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ON PURELY-MAXIMAL IDEALS WITH APPLICATIONS

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ABSTRACT. Let B be a ring of the form $B = A+J$ where A is a subring of B, J is an ideal of B such that $J \cap A = 0$ and $1 + J \subseteq \mathcal{U}(B)$ the set of units of B. Let C be a subring of B containing A. We prove that purely-maximal ideals of C are exactly IC where I ranges over purely-maximal ideals of A. We deduce that C is semi-Noetherian if and only if A is semi-Noetherian. We show that Tarizadeh and Aghajani's conjecture holds in C if and only if it holds in A. As an application, we generalize all results in [N. Ouni and A. Benhissi, Beitr. Algebra Geom., 65(1)(2024), 229-240] and we study purely-maximal ideals of an amalgamation ring along an ideal.

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1. Introduction

Throughout this paper all rings are commutative with identity. Let R be a ring and I an ideal of R. I is called pure if for every $a \in I$, there exists $b \in I$ such that $a = ab$ [2, page 141]. The ideal I is called purely-maximal if it is maximal (under inclusion) in the lattice of proper pure ideals of A [2, page 156]. The ideal I is called purely-prime if it is proper and if for any pure ideals I_1, I_2 of R with $I_1 \cap I_2 \subseteq I$, then $I_1 \subseteq I$ or $I_2 \subseteq I$ [2, page 156]. In [5], the authors studied the pure spectrum of a commutative ring R, denoted $\mathrm{Spp}(R)$ which consists of all purely-prime ideals. They build a new topological framework that complements the usual Zariski spectrum (there is a canonical correspondence between the idempotents of a ring and the clopens of its pure spectrum Spp(R)) and they found algebraic characterizations of key classes of rings (notely, Gelfand rings/reduced mp-rings) through the behavior of their pure spectrum.

Tarizadeh and Aghajani conjectured that each purely-prime ideal is purely-maximal [5, Conjecture 5.8] and they called a ring R to be semi-Noetherian if every pure ideal of R is finitely generated [5, page 834]. In [4], the authors studied purely-maximal ideals of power series rings of the form $A + XB[[X]]$ (where A is a subring of a ring B), polynomial rings of the form $A + XB[X]$, rings of the form

$A+I[[X]]$ (also $A[X]+I[X]$) and Nagata idealization ring. They also studied when each of the aformentioned ring is semi-Noetherian and they studied Tarizadeh and Aghajani's conjecture. The aim of this paper is to study purely-maximal ideals of the ring of the form $B = A + J$ where A is a subring of B, J is an ideal of B such that $J \cap A = 0$ and $1 + J \subseteq \mathcal{U}(B)$ the set of units of B. Let C be a subring of B containing A. We prove that purely-maximal ideals of C are exactly IC where I ranges over purely-maximal ideals of A (Theorem 2.7). We deduce that C is semi-Noetherian if and only if A is semi-Noetherian. Also we prove that Tarizadeh and Aghajani's conjecture holds in C if and only if it holds in A (Theorem 2.7). As an application, we deduce and generalize all results in [4] (Corollary 3.1 and Corollary 3.2). As another application, we study the case of an amalgamation ring along an ideal with respect to an homomorphism. Let A, B be two rings, J an ideal of B, $f : A \longrightarrow B$ be a ring homomorphism and $A \bowtie^f J = \{(a, f(a) + j) | a \in A, j \in J\}$ be the amalgamation ring of A with along J with respect to f. Let $\mathcal{C}$ be a subring of $A \bowtie^f J$ containing A. Assume that $J \subseteq \mathrm{Jac}(B)$ the Jacobson radical of B. We prove that purely-maximal ideals of the ring $\mathcal{C}$ are precisely $I\mathcal{C}$ where I ranges over purely-maximal ideals of A (Corollary 3.3). We deduce that $\mathcal{C}$ is semi-Noetherian if and only if A is semi-Noetherian. Also we show that Tarizadeh and Aghajani's conjecture holds in C if and only if it holds in A (Corollary 3.3).

2. Purely-maximal ideals of rings of the form $A + J$

Let B be a ring of the form $B = A + J$ where A is a subring of B, J is an ideal of B such that $J \cap A = 0$ and $1 + J \subseteq \mathcal{U}(B)$ the set of units of B.

Note that:

- If $a + b = a' + b'$, then $a = a'$ and $b = b'$, for all $a, a' \in A$ and $b, b' \in J$.
- If I, I' are ideals of A, then $I \subseteq I'$ if and only if $IB \subseteq I'B$.
- $IB \subseteq I + J$ for each ideal I of A.

Lemma 2.1. *Let $\mathcal{I}$ be a proper pure ideal of B and $I = \{a \in A \mid a + c \in \mathcal{I}$ for some $c \in J\}$. Then I is a proper pure ideal of A and $\mathcal{I} \subseteq IB$.*

Proof. I is a proper ideal of A because $1+t$ is a unit of B for each $t \in J$. If $a \in I$, then $a+c \in \mathcal{I}$ for some $c \in J$. Since $\mathcal{I}$ is pure, $a+c = (a+c)(r+b)$ for some $r+b \in \mathcal{I}$ (where $r \in A$ and $b \in J$). So $r \in I$ and $a - ar = -c + ab + c(r + b) \in J \cap A = 0$. Thus $a = ar$ and so I is a pure ideal of A. Let $x \in \mathcal{I}$. Since $\mathcal{I}$ is pure, $x = xy$ for some $y \in \mathcal{I}$. Let $a, r \in A$ and $c, b \in J$ such that $x = a + c$ and $y = r + b$. Thus $a, r \in I$ and so it suffices to show that $c \in IB$. Since $c = ab + c(r + b)$, $c(1 - b) = ab + rc \in IB$. But $1 - b$ is a unit of B and then $c \in IB$. □

Recall that each ideal I of a ring R contains a largest pure ideal (i.e., the sum of all pure ideals contained in I), denoted $\nu(I)$ (see [5, page 825]) (also denoted I° in [2, Chapter 7-Proposition 8]). Note that if R is a subring of a ring S and H is a purely-prime ideal of S, then $\nu(H \cap R)$ is a purely-prime ideal of R ([2, Chapter 7-Lemma 62], [5, Theorem 2.6]).

Lemma 2.2. *Let I be an ideal of A.*

(1) *IB is a proper pure ideal of B if and only if I is a proper pure ideal of A. In this case, for each $x \in IB$, $x = xa$ for some $a \in I$ (and hence $IB \cap A = I$).*
(2) *IB is purely-prime in B if and only if I is purely-prime in A.*
(3) *IB is purely-maximal in B if and only if I is purely-maximal in A.*

Proof. (1) Assume that I is a proper pure ideal of A. By [4, Lemma 2.2], IB is a proper pure ideal of B. Conversely, assume that IB is a proper (so is I) pure ideal of B and let $a \in I \subseteq IB$. Let $x \in IB$ such that $a = ax$. Let $a_1, \ldots, a_n \in I$ and $x_1, \ldots, x_n \in B$ such that $x = a_1x_1 + \cdots + a_nx_n$. Each $x_i = r_i + y_i$ for some $r_i \in A$ and $y_i \in J$. Since $J \cap A = 0$ and $a = ax$, $a = ar$ where $r = a_1r_1 + \cdots + a_nr_n \in I$. Thus I is pure. By [2, Chapter 7-Proposition 11], for each $x \in IB$, $x = xa$ for some $a \in I$.
(2) Assume that I is a purely-prime ideal of A and let $\mathcal{I}_1, \mathcal{I}_2$ be two pure ideals of B such that $\mathcal{I}_1\mathcal{I}_2 \subseteq IB$. For each i, $\mathcal{I}_i \subseteq I_iB$ for some proper pure ideal I_i of A, $I_1I_2 \subseteq IB \cap A = I$ and so $I_i \subseteq I$ for some i. Then $\mathcal{I}_i \subseteq I_iB \subseteq IB$. Conversely, assume that IB is purely-prime. By (1) and [2, Chapter 7-Lemma 62], $I = \nu(I) = \nu(IB \cap A)$ is a purely-prime ideal of A.
(3) Assume that I is a purely-maximal ideal of A. By (1), IB is a proper pure ideal of B. Let $\mathcal{I}$ be a proper pure ideal of B such that $IB \subseteq \mathcal{I}$. By Lemma 2.1, $\mathcal{I} \subseteq I'B$ for some proper pure ideal I' of A. Thus $IB \subseteq I'B$ and so $I \subseteq I'$. Therefore $I = I'$. So $IB = \mathcal{I}$. Conversely, assume that IB is a purely-maximal ideal of B. By (1), I is a proper pure ideal of A. Let I' be a proper pure ideal of A such that $I \subseteq I'$. Then $IB \subseteq I'B$. Thus $IB = I'B$. It follows that $I = I'$ and then I is a purely-maximal ideal of A. □

Theorem 2.3. *Purely-maximal ideals of the ring B are exactly IB where I ranges over purely-maximal ideals of A.*

Proof. By Lemma 2.2, if I is a purely-maximal ideal of A, then IB is a purely-maximal ideal of B. Conversely, let $\mathcal{I}$ be a purely-maximal ideal of B. By Lemma 2.1, $\mathcal{I} \subseteq IB$ for some proper pure ideal I of A. By Lemma 2.2, IB is a proper pure ideal of B and so $IB = \mathcal{I}$. Again by Lemma 2.2, I is a purely-maximal ideal of A. □

Tarizadeh and Aghajani proved that a ring is semi-Noetherian if and only if each purely-maximal ideal is finitely generated [5, Theorem 6.2]. We deduce that:

Corollary 2.4. *The ring A is semi-Noetherian if and only if the ring B is semi-Noetherian.*

Proof. The "only if" part follows from Theorem 2.3 and the fact that: if I is a finitely generated ideal of A, then IB is a finitely generated ideal of B. Conversely, assume that B is semi-Noetherian and let I be a purely-maximal ideal of A. Note that a finitely generated pure ideal is principal (see [5, page 834]). Then $IB = xB$ for some $x \in IB$. By Lemma 2.2, $x = xa$ for some $a \in I$ and so $IB = aB$. Hence $I = IB \cap A = aB \cap A = aA$. □

Tarizadeh and Aghajani noticed that in all known rings each purely-prime ideal is purely-maximal [5]. So, they asked if this fact holds for any ring. The following shows that Tarizadeh and Aghajani's conjecture holds in the ring B if and only if it holds in the ring A.

Corollary 2.5. *Every purely-prime ideal of B is purely-maximal if and only if every purely-prime ideal of A is purely-maximal.*

Proof. Assume that every purely-prime ideal of B is purely-maximal and let P be a purely-prime ideal of A. By Lemma 2.2, PB is a purely-prime ideal of B, so purely-maximal. Again by Lemma 2.2, P is purely-maximal ideal of A. Conversely, assume that every purely-prime ideal of A is purely-maximal and let $\mathcal{P}$ be a purely-prime ideal of B. By [2, Chapter 7-Lemma 62], $\nu(\mathcal{P} \cap A)$ is a purely-prime ideal of A. By hypothesis, $\nu(\mathcal{P} \cap A)$ is a purely-maximal ideal of A. Thus $\nu(\mathcal{P} \cap A)B$ is a purely-maximal ideal of B. Since $\nu(\mathcal{P} \cap A)B \subseteq \mathcal{P}$, $\mathcal{P} = \nu(\mathcal{P} \cap A)B$ and so $\mathcal{P}$ is a purely-maximal ideal of B. □

Lemma 2.6. *Let C be a subring of B containing A and I an ideal of A.*

(1) *IC is a proper pure ideal of C if and only if I is a proper pure ideal of A. In this case, for each $x \in IC$, $x = xa$ for some $a \in I$ (and so $IC \cap A = I$, in particular, $IC = IB \cap C$).*
(2) *IC is purely-prime in C if and only if I is purely-prime in A.*
(3) *IC is purely-maximal in C if and only if I is purely-maximal in A.*

Proof. (1) Assume that IC is a proper pure ideal of C. By [4, Lemma 2.2], $IB = (IC)B$ is a proper pure ideal of B. Then I is a proper pure ideal of A by Lemma 2.2. Conversely, assume that I is a proper pure ideal of A. By [4, Lemma 2.2], IC is a proper pure ideal of C.
(2) If IC is purely-prime in C, then $\nu(IC \cap A)$ is purely-prime in A by [2, Chapter 7-Lemma 62], and $I = IC \cap A$ is pure in A. Then I is purely-prime in A. Conversely, assume that I is a purely-prime ideal of A. Thus $IC = IB \cap C$ is pure in C. By Lemma 2.2, IB is purely-prime in B and so $\nu(IB \cap C)$ is purely-prime in C by [2, Chapter 7-Lemma 62]. Then $IC = \nu(IC)$ is purely-prime in C.
(3) Assume that IC is a purely-maximal ideal of C. Let I' be a proper pure ideal of A such that $I \subseteq I'$. Then $IC \subseteq I'C$ which is a pure ideal of C by (1). Thus $IC = I'C$ and so $I = I'$. Then I is a purely-maximal ideal of A. Conversely, assume that I is purely-maximal in A. By (1), IC is a proper pure ideal of C. Let $\mathcal{I}$ be a proper pure ideal of C such that $IC \subseteq \mathcal{I}$. By [4, Lemma 2.2], $\mathcal{I}B$ is a proper pure ideal of B and so $\mathcal{I}B \subseteq I'B$ for some proper pure ideal I' of A by Lemma 2.1. Thus $I = IC \cap A \subseteq \mathcal{I} \cap A \subseteq I'B \cap A = I'$ and so $I = I'$. Then $\mathcal{I}B \subseteq IB$. Therefore, $\mathcal{I} \subseteq \mathcal{I}B \cap C = IB \cap C = IC$ and so $\mathcal{I} = IC$. Then IC is a purely-maximal ideal of C. □

Theorem 2.7. *Let C be a subring of B containing A.*

(1) *Purely-maximal ideals of the ring C are precisely IC where I ranges over purely-maximal ideals of A.*
(2) *The ring C is semi-Noetherian if and only if the ring A is semi-Noetherian.*
(3) *Every purely-prime ideal of C is purely-maximal if and only if every purely-prime ideal of A is purely-maximal.*

Proof. (1) By Lemma 2.6, if I is a purely-maximal ideal of A, then IC is a purely-maximal ideal in C. Conversely, let $\mathcal{I}$ be a purely-maximal ideal of C. By [4, Lemma 2.2], $\mathcal{I}B$ is a proper pure ideal of B. So $\mathcal{I}B \subseteq IB$ for some proper pure ideal I of A by Lemma 2.1. Thus $\mathcal{I} \subseteq IB \cap C = IC$. Since IC is a proper pure ideal of C, $\mathcal{I} = IC$ (I is purely-maximal in A by Lemma 2.6).
(2) We can repeat the same argument used in Corollary 2.4.
(3) If I is a purely-prime ideal of A, then IC is a purely-prime ideal of C. Then IC is purely-maximal in C and so I is purely-maximal in A. Conversely, let $\mathcal{I}$ be a purely-prime ideal of C. Since $\nu(\mathcal{I} \cap A)$ is a purely-prime ideal of A, $\nu(\mathcal{I} \cap A)$ is a purely-maximal ideal of A and so $\nu(\mathcal{I} \cap A)C$ is a purely-maximal ideal of C. Since $\nu(\mathcal{I} \cap A)C \subseteq (\mathcal{I} \cap A)C \subseteq \mathcal{I}C = \mathcal{I}$ (which is proper and pure), $\mathcal{I} = \nu(\mathcal{I} \cap A)C$ is purely-maximal in C. □

3. Applications

Now, we show many consequences of Theorem 2.7. First, we deduce and generalize [4, Theorem 2.4, Corollary 2.6, Corollary 2.7, Corollary 3.3, Corollary 2.4 and Corollary 3.8] as follows:

Corollary 3.1. *Let A be a subring of a ring B, X an indeterminate over B. Let $\mathcal{C}$ be a subring of $A + XB[[X]]$ containing A.*

(1) *Purely-maximal ideals of the ring $\mathcal{C}$ are precisely $I\mathcal{C}$ where I ranges over purely-maximal ideals of A.*
(2) *The ring $\mathcal{C}$ is semi-Noetherian if and only if the ring A is semi-Noetherian.*
(3) *Every purely-prime ideal of $\mathcal{C}$ is purely-maximal if and only if every purely-prime ideal of A is purely-maximal.*

Proof. The ring $A + XB[[X]] = A + J$ where $J = XB[[X]]$ is an ideal of $B[[X]]$ and $J \cap A = 0$. It is well known that, for $f \in B[[X]]$, f is a unit of $B[[X]]$ if and only if the constant term of f is a unit of B [1, Chapitre 1 - Proposition 1.2] (in this case, the constant term of f^{-1} is the inverse of the constant term of f). Then $1 + J = 1 + XB[[X]] \subseteq \mathcal{U}(B[[X]])$. □

Let R be a ring and M be a unitary R-module. We recall that Nagata introduced the ring extension of R called the idealization of M in R, denoted here by $R(+)M$, as the R-module $R \oplus M$ endowed with a multiplicative structure defined by:

$(a, x)(b, y) = (ab, ay + bx)$ for all $a, b \in R$ and $x, y \in M$.

We deduce and generalize [4, Theorem 4.4 and Theorem 4.6] as follows:

Corollary 3.2. *Let R be a ring and M an R-module. Let $\mathcal{C}$ be a subring of $R(+)M$ containing R.*

(1) *Purely-maximal ideals of the ring $\mathcal{C}$ are precisely $I\mathcal{C}$ where I ranges over purely-maximal ideals of R.*
(2) *The ring $\mathcal{C}$ is semi-Noetherian if and only if the ring R is semi-Noetherian.*
(3) *Every purely-prime ideal of $\mathcal{C}$ is purely-maximal if and only if every purely-prime ideal of R is purely-maximal.*

Proof. We can write $R(+)M = A + J$ where $A = R(+)0$ (which is a subring of $R(+)M$) and $J = 0(+)M$. Clearly, $J \cap A = 0$. Also, J is an ideal of $R(+)M$ contained in its nilradical (so in its Jacobson radical). Then each element of $1 + J$ is a unit of $R(+)M$. □

We now study the case of an amalgamation ring along an ideal with respect to an homomorphism. Let A, B be two rings, J an ideal of B, $f : A \longrightarrow B$ be a ring homomorphism and $A \bowtie^f J = \{(a, f(a) + j) | a \in A, j \in J\}$ be the amalgamation ring of A with along J with respect to f. For more informations on the ring $A \bowtie^f J$, readers are referred to [3].

Corollary 3.3. *Let A, B be two rings, J an ideal of B, $f : A \longrightarrow B$ be a ring homomorphism and $A \bowtie^f J = \{(a, f(a) + j) | a \in A, j \in J\}$ be the amalgamation ring of A with along J with respect to f. Let $\mathcal{C}$ be a subring of $A \bowtie^f J$ containing A. If $J \subseteq Jac(B)$ the Jacobson radical of B, then:*

(1) *Purely-maximal ideals of the ring $\mathcal{C}$ are precisely $I\mathcal{C}$ where I ranges over purely-maximal ideals of A.*
(2) *The ring $\mathcal{C}$ is semi-Noetherian if and only if the ring A is semi-Noetherian.*
(3) *Every purely-prime ideal of $\mathcal{C}$ is purely-maximal if and only if every purely-prime ideal of A is purely-maximal.*

Proof. Note first that $i : A \longrightarrow A \bowtie^f J$ is a one-to-one ring homomorphism defined by $i(a) = (a, f(a))$ for all $a \in A$ (so i is an embedding making $A \bowtie^f J$ a ring extension of $A \cong i(A)$). Then $A \bowtie^f J = i(A) + \widetilde{J}$ where $i(A)$ is a subring of $A \bowtie^f J$ and $\widetilde{J} = 0 \times J$ is an ideal of $A \bowtie^f J$. Clearly, $i(A) \cap \widetilde{J} = 0$. It suffices to show that $(1, 1) + (0, j)$ is a unit of $A \bowtie^f J$ for each $j \in J$. Since $J \subseteq \mathrm{Jac}(B)$, $1 + j$ is a unit of B. Let $c = -(1 + j)^{-1} j \in J$. Thus $(1, 1 + c) \in A \bowtie^f J$ and $(1, 1+j)(1, 1+c) = (1, 1)$ because $(1+j)(1+c) = 1+j+(1+j)c = 1+j-j = 1$. $\square$

The following is an explicit example of a purely-maximal ideal:

Example 3.4. Consider the open real interval $(0, 1) \subseteq \mathbb{R}$ and A the quotient ring of the polynomial ring $\mathbb{F}_2\left[(X_r)_{0<r<1}\right]$ by the ideal H generated by elements of the form $X_r - X_r X_t$ with $0 < r < t < 1$. Let $J(0, 1)$ be the ideal of A generated by all $(x_r)_{0<r<1}$ where x_r is the class of X_r modulo H. We claim that $J(0, 1)$ is a purely-maximal ideal of A. For each $0 < r < 1, r < (1 + r)/2 < 1$ and $x_r = x_r x_{(1+r)/2}$. Thus $J(0, 1)$ is a pure ideal of A. Let J be a pure ideal of A such that $J(0, 1) \subseteq J \subseteq A$. The ideal of $\mathbb{F}_2\left[(X_r)_{0<r<1}\right]$ generated by all $(X_r)_{0<r<1}$ is a maximal ideal of $\mathbb{F}_2\left[(X_r)_{0<r<1}\right]$. So $J(0, 1)$ is a maximal ideal of A. Thus $J = J(0, 1)$ or $J = A$. Hence $J(0, 1)$ is a purely-maximal ideal of A.

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