\gamma,h,i_1,\ldots,i_t} \notin \operatorname{Ker} g'$ if and only if γ is the zero-length path ϵ_i and $r < h \leq s$. Thus we can write

$$\begin{aligned} \operatorname{Ker} g' &= (\theta_{\epsilon_i,h} : 1 \leq h \leq r) + (\theta_{\gamma,h,i_1,\ldots,i_t} : l(\gamma) > 0) \\ &= (\theta_{\gamma,h,i_1,\ldots,i_t} : 1 \leq h \leq r) \oplus (\theta_{\gamma,h,i_1,\ldots,i_t} : l(\gamma) > 0 \text{ and } r < h \leq s) \\ &= \mathcal{C}_i(\operatorname{Ker} g) \oplus L \end{aligned}$$

where $L \doteq (\theta_{\gamma,h,i_1,\ldots,i_t} : l(\gamma) > 0 \text{ and } r < h \leq s)$. Since the generators of L involve only paths of length strictly greater than zero, the only components of L that are non-zero are the ones over the successors of i, except for i itself. Therefore the condition about the support of L in the statement is satisfied. It remains to prove the other two assertions in the statement.

To prove (a), fix $j \in \Gamma_0$. If $j = i$ or if j is not a successor of i, then $L_j = 0$, so we may suppose this is not the case. Again using the equivalence given by Theorem 2.2,

$$\begin{aligned} L_j = L \cdot 1_j &= (\theta_{\gamma,h,i_1,\ldots,i_t} : \gamma : i \rightsquigarrow j \text{ and } r < h \leq s) \\ &= (p_h \otimes \overline{\gamma_1 a_{i_1}^{e(\gamma_1)} \gamma_2 a_{i_2}^{e(\gamma_2)} \ldots \gamma_t a_{i_t}^{j}} : \gamma : i \rightsquigarrow j \text{ and } r < h \leq s). \end{aligned}$$

So L_j is isomorphic to the free A_j-module whose basis is the set of all possible elements $p_h \otimes \overline{\gamma_1 a_{i_1}^{e(\gamma_1)} \gamma_2 a_{i_2}^{e(\gamma_2)} \ldots \gamma_t}$. In particular, L_j is free over A_j, and this proves (a).

(b) Assume that $I \setminus \{1, \ldots, i\} = I \setminus \{1, \ldots, i-1\}$ and let i^+ denote the set of immediate successors of i. Since, by hypothesis, there are no relations starting at i, we can write:

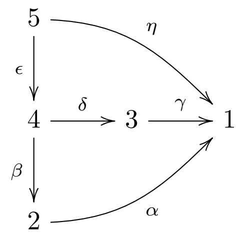
$$\begin{aligned} L &\doteq (\theta_{\gamma,h,i_1,\ldots,i_t} : l(\gamma) > 0 \text{ and } r < h \leq s) \\ &= (p_h \otimes \overline{\gamma_1 a_{i_1}^{e(\gamma_1)} \gamma_2 a_{i_2}^{e(\gamma_2)} \ldots \gamma_t a_{i_t}^{j}} : l(\gamma) > 0 \text{ and } r < h \leq s) \\ &= (p_h \otimes \gamma_1 a_{i_1}^{e(\gamma_1)} \otimes \overline{\gamma_2 a_{i_2}^{e(\gamma_2)} \ldots \gamma_t a_{i_t}^{j}} : l(\gamma) > 0 \text{ and } r < h \leq s) \\ &\cong \left(a_{i_1}^{e(\gamma_1)} \otimes \overline{\gamma_2 a_{i_2}^{e(\gamma_2)} \ldots \gamma_t a_{i_t}^{j}} : l(\gamma) > 0 \right)^{s-r} \\ &\cong \bigoplus_{i' \in i^+} \mathcal{C}_i(A_{i'})^{s-r}. \end{aligned}$$

Since $A_{i'}$ is projective over $A_{i'}$, $\mathcal{C}_i(A_{i'})$ is projective over Λ by Proposition 2.3. We have thus shown that L is isomorphic to a direct sum of projective Λ-modules, and therefore it is also projective, concluding the proof. □

3.5. A special kind of gbp-algebras. Before our next result, we need an additional definition. For a vertex j of Γ, denote by S_j the simple $k\Gamma/I$-module associated with j.

Definition 3.8. A gbp-algebra Λ is called *terraced* provided that for every $i \in \Gamma_0$ such that $I \setminus \{1, \ldots, i\} \neq I \setminus \{1, \ldots, i-1\}$ (i.e., every time there are relations starting at i), one has $\mathrm{pd}_{k\Gamma/I} S_i \geq \max\{\mathrm{pd}_{k\Gamma/I} S_j : j \text{ is a successor of } i\} + 1$.

Example 3.9. Observe that any gp-algebra (that is, when $I = 0$, which makes $k\Gamma$ hereditary) is terraced. An example of a non-terraced gbp-algebra is the following bound path algebra given by



and the relations $\beta\alpha = \delta\gamma$, $\epsilon\beta = 0$. Here the simple module S_5 associated to vertex 5 has projective dimension 2, and does not exceed by one the dimension of S_4, which is also 2.

Theorem 3.10. *Let $\Lambda = k(\Gamma, \mathcal{A}, I)$ be a terraced gbp-algebra. Then, for every representation M over Λ,*

$$\operatorname{pd}_\Lambda M \leq \max_{i \in \operatorname{supp} M} \{\operatorname{pd}_{A_i} M_i, \operatorname{pd}_{k\Gamma/I} S_i\}$$

where S_i denotes the simple $k\Gamma/I$-module associated with the vertex i.

Observe that if Λ is simply a terraced bound path algebra (i.e., $A_i = k$ for every $i \in \Gamma_0$), then each of the M_i are semisimple and we recover an inequality given by M. Auslander: $\operatorname{pd}_\Lambda M \leq \max\{\operatorname{pd}_\Lambda S : S$ is a simple composition factor of $M\}$.

Proof. The proof is done by induction. First, suppose $\operatorname{supp} M = \{n\}$. By the assumption on the numbering of the vertices, we know that n is a sink vertex of Γ_0. It follows from Lemma 3.1(b) that $\operatorname{pd}_\Lambda M = \operatorname{pd}_{A_n} M_n$. Since n is a sink vertex, the simple $k\Gamma/I$-module S_n is projective, and thus it holds that $\operatorname{pd}_\Lambda M = \max\{\operatorname{pd}_{A_n} M_n, \operatorname{pd}_{k\Gamma/I} S_n\}$. This proves the initial step of induction.

Now suppose that $\operatorname{supp} M \subseteq \{i, \ldots, n\}$ and that the statement is valid for representations whose support is contained in $\{i+1, \ldots, n\}$. Initially we are going to study the projective dimension of M_i over Λ. If i is a sink vertex, then, similarly to above, we have that $\operatorname{pd}_\Lambda M_i = \max\{\operatorname{pd}_{A_i} M_i, \operatorname{pd}_{k\Gamma/I} S_i\}$, so suppose i is not a sink vertex. Let (P, g) be a projective cover of M_i over A_i. Then, because of Lemma 3.7, there is an exact sequence in $\operatorname{mod} \Lambda$:

$$0 \longrightarrow \mathcal{C}_i(\operatorname{Ker} g) \oplus L \longrightarrow \mathcal{C}_i(P) \longrightarrow M_i \longrightarrow 0$$

where L satisfies the conditions given in the statement of the cited lemma. From this exact sequence, we deduce that

$$\operatorname{pd}_\Lambda M_i \leq \max\{\operatorname{pd}_\Lambda \mathcal{C}_i(P), \operatorname{pd}_\Lambda(\mathcal{C}_i(\operatorname{Ker} g) \oplus L) + 1\}.$$

Since P is projective over A_i, Proposition 2.3 implies that $\mathrm{pd}_\Lambda \mathcal{C}_i(P) = 0$. Thus

$$\mathrm{pd}_\Lambda M_i \leq \mathrm{pd}_\Lambda(\mathcal{C}_i(\mathrm{Ker}\, g) \oplus L) + 1 \leq \max\{\mathrm{pd}_\Lambda \mathcal{C}_i(\mathrm{Ker}\, g), \mathrm{pd}_\Lambda L\} + 1.$$

Using Corollary 3.3, we have

$$\mathrm{pd}_\Lambda M_i \leq \max\{\mathrm{pd}_{A_i} \mathrm{Ker}\, g, \mathrm{pd}_\Lambda L\} + 1. \tag{3.1}$$

Now we divide our analysis in cases:

Case 1: $\mathrm{pd}_{A_i} \mathrm{Ker}\, g \geq \mathrm{pd}_\Lambda L$.

In this case, Equation 3.1 implies that $\mathrm{pd}_\Lambda M_i \leq \mathrm{pd}_{A_i} \mathrm{Ker}\, g + 1 = \mathrm{pd}_{A_i} M_i$, because (P, g) is the projective cover of M_i.

Case 2: $\mathrm{pd}_{A_i} \mathrm{Ker}\, g \leq \mathrm{pd}_\Lambda L$.

Now, from Equation 3.1, $\mathrm{pd}_\Lambda M_i \leq \mathrm{pd}_\Lambda L + 1$. In case $I \setminus \{1, \ldots, i\} = I \setminus \{1, \ldots, i-1\}$, from Lemma 3.7, we get that $\mathrm{pd}_\Lambda L = 0$. Since we have already supposed in this case that $\mathrm{pd}_{A_i} \mathrm{Ker}\, g \leq \mathrm{pd}_\Lambda L$, we have $\mathrm{pd}_{A_i} \mathrm{Ker}\, g = 0$. Again from Equation 3.1, $\mathrm{pd}_\Lambda M_i \leq 1$. Since i is not a sink, we know that S_i is not projective over $k\Lambda/I$ and so $\mathrm{pd}_{k\Lambda/I} S_i \geq 1$. Thus $\mathrm{pd}_\Lambda M_i \leq \mathrm{pd}_{k\Lambda/I} S_i$.

Assume now $I \setminus \{1, \ldots, i\} \neq I \setminus \{1, \ldots, i-1\}$. By Lemma 3.7, $\mathrm{pd}_{A_j} L_j = 0$ for every j, and since the support of L is contained in $\{i+1, \ldots, n\}$, by the induction hypothesis and because Λ is terraced:

$$\mathrm{pd}_\Lambda L \leq \max_{j \in \mathrm{supp}\, L} \{\mathrm{pd}_{k\Gamma/I} S_j\} \leq \mathrm{pd}_{k\Gamma/I} S_i - 1.$$

Then $\mathrm{pd}_\Lambda M_i \leq \mathrm{pd}_\Lambda L + 1 \leq \mathrm{pd}_{k\Gamma/I} S_i - 1 + 1 = \mathrm{pd}_{k\Gamma/I} S_i$.

Putting together all cases discussed above, we conclude that

$$\mathrm{pd}_\Lambda M_i \leq \max\{\mathrm{pd}_{A_i} M_i, \mathrm{pd}_{k\Gamma/I} S_i\}.$$

Now, using Proposition 3.5, we have that

$$\mathrm{pd}_\Lambda M \leq \max_{j \in \mathrm{supp}\, M} \mathrm{pd}_\Lambda M_j \leq \max_{j \in \mathrm{supp}\, M} \{\mathrm{pd}_{A_j} M_j, \mathrm{pd}_{k\Gamma/I} S_j\},$$

which proves the theorem. □

Corollary 3.11. *Let $\Lambda = k(\Gamma, \mathcal{A}, I)$ be a terraced gbp-algebra. Then, for every $j \in \Gamma_0$, gl.dim $A_j \leq$ gl.dim Λ, and the following inequality holds:*

$$\mathit{gl.dim}\, \Lambda \leq \max_{j \in \Gamma_0} \left\{ \mathit{gl.dim}\, \frac{k\Gamma}{I}, \mathit{gl.dim}\, A_j \right\}.$$

3.6. The dual result. Using the fact that the duality functor $D = \operatorname{Hom}_k(-, k)$ anti-preserves homological properties and [6, Proposition 3], we obtain the following result, which is dual to Theorem 3.10.

Corollary 3.12. *Let $\Lambda = k(\Gamma, \mathcal{A}, I)$ be a terraced gbp-algebra, and let M be a representation over Λ. Then $\operatorname{id}_\Lambda M = \max_{i \in \operatorname{supp} M}\{\operatorname{id}_{A_i} M_i, \operatorname{id}_{k\Gamma/I} S_i\}$ where S_i denotes the simple $k\Gamma/I$-module associated with the vertex i.*

3.7. Finitistic dimension. Given an algebra A, its *finitistic dimension* is given by:

$$\operatorname{fin.dim} A = \sup\{\operatorname{pd}_A M : M \text{ is an } A\text{-module of finite projective dimension}\}.$$

A still open conjecture, called the *Finitistic Dimension Conjecture*, states that every algebra has finite finitistic dimension.

Proposition 3.13. *Let $\Lambda = k(\Gamma, \mathcal{A}, I)$ be a terraced gbp-algebra. Then*

$$\textit{fin.dim}\, \Lambda \leq \max_{i \in \Gamma_0} \left\{ \textit{gl.dim}\, \frac{k\Gamma}{I}, \textit{fin.dim}\, A_i \right\}.$$

In particular, if the bound path algebra $k\Gamma/I$ has finite global dimension and fin.dim $A_i < \infty$ for each i, then also fin.dim $\Lambda < \infty$.

Proof. Let $M = ((M_i)_{i \in \Gamma_0}, (\phi_\alpha)_{\alpha \in \Gamma_1})$ be a representation of finite projective dimension over Λ. From Lemma 3.1, for every $i \in \Gamma_0$, $\operatorname{pd}_{A_i} M_i \leq \operatorname{pd}_\Lambda M$, so M_i has finite projective dimension over A_i, and thus $\operatorname{pd}_{A_i} M_i \leq \operatorname{fin.dim} A_i$. Using Theorem 3.10,

$$\operatorname{pd}_\Lambda M \;\leq\; \max_{i \in \Gamma_0}\{\operatorname{pd}_{k\Gamma/I} S_i, \operatorname{pd}_{A_i} M_i\} \;\leq\; \max_{i \in \Gamma_0}\{\operatorname{gl.dim} k\Gamma/I, \operatorname{fin.dim} A_i\}.$$

Since M is arbitrary, the statement follows. □

4. Homological dimensions for gp-algebras

We now focus on gp-algebras, which are, as observed above, terraced gbp-algebras. We start with the following result which is a direct consequence of the above considerations.

Theorem 4.1. *Let $\Lambda = k(\Gamma, \mathcal{A})$ be a gp-algebra, with Γ having at least one arrow. Then gl.dim $\Lambda = \max_{j \in \Gamma_0}\{1, \textit{gl.dim}\, A_j\}$.*

Proof. Observe that $\operatorname{gl.dim} k\Gamma = 1$ in this case and hence, by Corollary 3.11, $\operatorname{gl.dim} \Lambda \leq \max_{j \in \Gamma_0}\{1, \operatorname{gl.dim} A_j\}$. The equality now follows using Corollary 3.2 and the fact that Λ is not semisimple (since $k\Gamma$ is not). □

Remark 4.2. Theorem 4.1 may be considered a slight improvement from basic formulas for calculating the global dimension of a tensor algebra. For example, if we had used [11, Theorem 2.2.11], then we could only affirm that $\operatorname{gl.dim} k(\Gamma, \mathcal{A}) \leq \max_{j\in\Gamma_0}\{\operatorname{gl.dim} A_j\} + 1$.

4.1. Shod and quasitilted algebras. The next result is an application to the study of shod and quasitilted algebras. Quasitilted algebras were introduced in [10] as a generalization of tilted algebras, by considering tilting objects in abelian categories. We shall, however, use a characterization of quasitilted algebras, also proven in [10], which suits better our purpose here. The shod algebras were introduced in [7] in order to generalize the concept of quasitilted. The acronym shod stands for small homological dimension, as it is clear from the definition below. We refer to [7,10] for more details.

Definition 4.3. Let A be an algebra. We say that A is a *shod* algebra if, for every indecomposable A-module M, either $\operatorname{pd}_A M \leq 1$ or $\operatorname{id}_A M \leq 1$. If, besides from being shod, A has global dimension of at most two, we say that A is *quasitilted*.

Our next result allows us to produce a quasitilted or shod gp-algebra from other algebras. It is worth mentioning that it is not intended as a complete description of which generalized (bound) path algebras are quasitilted or shod. Before stating it, please note that every hereditary algebra is quasitilted, and thus also shod.

Proposition 4.4. *Let $\Lambda = k(\Gamma, \mathcal{A})$ be a gp-algebra, with Γ acyclic. Suppose that A_j is hereditary for every $j \in \Gamma_0$, except possibly for a single vertex $i \in \Gamma_0$. Then:*

(a) *If A_i is shod, then Λ is shod.*
(b) *If A_i is quasitilted, then Λ is quasitilted.*

Proof. (a) Let $M = ((M_j)_{j\in\Gamma_0}, (\phi_\alpha)_{\alpha\in\Gamma_1})$ be an indecomposable representation over Λ. Since Γ is acyclic, we infer that the algebra $k\Gamma$ is hereditary and so every simple module over it will have projective and injective dimension of at most one. Observe also that, since A_j is hereditary for $j \neq i$, we also have $\operatorname{pd}_{A_j} M_j \leq 1$ and $\operatorname{id}_{A_j} M_j \leq 1$ if $j \neq i$.

Now, since A_i is shod, either $\operatorname{pd}_{A_i} M_i \leq 1$ or $\operatorname{id}_{A_i} M_i \leq 1$. In the former case, from Theorem 3.10, we have that $\operatorname{pd}_\Lambda M \leq \max_{j\in\Gamma_0}\{\operatorname{pd}_{A_j} M_j, \operatorname{pd}_{k\Gamma} S_j\} \leq 1$, and in the latter, using Corollary 3.12 in an analogous manner, one obtains that $\operatorname{id}_\Lambda M \leq 1$. Thus Λ is shod.

(b) Since A_i is quasitilted, it is shod and from the previous item we get that Λ is shod. It remains to prove that $\mathrm{gl.dim}\,\Lambda \leq 2$. Applying Corollary 3.11,

$$\mathrm{gl.dim}\,\Lambda \leq \max_{j\in\Gamma_0}\{k\Gamma, \mathrm{gl.dim}\,A_j\} \leq 2,$$

using that A_i is quasitilted and that the other algebras are hereditary. □

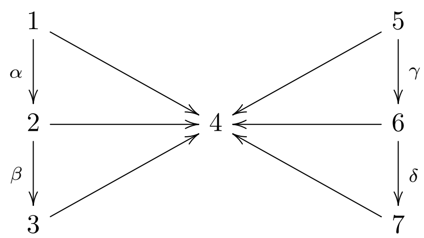
Example 4.5. This example will show that the converse of proposition above could not hold. Let A be the bound path algebra over the quiver

$$1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3$$

bound by $\alpha\beta = 0$, and let Λ be the generalized path algebra given by

$$A \longrightarrow k \longleftarrow A\,.$$

We have that, with this setting, Λ does not satisfy the hypothesis from the last proposition: there is more than one vertex upon which the algebra is quasitilted and non-hereditary. However, using [12, Theorem 3.3] or [5, Theorem 3.9], we see that Λ is isomorphic to the bound path algebra over the quiver



bound by $\alpha\beta = \gamma\delta = 0$. Then a routine calculation shows that Λ is a quasitilted algebra. The same example shows that the converse of the above proposition also does not hold for shod algebras.

We finish our considerations with a result which is a direct consequence of Proposition 3.13.

Proposition 4.6. *Let $\Lambda = k(\Gamma, \mathcal{A})$ be a gp-algebra, with Γ having at least one arrow. Then*

$$\textit{fin.dim}\,\Lambda = \max_{i\in\Gamma_0}\{1, \textit{fin.dim}\,A_i\}\,.$$

In particular, if fin.dim $A_i < \infty$ for each i, then also fin.dim $\Lambda < \infty$.

Proof. Just observe that $\mathrm{gl.dim}\,k\Gamma = 1$, and use Proposition 3.13. □

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