International Electronic Journal of Algebra
Volume 39 (2026) 99-115
DOI: 10.24330/ieja.1747239

ON COHOMOLOGY GROUPS OF CURRENT LIE ALGEBRAS

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Received: 11 January 2025; Revised: 17 June 2025; Accepted: 23 June 2025
Communicated by Abdullah Harmancı

Abstract. In this work we state a result that relates the cohomology groups of a Lie algebra $\mathfrak{g}$ and a current Lie algebra $\mathfrak{g}\otimes\mathcal{S}$, by means of a short exact sequence similar to the universal coefficients theorem for modules, where $\mathcal{S}$ is a finite dimensional, commutative and associative algebra with unit over a field $\mathbb{F}$. Using this result we determine the cohomology group of $\mathfrak{g}\otimes\mathcal{S}$ where $\mathfrak{g}$ is a semisimple Lie algebra.

Mathematics Subject Classification (2020): 17B05, 17B56, 17B60, 17B10
Keywords: Lie algebra cohomology, current Lie algebra, tensor product, associative and commutative algebra, semisimple Lie algebra

1. Introduction

Let $\mathfrak{g}$ be a Lie algebra with bracket $[\,\cdot\,,\,\cdot\,]$ and let $\mathcal{S}$ be an associative and commutative algebra over a field $\mathbb{F}$ with product $(s,t)\mapsto s\,t$, for all s,t in $\mathcal{S}$. The skew-symmetric and bilinear map $[\,\cdot\,,\,\cdot\,]_{\mathfrak{g}\otimes\mathcal{S}}$ defined on $\mathfrak{g}\otimes\mathcal{S}$, by

$$[x\otimes s, y\otimes t]_{\mathfrak{g}\otimes\mathcal{S}}=[x,y]\otimes s\,t,\quad \text{for all}\;\; x,y\in\mathfrak{g},\;\;\text{and}\;\; s,t\in\mathcal{S},$$

yields a Lie algebra in $\mathfrak{g}\otimes\mathcal{S}$, which is called the current Lie algebra of $\mathfrak{g}$ by $\mathcal{S}$.

Let $\rho:\mathfrak{g}\to\mathfrak{gl}(V)$ be a representation of $\mathfrak{g}$ on a vector space V, then V is said to be a $\mathfrak{g}$-module. The representation ρ can be extended to a representation R of $\mathfrak{g}\otimes\mathcal{S}$ on the vector space $V\otimes\mathcal{S}$ by means of

$$R(x\otimes s)(v\otimes t)=\rho(x)(v)\otimes s\,t,\;\text{for all}\; x,y\in\mathfrak{g},\; v\in V,\; s,t\in\mathcal{S}. \tag{1}$$

Let $C(\mathfrak{g};V)=C^0(\mathfrak{g};V)\oplus\ldots\oplus C^p(\mathfrak{g};V)\oplus\ldots$ be the space of cochains from $\mathfrak{g}$ into V, where $C^0(\mathfrak{g},V)=V$ and $C^p(\mathfrak{g};V)$ is the space of the alternating p-multilinear maps of $\mathfrak{g}$ with values in V. For any $\mathfrak{g}$-mdule V and $p\geq 0$, let $\mathrm{d}:C^p(\mathfrak{g};V)\to C^{p+1}(\mathfrak{g};V)$

be the differential map given by

$$\begin{aligned} \mathrm{d}\,\lambda(x_1,\dots,x_{p+1}) &= \sum_{j=1}^{p+1}(-1)^{j-1}\rho(x_j)(\lambda(x_1,\dots,x_{\hat{j}},\dots,x_{p+1})) \\ &+ \sum_{j<k}(-1)^{j+k}\lambda([x_j,x_k],x_1,\dots,x_{\hat{j}},\dots,x_{\hat{k}},\dots,x_{p+1}), \qquad p>0, \end{aligned} \tag{2}$$

where λ is in $C^p(\mathfrak{g};V)$ and $x_1,\dots,x_{p+1}$ are in $\mathfrak{g}$. For $p=0$, we let $\mathrm{d}(v)(x)=\rho(x)(v)$ where v is in V and x is in $\mathfrak{g}$.

The aim of this work is to set a result that relates the cohomology groups $\mathcal{H}(\mathfrak{g}\otimes\mathcal{S};V\otimes\mathcal{S})$ and $\mathcal{H}(\mathfrak{g};V)$, similar to the Universal coefficient theorems for modules (see [2, Chapter VI, §3, Theorem 3.3]).

To achieve our goal, in Proposition 3.2 we introduce a map $\mathcal{T}$ between the set of cochains of $\mathfrak{g}$ and cochains of $\mathfrak{g}\otimes\mathcal{S}$, that is sort like a functor except that $\mathcal{T}(\mathrm{Id})$ is not the identity map Id (see §2 and Remark 3.1). Next we prove that there exists a surjective linear map α between $\mathcal{H}(\mathfrak{g}\otimes\mathcal{S};V\otimes\mathcal{S})$ and $\mathcal{H}(\mathfrak{g};V)\otimes\mathcal{S}$ (see Proposition 4.2). In Theorem 4.3 we determine the kernel of α and we state a result that relates the cohomology groups $\mathcal{H}(\mathfrak{g}\otimes\mathcal{S};V\otimes\mathcal{S})$ and $\mathcal{H}(\mathfrak{g};V)\otimes\mathcal{S}$ by means of a short exact sequence.

It is a well known result that if V is an irreducible $\mathfrak{g}$-module and $\mathfrak{g}$ is semisimple, then $\mathcal{H}(\mathfrak{g};V)=\{0\}$ (see [4, Theorem 24.1]). In order to illustrate the results of this work, we use this and the fact that $\mathcal{T}(\mathrm{Id})\neq\mathrm{Id}$ to determine the cohomology group $\mathcal{H}(\mathfrak{g}\otimes\mathcal{S};V\otimes\mathcal{S})$, where $\mathfrak{g}$ is a semisimple Lie algebra and V is an irreducible $\mathfrak{g}$-module (see Proposition 4.5).

The results obtained in this work are focused at knowing the cohomology group $\mathcal{H}(\mathfrak{g}\otimes\mathcal{S};V\otimes\mathcal{S})$, based on the cohomology group $\mathcal{H}(\mathfrak{g};V)$. Results in the literature include those given in [6] for the first and second cohomology groups of a current Lie algebra $\mathfrak{g}\otimes\mathcal{S}$ with coefficients in a module $V\otimes\mathcal{A}$, where V is a $\mathfrak{g}$-module and $\mathcal{A}$ is an $\mathcal{S}$-module. Other results are given in [7] (Theorem 2.1) for the second cohomology group of $\mathfrak{g}\otimes\mathcal{S}$ with coefficients in the trivial module and $\mathcal{S}$ has no unit. A description of the cohomology group $\mathcal{H}(\mathfrak{g}\otimes\mathcal{S};\mathcal{V})$, where $\mathcal{V}$ is a trivial $\mathfrak{g}\otimes\mathcal{S}$ is given in [5]. It seems that one of the first results with this focus appears in [1], where it is shown that cohomology of $\mathfrak{g}\otimes\mathcal{S}$, where $\mathcal{S}$ is a local algebra, can be reduced to cohomology of $\mathfrak{g}$. On the other hand, it is unknown if there exists a criterion for recognizing whether an arbitrary Lie algebra is a current Lie algebra. A step in this direction can be found in [3], where examples in 4-dimensional current Lie algebras are given. All vector spaces considered in this work are finite dimensional over a unique field $\mathbb{F}$ of zero characteristic.

2. The map $\mathcal{L} : C(\mathfrak{g} \otimes \mathcal{S}; V) \to C(\mathfrak{g}; V)$

The proof of the following result is standard and we omit it.

Proposition 2.1. *Let V and $\mathcal{S}$ be finite dimensional vector spaces over $\mathbb{F}$. Let $\{s_1, \ldots, s_m\}$ be a basis of $\mathcal{S}$. For any X in $V \otimes \mathcal{S}$, there are unique elements $v_1, \ldots, v_m$ in V such that $X = v_1 \otimes s_1 + \ldots + v_m \otimes s_m$.*

Let $\mathfrak{g} \otimes \mathcal{S}$ be the current Lie algebra of $\mathfrak{g}$ by $\mathcal{S}$, where $\mathcal{S}$ is an m-dimensional commutative and associative algebra with unit 1 over $\mathbb{F}$. We use the same symbol for the bracket on $\mathfrak{g} \otimes \mathcal{S}$ and the bracket on $\mathfrak{g}$, i.e., $[x \otimes s, y \otimes t] = [x, y] \otimes st$ for all x, y in $\mathfrak{g}$ and s, t in $\mathcal{S}$.

We fix a basis $\{s_1, \ldots, s_m\}$ of $\mathcal{S}$, where $s_1 = 1$. Let $X_1, \ldots, X_p$ be in $\mathfrak{g} \otimes \mathcal{S}$ and Λ in $C^p(\mathfrak{g} \otimes \mathcal{S}; V \otimes \mathcal{S})$, where $p > 0$. Since $\Lambda(X_1, \ldots, X_p)$ lies in $V \otimes \mathcal{S}$, by Proposition 2.1, we write $\Lambda(X_1, \ldots, X_p)$ as follows:

$$\Lambda(X_1, \ldots, X_p) = \Lambda_1(X_1, \ldots, X_p) \otimes s_1 + \ldots + \Lambda_m(X_1, \ldots, X_p) \otimes s_m, \tag{3}$$

where $\Lambda_j(X_1, \cdots, X_p)$ belongs to V for all j. As Λ is in $C^p(\mathfrak{g} \otimes \mathcal{S}; V \otimes \mathcal{S})$, the map $(X_1, \ldots, X_p) \mapsto \Lambda_j(X_1, \ldots, X_p)$ belongs to $C^p(\mathfrak{g} \otimes \mathcal{S}; V)$. We denote this map by Λ_j for all $1 \leq j \leq m$.

Let $\{\omega_1, \ldots, \omega_m\} \subset \mathcal{S}^*$ be the dual basis of $\{s_1, \ldots, s_m\}$. For each j, the bilinear map $(v, s) \mapsto \omega_j(s)v$ yields the linear map $\widehat{\omega}_j : V \otimes \mathcal{S} \to V$, $v \otimes s \mapsto \omega_j(s)\, v$. By Proposition 2.1, we can write any X in $V \otimes \mathcal{S}$, as $X = v_1 \otimes s_1 + \ldots + v_m \otimes s_m$, then $v_j = \widehat{\omega}_j(X)$. Similarly, if Λ is in $C^p(\mathfrak{g} \otimes \mathcal{S}; V \otimes \mathcal{S})$, where $p > 0$, by (3) it follows $\Lambda_j = \widehat{\omega}_j \circ \Lambda$ for all $1 \leq j \leq m$.

For each j, define the map $\chi_j : C(\mathfrak{g} \otimes \mathcal{S}; V \otimes \mathcal{S}) \to C(\mathfrak{g} \otimes \mathcal{S}; V)$ by

$$\begin{aligned} &\chi_j(\Lambda) = \widehat{\omega}_j \circ \Lambda, \quad \text{for } \Lambda \in C^p(\mathfrak{g} \otimes \mathcal{S}; V \otimes \mathcal{S}),\ p > 0, \text{ and} \\ &\chi_j(v \otimes s) = \widehat{\omega}_j(v \otimes s), \quad \text{for } v \in V \text{ and } s \in \mathcal{S}. \end{aligned} \tag{4}$$

Then for any Λ in $C^p(\mathfrak{g} \otimes \mathcal{S}; V \otimes \mathcal{S})$, where $p \geq 0$, we have

$$\begin{aligned} &\text{If } \Lambda(X_1, \ldots, X_p) = \Lambda_1(X_1, \ldots, X_p) \otimes s_1 + \ldots + \Lambda_m(X_1, \ldots, X_p) \otimes s_m, \\ &\text{then } \Lambda_j = \chi_j(\Lambda) \quad \text{for each } 1 \leq j \leq m. \end{aligned} \tag{5}$$

Let $\mathcal{L} : C(\mathfrak{g} \otimes \mathcal{S}; V) \to C(\mathfrak{g}; V)$ be the map defined by

$$\begin{aligned} &\mathcal{L}(v) = v, \text{ for all } v \in V, \text{ and} \\ &\mathcal{L}(\lambda)(x_1, \ldots, x_p) = \lambda(x_1 \otimes 1, \ldots, x_p \otimes 1), \end{aligned} \tag{6}$$

where λ is in $C^p(\mathfrak{g} \otimes \mathcal{S}; V)$, and $x_1, \ldots, x_p$ are in $\mathfrak{g}$. Let D be the differential in $C(\mathfrak{g} \otimes \mathcal{S}; V \otimes \mathcal{S})$ (see (2)). In the next result, we will prove that D, d, $\mathcal{L}$ and χ_j can be inserted into a commutative diagram.

Proposition 2.2. *For each j, the following diagram is commutative*

$$\begin{array}{ccccc} C^p(\mathfrak{g}\otimes\mathcal{S};V\otimes\mathcal{S}) & \xrightarrow{\chi_j} & C^p(\mathfrak{g}\otimes\mathcal{S};V) & \xrightarrow{\mathcal{L}} & C^p(\mathfrak{g};V) \\ \downarrow{\scriptstyle \mathrm{D}} & & & & \downarrow{\scriptstyle \mathrm{d}} \\ C^{p+1}(\mathfrak{g}\otimes\mathcal{S};V\otimes\mathcal{S}) & \xrightarrow{\chi_j} & C^{p+1}(\mathfrak{g}\otimes\mathcal{S};V) & \xrightarrow{\mathcal{L}} & C^{p+1}(\mathfrak{g};V) \end{array} \tag{7}$$

That is $\mathrm{d}\circ\mathcal{L}\circ\chi_j=\mathcal{L}\circ\chi_j\circ\mathrm{D}$. *By* (5), *this is equivalent to*

$$\mathcal{L}\left((\mathrm{D}\,\Lambda)_j\right)=\mathrm{d}\,\mathcal{L}(\Lambda_j),\ \ \textit{for each } 1\leq j\leq m. \tag{8}$$

Proof. Let Λ be in $C^p(\mathfrak{g}\otimes\mathcal{S};V\otimes\mathcal{S})$ as in (5). Applying (5) to $\mathrm{D}\,\Lambda$, we obtain $(\mathrm{D}\,\Lambda)_j=\chi_j(\mathrm{D}\,\Lambda)$ for all j. Then (8) holds if and only if the diagram (7) is commutative, that is

$$\begin{aligned} &\mathcal{L}\left((\mathrm{D}\,\Lambda)_j\right)=\mathcal{L}\left(\chi_j(\mathrm{D}\,\Lambda)\right)=\mathcal{L}\circ\chi_j\circ\mathrm{D}\,\Lambda,\ \text{ and}\\ &\mathrm{d}\left(\mathcal{L}(\Lambda_j)\right)=\mathrm{d}\,\mathcal{L}(\chi_j\circ\Lambda)=\mathrm{d}\circ\mathcal{L}\circ\chi_j(\Lambda). \end{aligned} \tag{9}$$

We shall prove that $\mathcal{L}\left((\mathrm{D}\,\Lambda)_j\right)=\mathrm{d}\left(\mathcal{L}(\Lambda_j)\right)$. Let $X_i=x_i\otimes 1$, where x_i belongs to $\mathfrak{g}$ for all $1\leq i\leq p+1$. By (2), we have

$$\begin{aligned} &\mathrm{D}\,\Lambda(X_1,\dots,X_{p+1})=\sum_{i=1}^{p+1}(-1)^{i-1}R(X_i)(\Lambda(X_1,\dots,X_{\bar{i}},\dots,X_{p+1}))\\ &+\sum_{i<k}(-1)^{i+k}\Lambda([X_i,X_k],X_1,\dots,X_{\hat{i}},\dots,X_{\hat{k}},\dots,X_{p+1}). \end{aligned} \tag{10}$$

Applying (5) to Λ in (10) above, it follows

$$\begin{aligned} &\mathrm{D}\,\Lambda(X_1,\dots,X_{p+1})\\ &=\sum_{i=1}^{p+1}\sum_{j=1}^{m}(-1)^{i-1}R(X_i)\left(\Lambda_j(X_1,\dots,X_{\bar{i}},\dots,X_{p+1})\otimes s_j\right)\\ &+\sum_{i<k}\sum_{j=1}^{m}(-1)^{i+k}\Lambda_j([X_i,X_k],X_1,\dots,X_{\bar{i}},\dots,X_{\bar{k}},\dots,X_{p+1})\otimes s_j. \end{aligned} \tag{11}$$

Let us analyze each of the terms

$$\begin{aligned} &R(X_i)\left(\Lambda_j(X_1,\dots,X_{\bar{i}},\dots,X_{p+1})\otimes s_j\right),\ \ \text{and}\\ &\Lambda_j([X_i,X_k],X_1,\dots,X_{\bar{i}},\dots,X_{\bar{k}},\dots,X_{p+1})\otimes s_j \end{aligned}$$

given in (11). Applying the representation R (see (1)) and $\mathcal{L}$ (see (6)), we obtain

$$\begin{aligned} &R(X_i)\left(\Lambda_j(X_1,\dots,X_{\hat{i}},\dots,X_{p+1})\otimes s_j\right)\\ &=\rho(x_i)\left(\Lambda_j(X_1,\dots,X_{\hat{i}},\dots,X_{p+1})\right)\otimes s_j\\ &=\rho(x_i)\left(\mathcal{L}(\Lambda_j)(x_1,\dots,x_{\hat{i}},\dots,x_{p+1})\right)\otimes s_j \end{aligned} \tag{12}$$

as $\Lambda_j(X_1,\ldots,X_{\hat{i}},\ldots,X_{p+1}){=}\mathcal{L}(\Lambda_j)(x_1,\ldots,x_{\hat{i}},\ldots,x_{p+1})$. In addition,

$$\begin{aligned}&\Lambda_j([X_i,X_k],X_1,\ldots,X_{\hat{i}},\ldots,X_{\hat{k}},\ldots,X_{p+1})\\&=\mathcal{L}(\Lambda_j)([x_i,x_k],\ldots,x_{\hat{i}},\ldots,x_{\hat{k}},\ldots,x_{p+1}),\ \text{ for all } 1\le j\le m,\end{aligned}\tag{13}$$

as $[X_i,X_k]{=}[x_i,x_k]{\otimes}1$. We substitute (12)-(13) in (11), to get

$$\begin{aligned}&\mathrm{D}\,\Lambda(X_1,\ldots,X_{p+1})=\mathrm{D}\,\Lambda(x_1\otimes 1,\ldots,x_p\otimes 1)\\&=\sum_{i=1}^{p+1}\sum_{j=1}^{m}(-1)^{i-1}\rho(x_i)\left(\mathcal{L}(\Lambda_j)(x_1,\ldots,x_{\hat{i}},\ldots,x_{p+1})\right)\otimes s_j\\&+\sum_{i<k}\sum_{j=1}^{m}(-1)^{i+k}\mathcal{L}(\Lambda_j)([x_i,x_k],\ldots,x_{\hat{i}},\ldots,x_{\hat{k}},\ldots,x_{p+1})\otimes s_j.\end{aligned}\tag{14}$$

In (14) we gather the terms corresponding at each s_j and we obtain

$$\mathrm{D}\,\Lambda(x_1\otimes 1,\ldots,x_p\otimes 1)=\sum_{j=1}^{m}\mathrm{d}\left(\mathcal{L}(\Lambda_j)\right)(x_1,\ldots,x_{p+1})\otimes s_j.\tag{15}$$

On the other hand, applying (5) to $\mathrm{D}\,\Lambda$, we obtain

$$\mathrm{D}\,\Lambda(x_1\otimes 1,\ldots,x_p\otimes 1)=\sum_{j=1}^{m}\left(\mathrm{D}(\Lambda)\right)_j(x_1\otimes 1,\ldots,x_p\otimes 1)\otimes s_j.\tag{16}$$

By (6), $(\mathrm{D}\,\Lambda)_j(x_1\otimes 1,\ldots,x_p\otimes 1)=\mathcal{L}\left((\mathrm{D}\,\Lambda)_j\right)(x_1,\ldots,x_p)$. Then from (15) and (16), it follows $\mathcal{L}\left((\mathrm{D}\,\Lambda)_j\right)=\mathrm{d}\,\mathcal{L}(\Lambda_j)$ for each $1\le j\le m$. Therefore by (9), the diagram (7) is commutative. □

3. The map $\mathcal{T}:\mathcal{C}(\mathfrak{g})\to\mathcal{C}(\mathfrak{g}\otimes\mathcal{S})$

Let $\mathcal{C}(\mathfrak{g})$ be the set of cochains of $\mathfrak{g}$, i.e., $\mathcal{C}(\mathfrak{g})=\{C(\mathfrak{g};V)\,|\,V \text{ is a } \mathfrak{g}\text{-module}\}$. We define a map $\mathcal{T}:\mathcal{C}(\mathfrak{g})\to\mathcal{C}(\mathfrak{g}\otimes\mathcal{S})$ by

$$\begin{aligned}\mathcal{T}(C^p(\mathfrak{g};V))&=C^p(\mathfrak{g}\otimes\mathcal{S};V\otimes\mathcal{S}),\qquad \text{for } p>0,\ \text{ and}\\ \mathcal{T}(V)&=V\otimes\mathcal{S},\qquad \text{where } V \text{ is a } \mathfrak{g}\text{-module}.\end{aligned}\tag{17}$$

From now on, we assume that $x,x_1,\ldots,x_{p+1}$ are in $\mathfrak{g}$; $s,t,t_1,\ldots,t_{p+1}$ are in $\mathcal{S}$; u is in U, v is in V, w is in W; U,V,W are finite dimensional $\mathfrak{g}$-modules. We also consider any cochain Λ as in (5). For $t_1,\ldots,t_p$ in $\mathcal{S}$, we write $\widetilde{t}=t_1\cdots t_p$.

Given a linear map $f:\mathcal{V}\to\mathcal{W}$, where $\mathcal{V},\mathcal{W}$ are in $\mathcal{C}(\mathfrak{g})$, we shall define a linear map $\mathcal{T}(f):\mathcal{T}(\mathcal{U})\to\mathcal{T}(\mathcal{V})$. We shall consider four cases.

Case 1: Let $f:C^p(\mathfrak{g};V)\to C^p(\mathfrak{g};W)$ be a linear map. We define the linear map $\mathcal{T}(f):C^p(\mathfrak{g}\otimes\mathcal{S};V\otimes\mathcal{S})\to C^p(\mathfrak{g}\otimes\mathcal{S};W\otimes\mathcal{S})$ by

$$\mathcal{T}(f)(\Lambda)(x_1\otimes t_1,\ldots,x_p\otimes t_p)=\sum_{j=1}^{m}f\left(\mathcal{L}(\Lambda_j)\right)(x_1,\ldots,x_p)\otimes s_j\,\widetilde{t}.\tag{18}$$

Observe that by (5), we can write Λ as

$$\Lambda(x_1 \otimes t_1, \ldots, x_p \otimes t_p) = \sum_{j=1}^{m} \Lambda_j(x_1 \otimes t_1, \ldots, x_p \otimes t_p) \otimes s_j,$$

where Λ_j belongs to $C^p(\mathfrak{g} \otimes \mathcal{S}, V)$ for all j. Since $\mathcal{L}(\Lambda_j)$ belongs to $C^p(\mathfrak{g}, V)$, it makes sense to consider $f(\mathcal{L}(\Lambda_j))$ in (18) above.
Case 2: Now consider $p = 0$, $f : V \to W$ a linear map and $v \otimes s$ in $C^0(\mathfrak{g} \otimes \mathcal{S}; V \otimes \mathcal{S}) = V \otimes \mathcal{S}$. We define $\mathcal{T}(f) : V \otimes \mathcal{S} \to W \otimes \mathcal{S}$ by

$$\mathcal{T}(f)(v \otimes s) = f(v) \otimes s. \tag{19}$$

In this case we also denote $\mathcal{T}(f)$ by $f \otimes \mathcal{S}$.
Case 3: Let $f : V \to C^p(\mathfrak{g}; W)$ be a linear map. We define the linear map $\mathcal{T}(f) : V \otimes \mathcal{S} \to C^p(\mathfrak{g} \otimes \mathcal{S}; V \otimes \mathcal{S})$ by

$$\mathcal{T}(f)(v \otimes s)(x_1 \otimes t_1, \ldots, x_p \otimes t_p) = f(v)(x_1, \ldots, x_p) \otimes s\,\widetilde{t}. \tag{20}$$

Case 4: Let $f : C^p(\mathfrak{g}; V) \to W$ be a linear map. We define the linear map $\mathcal{T}(f) : C^p(\mathfrak{g} \otimes \mathcal{S}; V \otimes \mathcal{S}) \to W \otimes \mathcal{S}$ by

$$\mathcal{T}(f)(\Lambda) = f(\mathcal{L}(\Lambda_1)) \otimes s_1 + \ldots + f(\mathcal{L}(\Lambda_m)) \otimes s_m. \tag{21}$$

Remark 3.1. If Id is the identity map on $C^p(\mathfrak{g}; V)$, then $\mathcal{T}(\mathrm{Id})$ is not the identity map on $C^p(\mathfrak{g} \otimes \mathcal{S}; V \otimes \mathcal{S})$. Indeed, let Λ be in $C^p(\mathfrak{g} \otimes \mathcal{S}; V \otimes \mathcal{S})$. By definition of $\mathcal{L}$ and (18), it follows

$$\begin{aligned} \mathcal{T}(\mathrm{Id})(\Lambda)(x_1 \otimes t_1, \ldots, x_p \otimes t_p) &= \sum_{j=1}^{m} \mathrm{Id}\left(\mathcal{L}(\Lambda_j)\right)(x_1, \ldots, x_p) \otimes s_j\,\widetilde{t} \\ = \sum_{j=1}^{m} \mathcal{L}(\Lambda_j)(x_1, \ldots, x_p) \otimes s_j \widetilde{t} &= \sum_{j=1}^{m} \Lambda_j(x_1 \otimes 1, \ldots, x_p \otimes 1) \otimes s_j\,\widetilde{t}. \end{aligned} \tag{22}$$

Then $\mathcal{T}(\mathrm{Id})(\Lambda) = \Lambda$ if and only if

$$\Lambda(x_1 \otimes t_1, \ldots, x_p \otimes t_p) = \sum_{j=1}^{m} \Lambda_j(x_1 \otimes 1, \ldots, x_p \otimes 1) \otimes s_j\,\widetilde{t}. \tag{23}$$

As we mentioned in the introduction, we will determine the cohomology group $\mathcal{H}(\mathfrak{g} \otimes \mathcal{S}; V \otimes \mathcal{S})$, where V is an irreducible $\mathfrak{g}$-module and $\mathfrak{g}$ is a semisimple Lie algebra (see Proposition 4.5). Apart from the fact that in this case $\mathcal{H}(\mathfrak{g}; V) = \{0\}$, we find this case interesting to apply our results since the condition given in (23), is exactly the fact that helps to determine the cohomology group $\mathcal{H}(\mathfrak{g} \otimes \mathcal{S}; V \otimes \mathcal{S})$.

In the next result, we prove that $\mathcal{T}$ preserves the composition of maps.

Proposition 3.2. *The cochain complexes maps* $f : C^p(\mathfrak{g}; V) \to C^p(\mathfrak{g}; V)$ *and* $g : C^p(\mathfrak{g}; V) \to C^p(\mathfrak{g}; W)$ *yield a map* $\mathcal{T}(g \circ f)$ *from* $C^p(\mathfrak{g} \otimes \mathcal{S}; U \otimes \mathcal{S})$ *to* $C^p(\mathfrak{g} \otimes \mathcal{S}; W \otimes \mathcal{S})$ *satisfying* $\mathcal{T}(f \circ g) = \mathcal{T}(f) \circ \mathcal{T}(g)$.

We shall verify that if $f : \mathcal{U} \to \mathcal{V}$ and $g : \mathcal{V} \to \mathcal{W}$ are maps in $\mathcal{C}(\mathfrak{g})$, then $\mathcal{T}(g \circ f) = \mathcal{T}(g) \circ \mathcal{T}(f) : \mathcal{T}(\mathcal{U}) \to \mathcal{T}(\mathcal{W})$. Several cases should be considered and we only will prove one of them. The proof of the remaining cases uses the same arguments.

Claim 1. *Let* $f : C^p(\mathfrak{g}; U) \to C^p(\mathfrak{g}; V)$ *and* $g : C^p(\mathfrak{g}; V) \to C^p(\mathfrak{g}; W)$ *be maps, then* $\mathcal{T}(g \circ f) = \mathcal{T}(g) \circ \mathcal{T}(f)$ *is a map between* $C^p(\mathfrak{g} \otimes \mathcal{S}; U \otimes \mathcal{S})$ *and* $C^p(\mathfrak{g} \otimes \mathcal{S}; W \otimes \mathcal{S})$.

Proof. Let Λ be in $C^p(\mathfrak{g} \otimes \mathcal{S}; U \otimes \mathcal{S})$, and $\Theta = \mathcal{T}(f)(\Lambda)$. By (18), we have

$$\mathcal{T}(g)(\Theta)(x_1 \otimes t_1, \cdots, x_p \otimes t_p) = \sum_{j=1}^{m} g(\mathcal{L}(\Theta_j))(x_1, \ldots, x_p) \otimes s_j \,\widetilde{t}, \tag{24}$$

where $\Theta_j = \widehat{\omega}_j \circ \mathcal{T}(f)(\Lambda)$ (see (5)). We claim that $\mathcal{L}(\Theta_j) = f(\mathcal{L}(\Lambda_j))$. Indeed, using the definition of $\mathcal{L}$ and applying (18) to $\mathcal{T}(f)(\Lambda)$, we get

$$\begin{aligned} &\mathcal{L}(\Theta_j)(x_1, \ldots, x_p) = \Theta_j(x_1 \otimes 1, \ldots, x_p \otimes 1) \\ &= \widehat{\omega}_j \circ \mathcal{T}(f)(\Lambda)(x_1 \otimes 1, \ldots, x_p \otimes 1) \\ &= \widehat{\omega}_j \left(\sum_{k=1}^{m} f\left(\mathcal{L}(\Lambda_k)\right)(x_1, \ldots, x_p) \otimes s_k \right) = f(\mathcal{L}(\Lambda_j))(x_1, \ldots, x_p). \end{aligned}$$

Then $\mathcal{L}(\Theta_j) = f(\mathcal{L}(\Lambda_j))$, for all j. Substituting this in (24), we obtain

$$\begin{aligned} \mathcal{T}(g) \circ \mathcal{T}(f)(\Lambda)(x_1 \otimes t_1, \ldots, x_p \otimes t_p) &= \sum_{j=1}^{m} (g \circ f)(\mathcal{L}(\Lambda_j))(x_1, \ldots, x_p) \otimes s_j \,\widetilde{t}, \\ &= \mathcal{T}(g \circ f)(\Lambda)(x_1 \otimes t_1, \cdots, x_p \otimes t_p). \end{aligned}$$

In the last step above we use (18). Thus $\mathcal{T}(g \circ f) = \mathcal{T}(g) \circ \mathcal{T}(f)$. $\square$

Proposition 3.3. *Let* $f : C(\mathfrak{g}; V) \to C(\mathfrak{g}; W)$ *be a map of complexes, that is* $\mathrm{d} \circ f = f \circ \mathrm{d}$ *and* $f(C^p(\mathfrak{g}; V)) \subset C^p(\mathfrak{g}; W)$ *for all* $p \geq 0$. *Then* $\mathcal{T}(f) : C(\mathfrak{g} \otimes \mathcal{S}; V \otimes \mathcal{S}) \to C(\mathfrak{g} \otimes \mathcal{S}; W \otimes \mathcal{S})$ *is a map of complexes.*

Proof. To shorten the length of expressions, we will use the notation:

$$\begin{aligned} &\mathbf{x}\,\mathbf{t} = (x_1 \otimes t_1, \ldots, x_{p+1} \otimes t_{p+1}), \\ &(\mathbf{x}\,\mathbf{t})_i = (x_1 \otimes t_1, \ldots, x_{\hat{i}} \otimes t_{\hat{i}}, \ldots, x_{p+1} \otimes t_{p+1}), \\ &(\mathbf{x}\,\mathbf{t})_{i,j} = (x_1 \otimes t_1, \ldots, x_{\hat{i}} \otimes t_{\hat{i}}, \ldots, x_{\hat{j}} \otimes t_{\hat{j}}, \ldots, x_{p+1} \otimes t_{p+1}), \\ &\mathbf{x} = (x_1, \ldots, x_{p+1}), \qquad \mathbf{x}_i = (x_1, \ldots, x_{\hat{i}}, \ldots, x_{p+1}), \\ &\mathbf{x}_{i,j} = (x_1, \ldots, x_{\hat{i}}, \ldots, x_{\hat{j}}, \ldots, x_{p+1}). \end{aligned} \tag{25}$$

Let Λ be in $C^p(\mathfrak{g}\otimes\mathcal{S}; V\otimes\mathcal{S})$, $p>0$. We shall prove that $\mathrm{D}\circ\mathcal{T}(f)(\Lambda)=\mathcal{T}(f)\circ\mathrm{D}\,\Lambda$. Indeed, first we apply D to $\mathcal{T}(f)(\Lambda)$:

$$\begin{aligned}\mathrm{D}\,\mathcal{T}(f)(\Lambda)(\mathbf{x}\,\mathbf{t}) &= \sum_{i=1}^{p+1}(-1)^{i-1}R(x_i\otimes t_i)\,(\mathcal{T}(f)(\Lambda))\,((\mathbf{x}\,\mathbf{t})_i)\\ &+\sum_{i<j}(-1)^{i+j}\mathcal{T}(f)(\Lambda)\,([x_i\otimes t_i, x_j\otimes t_j],(\mathbf{x}\,\mathbf{t})_{i,j}).\end{aligned}\tag{26}$$

We write A and B to denote the first and second term in (26), respectively, i.e., $\mathrm{D}\,\mathcal{T}(f)(\Lambda)(\mathbf{x}\,\mathbf{t})=A+B$. Applying (18) to $\mathcal{T}(f)(\Lambda)$ in A, we obtain

$$A=\sum_{i=1}^{p+1}\sum_{k=1}^{m}(-1)^{i-1}R(x_i\otimes t_i)\left(f(\mathcal{L}(\Lambda_k))(\mathbf{x}_i)\otimes s_k\,\hat{t}_i\right),\tag{27}$$

where $\hat{t}_i=t_1\cdots t_{i-1}t_{i+1}\cdots t_{p+1}$. In (27) above, we apply $R(x_i\otimes t_i)$ to $f(\mathcal{L}(\Lambda_k))(\mathbf{x}_i)\otimes s_k\,\hat{t}_i$ (see (1)), and we get

$$A=\sum_{i=1}^{p+1}\sum_{k=1}^{m}(-1)^{i-1}\rho(x_i)\,(f\,(\mathcal{L}(\Lambda_k))\,(\mathbf{x}_i)\otimes s_k\,\widetilde{t},\tag{28}$$

because $\widetilde{t}=t_i\hat{t}_i$. Regarding to B, we fix $i<j$; by (25), we have

$$\begin{aligned}&\mathcal{T}(f)(\Lambda)\,([x_i\otimes t_i, x_j\otimes t_j],(\mathbf{x}\,\mathbf{t})_{i,j})\\ &=\mathcal{T}(f)(\Lambda)\left([x_i,x_j]\otimes t_it_j, x_1\otimes t_1,\dots,x_{\hat{i}}\otimes t_{\hat{i}},\dots,x_{\hat{j}}\otimes t_{\hat{j}},\dots,x_{p+1}\otimes t_{p+1}\right)\\ &=\sum_{k=1}^{m}f\,(\mathcal{T}(\Lambda_k))\,([x_i,x_j],\mathbf{x}_{i,j})\otimes s_k\,\widetilde{t}\quad\text{(we use (18))}\end{aligned}$$

because $\widetilde{t}=(t_it_j)(t_1\cdots t_{\hat{i}}\cdots t_{\hat{j}}\cdots t_{p+1})$. Hence,

$$B=\sum_{i<j}\sum_{k=1}^{m}(-1)^{i+j}f\,(\mathcal{L}(\Lambda_k))\,([x_i,x_j],\mathbf{x}_{i,j})\otimes s_k\,\widetilde{t}.\tag{29}$$

From (28) and (29), it follows:

$$\mathrm{D}\,\mathcal{T}(f)(\Lambda)(\mathbf{x}\,\mathbf{t})=A+B=\sum_{k=1}^{m}\mathrm{d}\,(f\,(\mathcal{L}(\Lambda_k)))\,(\mathbf{x})\otimes s_k\widetilde{t}.\tag{30}$$

By hypothesis, f is a map of complex, then $f\circ\mathrm{d}=\mathrm{d}\circ f$. By (8), $\mathrm{d}(\mathcal{L}(\Lambda_k))=\mathcal{L}((\mathrm{D}\,\Lambda)_k)$, then by (30), we get

$$\begin{aligned}\mathrm{D}\,\mathcal{T}(f)(\Lambda)(\mathbf{x}\,\mathbf{t}) &= \sum_{k=1}^{m}f(\mathrm{d}(\mathcal{L}(\Lambda_k)))(\mathbf{x})\otimes s_k\,\widetilde{t}\\ =\sum_{k=1}^{m}f\,((\mathcal{L}(\mathrm{D}\,\Lambda)_k))\,(\mathbf{x})\otimes s_k\,\widetilde{t} &= \mathcal{T}(f)(\mathrm{D}\,\Lambda)(\mathbf{x}\,\mathbf{t}).\ \text{(We use (18).)}\end{aligned}$$

As Λ is arbitrary, it follows $\mathrm{D} \circ \mathcal{T}(f) = \mathcal{T}(f) \circ \mathrm{D}$. The proof of the case $p = 0$ uses the same arguments. □

4. The map $\alpha : \mathcal{H}(\mathfrak{g} \otimes \mathcal{S}; V \otimes \mathcal{S}) \to \mathcal{H}(\mathfrak{g}; V) \otimes \mathcal{S}$

Let V be a $\mathfrak{g}$-module. We denote the group of cocycles and coboundaries of $C(\mathfrak{g}; V)$, by $\mathcal{Z}$ and $\mathcal{B}$, respectively. The cohomology group of $\mathfrak{g}$ with coefficients in V is denoted by $\mathcal{H}(\mathfrak{g}; V)$. The quotient $C(\mathfrak{g}; V)/\mathcal{B}$ is denoted by $\mathcal{Z}'$ and $C(\mathfrak{g}; V)/\mathcal{Z}$ is denoted by $\mathcal{B}'$.

By [4, Chapter IV, §23], $\mathcal{Z}$ and $\mathcal{B}$ are $\mathfrak{g}$-modules, then $\mathcal{Z}', \mathcal{B}'$ and $\mathcal{H}(\mathfrak{g}; V)$ are $\mathfrak{g}$-modules. Moreover, as in the classical and standard way, the $\mathfrak{g}$-modules $\mathcal{Z}, \mathcal{B}, \mathcal{Z}', \mathcal{B}'$ and $\mathcal{H}(\mathfrak{g}; V)$ will be regarded as modules with zero differentiation (see [2, Chapter IV, §1]).

Lemma 4.1. *For $p = 0$, $\mathcal{H}^0(\mathfrak{g} \otimes \mathcal{S}; V \otimes \mathcal{S}) = \mathcal{H}^0(\mathfrak{g}; V) \otimes \mathcal{S}$.*

Proof. For $p = 0$, we have $\mathcal{Z}^0 = \mathcal{H}^0(\mathfrak{g}; V)$. Let $\bar{v}$ be an element in $\mathcal{H}^0(\mathfrak{g} \otimes \mathcal{S}; V \otimes \mathcal{S}) \subset V \otimes \mathcal{S}$. We write $\bar{v} = v_1 \otimes s_1 + \ldots + v_m \otimes s_m$, where v_j belongs to V for all $1 \leq j \leq m$ (see Proposition 2.1). Then

$$\begin{aligned} 0 = \mathrm{D}(\bar{v})(x \otimes 1) = R(x \otimes 1)(\bar{v}) & \\ = \rho(x)(v_1) \otimes s_1 + \ldots + \rho(x)(v_m) \otimes s_m & \qquad (31) \\ = \mathrm{d}(v_1)(x) \otimes s_1 + \ldots + \mathrm{d}(v_m)(x) \otimes s_m. & \end{aligned}$$

Whence $\mathrm{d}(v_j) = 0$ and v_j belongs to $\mathcal{H}^0(\mathfrak{g}; V)$ for all j, which implies that $\bar{v}$ belongs to $\mathcal{H}^0(\mathfrak{g}; V) \otimes \mathcal{S}$. Hence, $\mathcal{H}^0(\mathfrak{g} \otimes \mathcal{S}; V \otimes \mathcal{S}) \subset \mathcal{H}^0(\mathfrak{g}; V) \otimes \mathcal{S}$.

Let $\bar{v}$ be in $\mathcal{H}^0(\mathfrak{g}; V) \otimes \mathcal{S} \subset V \otimes \mathcal{S}$. By Proposition 2.1, there are $v_1, \ldots, v_m$ in $\mathcal{H}^0(\mathfrak{g}; V)$ such that $\bar{v} = v_1 \otimes s_1 + \ldots + v_m \otimes s_m$. As each v_j belongs to $\mathcal{H}^0(\mathfrak{g}; v) = \mathcal{Z}^0$, then $\mathrm{d}(v_j) = 0$. Thus,

$$\begin{aligned} \mathrm{D}(\bar{v})(x \otimes s) = R(x \otimes s)(\bar{v}) = \sum_{j=1}^{m} R(x \otimes s)(v_j \otimes s_j) \\ = \sum_{j=1}^{m} \rho(x)(v_j) \otimes s s_j = \sum_{j=1}^{m} \mathrm{d}(v_j)(x) \otimes s s_j = 0. \end{aligned}$$

Hence $\mathrm{D}(\bar{v}) = 0$, and $\bar{v}$ is in $\mathcal{H}^0(\mathfrak{g} \otimes \mathcal{S}; V \otimes \mathcal{S})$. Therefore $\mathcal{H}^0(\mathfrak{g} \otimes \mathcal{S}; V \otimes \mathcal{S}) = \mathcal{H}^0(\mathfrak{g}; V) \otimes \mathcal{S}$. □

Let $\iota : \mathcal{Z} \to C(\mathfrak{g}; V)$ be the inclusion map. By (20), we get a map $\mathcal{T}(\iota) : \mathcal{Z} \otimes \mathcal{S} \to C(\mathfrak{g} \otimes \mathcal{S}; V \otimes \mathcal{S})$. Since $\mathcal{Z} \otimes \mathcal{S}$ has zero differential, we can define a map Φ from

$\mathcal{Z}\otimes\mathcal{S}$ into $\mathcal{H}(\mathfrak{g}\otimes\mathcal{S};V\otimes\mathcal{S})$, by

$$\begin{array}{rccl} \Phi: & \mathcal{Z}\otimes\mathcal{S} & \longrightarrow & \mathcal{H}(\mathfrak{g}\otimes\mathcal{S};V\otimes\mathcal{S}) \\ & \bar{x} & \mapsto & \mathcal{T}(\iota)(\bar{x})+\mathcal{B}(\mathfrak{g}\otimes\mathcal{S};V\otimes\mathcal{S}). \end{array} \tag{32}$$

Consider $\pi' : C(\mathfrak{g};V)\to\mathcal{Z}'$ defined by $\pi'(\lambda)=\lambda+\mathcal{B}$. By (21), we get a map $\mathcal{T}(\pi') : C(\mathfrak{g}\otimes\mathcal{S};V\otimes\mathcal{S})\to\mathcal{Z}'\otimes\mathcal{S}$. As $\mathcal{Z}'\otimes\mathcal{S}$ has zero differential, we can define a map Ψ from $\mathcal{H}(\mathfrak{g}\otimes\mathcal{S};V\otimes\mathcal{S})$ into $\mathcal{Z}'\otimes\mathcal{S}$ by

$$\begin{array}{rccl} \Psi: & \mathcal{H}(\mathfrak{g}\otimes\mathcal{S};V\otimes\mathcal{S}) & \longrightarrow & \mathcal{Z}'\otimes\mathcal{S} \\ & \Lambda+\mathcal{B}(\mathfrak{g}\otimes\mathcal{S};V\otimes\mathcal{S}) & \mapsto & \mathcal{T}(\pi')(\Lambda)=\sum_{j=1}^{m}(\mathcal{L}(\Lambda_j)+\mathcal{B})\otimes s_j. \end{array} \tag{33}$$

We shall prove that Ψ is well-defined. Let σ be in $C(\mathfrak{g}\otimes\mathcal{S};V\otimes\mathcal{S})$. Using (8) and (21), as well as $\pi'\circ\mathrm{d}=0$, we obtain

$$\begin{aligned} \mathcal{T}(\pi')(\mathrm{D}\,\sigma) &= \pi'\left(\mathcal{L}((\mathrm{D}\,\sigma)_1)\right)\otimes s_1+\ldots+\pi'\left(\mathcal{L}((\mathrm{D}\,\sigma)_m)\right)\otimes s_m \\ &= \pi'\left(\mathrm{d}\,\mathcal{L}(\sigma_1)\right)\otimes s_1+\ldots+\pi'\left(\mathrm{d}\,\mathcal{L}(\sigma_m)\right)\otimes s_m=0. \end{aligned}$$

Then $\mathcal{B}(\mathfrak{g}\otimes\mathcal{S};V\otimes\mathcal{S})\subset\mathrm{Ker}(\mathcal{T}(\pi'))$, hence Ψ is well-defined.

Let $\pi : \mathcal{Z}\to\mathcal{H}(\mathfrak{g};V)$ be the projection map and $\iota' : \mathcal{H}(\mathfrak{g};V)\to\mathcal{Z}'$ be the inclusion map. In the next result we will prove that there exists a surjective linear map α between $\mathcal{H}(\mathfrak{g}\otimes\mathcal{S};V\otimes\mathcal{S})$ and $\mathcal{H}(\mathfrak{g};V)\otimes\mathcal{S}$.

Proposition 4.2. *Let $\mathfrak{g}\otimes\mathcal{S}$ be the current Lie algebra of $\mathfrak{g}$ by $\mathcal{S}$.*

(i) *For any $\mathfrak{g}$-module V, there exists a unique surjective linear map $\alpha:\mathcal{H}(\mathfrak{g}\otimes\mathcal{S};V\otimes\mathcal{S})\to\mathcal{H}(\mathfrak{g};V)\otimes\mathcal{S}$ that makes commutative the diagram*

$$\begin{array}{ccc} \mathcal{Z}\otimes\mathcal{S} & \xrightarrow{\ \pi\otimes\mathcal{S}\ } & \mathcal{H}(\mathfrak{g};V)\otimes\mathcal{S} \\ {\scriptstyle\Phi}\downarrow & \nearrow{\scriptstyle\alpha} & \downarrow{\scriptstyle\iota'\otimes\mathcal{S}} \\ \mathcal{H}(\mathfrak{g}\otimes\mathcal{S};V\otimes\mathcal{S}) & \xrightarrow{\ \Psi\ } & \mathcal{Z}'\otimes\mathcal{S} \end{array} \tag{34}$$

(ii) *Let $f : C(\mathfrak{g};V)\to C(\mathfrak{g};W)$ be a map of complexes and consider $\mathcal{H}(f) : \mathcal{H}(\mathfrak{g};V)\to\mathcal{H}(\mathfrak{g};W)$ the map induced by f, i.e., $\mathcal{H}(f)(\lambda+\mathcal{B})=f(\lambda)+\mathcal{B}$. Then the following diagram is commutative*

$$\begin{array}{ccc} \mathcal{H}(\mathfrak{g}\otimes\mathcal{S};V\otimes\mathcal{S}) & \xrightarrow{\ \mathcal{H}(\mathcal{T}(f))\ } & \mathcal{H}(\mathfrak{g}\otimes\mathcal{S};W\otimes\mathcal{S}) \\ {\scriptstyle\alpha_V}\downarrow & & \downarrow{\scriptstyle\alpha_W} \\ \mathcal{H}(\mathfrak{g};V)\otimes\mathcal{S} & \xrightarrow{\ \mathcal{H}(f)\otimes\mathcal{S}=\mathcal{T}(\mathcal{H}(f))\ } & \mathcal{H}(\mathfrak{g};W)\otimes\mathcal{S} \end{array} \tag{35}$$

where $\mathcal{H}(\mathcal{T}(f))$ is the map induced by $\mathcal{T}(f)$.

Proof. (i) Let $\eta : \mathcal{B} \to \mathcal{Z}$ be the inclusion map and $\zeta : \mathcal{Z}' \to \mathcal{B}'$ be the map defined by $\zeta(\lambda + \mathcal{B}) = \lambda + \mathcal{Z}$. We have the following short exact sequences

$$0 \longrightarrow \mathcal{H}(\mathfrak{g}; V) \xrightarrow{\iota'} \mathcal{Z}' \xrightarrow{\zeta} \mathcal{B}' \longrightarrow 0,$$

$$0 \longrightarrow \mathcal{B} \xrightarrow{\eta} \mathcal{Z} \xrightarrow{\pi} \mathcal{H}(\mathfrak{g}; V) \longrightarrow 0.$$

Since $\mathcal{S}$ is finite dimensional, the following sequences are exact

$$0 \longrightarrow \mathcal{H}(\mathfrak{g}; V) \otimes \mathcal{S} \xrightarrow{\iota' \otimes \mathcal{S}} \mathcal{Z}' \otimes \mathcal{S} \xrightarrow{\zeta \otimes \mathcal{S}} \mathcal{B}' \otimes \mathcal{S} \longrightarrow 0,$$

$$0 \longrightarrow \mathcal{B} \otimes \mathcal{S} \xrightarrow{\eta \otimes \mathcal{S}} \mathcal{Z} \otimes \mathcal{S} \xrightarrow{\pi \otimes \mathcal{S}} \mathcal{H}(\mathfrak{g}; V) \otimes \mathcal{S} \longrightarrow 0.$$

Whence, $\pi \otimes \mathcal{S}$ is surjective and $\iota' \otimes \mathcal{S}$ is injective. Since $\iota' \circ \pi = \pi' \circ \iota$, Proposition 3.2 leads to the commutativity of the following diagram:

$$\begin{array}{ccc} \mathcal{Z} \otimes \mathcal{S} & \xrightarrow{\pi \otimes \mathcal{S}} & \mathcal{H}(\mathfrak{g}; V) \otimes \mathcal{S} \\ {\scriptstyle \mathcal{T}(\iota)} \downarrow & & \downarrow {\scriptstyle \iota' \otimes \mathcal{S}} \\ C(\mathfrak{g} \otimes \mathcal{S}; V \otimes \mathcal{S}) & \xrightarrow{\mathcal{T}(\pi')} & \mathcal{Z}' \otimes \mathcal{S} \end{array} \tag{36}$$

By (32) and (33), we have the commutative diagram

$$\begin{array}{cccc} & & 0 & \\ & & \downarrow & \\ \mathcal{Z} \otimes \mathcal{S} & \xrightarrow{\pi \otimes \mathcal{S}} & \mathcal{H}(\mathfrak{g}; V) \otimes \mathcal{S} & \longrightarrow 0 \\ {\scriptstyle \Phi} \downarrow & & \downarrow {\scriptstyle \iota' \otimes \mathcal{S}} & \\ \mathcal{H}(\mathfrak{g} \otimes \mathcal{S}; V \otimes \mathcal{S}) & \xrightarrow{\Psi} & \mathcal{Z}' \otimes \mathcal{S} & \end{array} \tag{37}$$

If α and α' make commutative the diagram (37), then $(\iota' \otimes \mathcal{S}) \circ \alpha = (\iota' \otimes \mathcal{S}) \circ \alpha'$. As $\iota' \otimes \mathcal{S}$ is injective, it follows $\alpha = \alpha'$.

Claim 2. *For $p = 0$, $\alpha : \mathcal{H}(\mathfrak{g} \otimes \mathcal{S}; V \otimes \mathcal{S}) \to \mathcal{H}(\mathfrak{g}, V) \otimes \mathcal{S}$ is the identity map.*

Proof. For $p = 0$, $T(\iota) = \iota \otimes \mathcal{S}$ because $\iota : \mathcal{Z}^0 \to C^0(\mathfrak{g}; V)$ and $V = C^0(\mathfrak{g}; V)$ (see (19)). Then $\mathrm{Im}(\mathcal{T}(\iota)) = \mathrm{Im}(\iota \otimes \mathcal{S}) = \mathcal{Z}^0 \otimes \mathcal{S} = \mathcal{H}^0(\mathfrak{g}; V) \otimes \mathcal{S} = \mathcal{H}^0(\mathfrak{g} \otimes \mathcal{S}; V \otimes \mathcal{S})$. Hence, by (32), $\Phi^0 : \mathcal{Z}^0 \otimes \mathcal{S} \to \mathcal{H}^0(\mathfrak{g} \otimes \mathcal{S}; V \otimes \mathcal{S})$ is the identity map.

Similarly for $p = 0$, $\pi' : C(\mathfrak{g}; V) \to \mathcal{Z}'$ is the identity, as $C^0(\mathfrak{g}, V) = V$ and $\mathcal{Z}'^0 = V$. Since both V and $\mathcal{Z}'$ have zero differential, by (19), $\mathcal{T}(\pi') = \mathrm{Id}_V \otimes \mathcal{S} = \mathrm{Id}_{V \otimes \mathcal{S}}$ is the identity on $V \otimes \mathcal{S}$. Therefore, $\Psi^0 : \mathcal{H}^0(\mathfrak{g} \otimes \mathcal{S}; V \otimes \mathcal{S}) \to \mathcal{Z}'^0 \otimes \mathcal{S}$ is the inclusion (see (33)).

For $p = 0$, the map $\pi \otimes \mathcal{S} : \mathcal{Z}^0 \otimes \mathcal{S} \to \mathcal{H}^0(\mathfrak{g}; V) \otimes \mathcal{S}$ is the identity, as $\mathcal{Z}^0 = \mathcal{H}^0(\mathfrak{g}, V)$. Similarly, $\iota' \otimes \mathcal{S} : \mathcal{H}^0(\mathfrak{g}; V) \otimes \mathcal{S} \to \mathcal{Z}'^0 \otimes \mathcal{S}$ is the inclusion map, as $\mathcal{Z}'^0 = C^0(\mathfrak{g}, V)/\mathcal{B}^0(\mathfrak{g}, V) = V$.

In summary, for $p = 0$, we have that Φ^0 and $\pi \otimes \mathcal{S}$ are the identity maps while Ψ^0 and $\iota' \otimes \mathcal{S}$ are the inclusion maps. Since any map that makes commutative (34) is unique, we deduce that α is the identity for $p = 0$. □

Now we shall consider $p > 0$. If $\mathrm{Im}(\Psi) \subset \mathrm{Im}\,(\iota' \otimes \mathcal{S}) = \mathrm{Ker}\,(\zeta \otimes \mathcal{S})$, then there exists a map α between $\mathcal{H}(\mathfrak{g} \otimes \mathcal{S}; V \otimes \mathcal{S})$ and $\mathcal{H}(\mathfrak{g}; V) \otimes \mathcal{S}$. We shall now prove this assertion.

Claim 3. *The composition* $(\zeta \otimes \mathcal{S}) \circ \Psi$ *is zero.*

Proof. For $p > 0$, let Λ be in $C^p(\mathfrak{g} \otimes \mathcal{S}; V \otimes \mathcal{S})$ such that $\mathrm{D}\,\Lambda = 0$. By (8), $\mathcal{L}(\Lambda_j)$ belongs to $\mathcal{Z}$ for all j. By (19) and (33), we have

$$
\begin{aligned}
&(\zeta \otimes \mathcal{S}) \circ \Psi\,(\Lambda + \mathcal{B}(\mathfrak{g} \otimes \mathcal{S}; V \otimes \mathcal{S})) \\
&= (\zeta \otimes \mathcal{S})\,((\mathcal{L}(\Lambda_1) + \mathcal{B}) \otimes s_1 + \ldots + (\mathcal{L}(\Lambda_m) + \mathcal{B}) \otimes s_m) \\
&= (\mathcal{L}(\Lambda_1) + \mathcal{Z}) \otimes s_1 + \ldots + (\mathcal{L}(\Lambda_m) + \mathcal{Z}) \otimes s_m = 0.
\end{aligned}
$$

Then the composition $(\zeta \otimes \mathcal{S}) \circ \Psi$ is zero for $p > 0$. Using a similar argument, it is proved that $(\zeta \otimes \mathcal{S}) \circ \Psi = 0$ for $p = 0$. □

Claim 4. *There exists a linear map* $\alpha : \mathcal{H}(\mathfrak{g} \otimes \mathcal{S}; V \otimes \mathcal{S}) \to \mathcal{H}(\mathfrak{g}; V) \otimes \mathcal{S}$ *that makes commutative the diagram* (34).

Proof. Since $\mathrm{Im}(\Psi) \subset \mathrm{Ker}(\zeta \otimes \mathcal{S}) = \mathrm{Im}(\iota' \otimes \mathcal{S})$ (see Claim 3), for each $\overline{\Lambda}$ in $\mathcal{H}(\mathfrak{g} \otimes \mathcal{S}; V \otimes \mathcal{S})$, there exists θ in $\mathcal{H}(\mathfrak{g}; V) \otimes \mathcal{S}$ such that $\Psi(\overline{\Lambda}) = (\iota' \otimes \mathcal{S})\,(\theta)$. Since $\iota' \otimes \mathcal{S}$ is injective, θ is unique.

Define $\alpha : \mathcal{H}(\mathfrak{g} \otimes \mathcal{S}; V \otimes \mathcal{S}) \to \mathcal{H}(\mathfrak{g}; V) \otimes \mathcal{S}$, by $\alpha(\overline{\Lambda}) = \theta$, then $\Psi = (\iota' \otimes \mathcal{S}) \circ \alpha$. Since $\iota' \otimes \mathcal{S}$ is injective, $\mathrm{Ker}(\alpha) = \mathrm{Ker}(\Psi)$. As (37) is commutative, $(\iota' \otimes \mathcal{S}) \circ (\alpha \circ \Phi) = (\iota' \otimes \mathcal{S}) \circ (\pi \otimes \mathcal{S})$. Hence, $\alpha \circ \Phi = \pi \otimes \mathcal{S}$ and α makes commutative the diagram (34). □

Claim 5. *The map* $\alpha : \mathcal{H}(\mathfrak{g} \otimes \mathcal{S}; V \otimes \mathcal{S}) \to \mathcal{H}(\mathfrak{g}; V) \otimes \mathcal{S}$ *is surjective.*

Proof. Let θ be in $\mathcal{H}(\mathfrak{g}; V) \otimes \mathcal{S}$. Since $\pi \otimes \mathcal{S}$ is surjective, there exists μ in $\mathcal{Z} \otimes \mathcal{S}$ such that $(\pi \otimes \mathcal{S})\,(\mu) = \theta$. Let $\bar{\Lambda} = \Phi(\mu)$, then $\alpha(\bar{\Lambda}) = (\alpha \circ \Phi)(\mu) = (\pi \otimes \mathcal{S})(\mu) = \theta$. Whence, α is surjective. □

For $p > 0$, we shall give an explicit description of the map α. Let $\Lambda + \mathcal{B}(\mathfrak{g} \otimes \mathcal{S}; V \otimes \mathcal{S})$ be in $\mathcal{H}(\mathfrak{g} \otimes \mathcal{S}; V \otimes \mathcal{S})$, where Λ is in $C^p(\mathfrak{g} \otimes \mathcal{S}; V \otimes \mathcal{S})$. As $\alpha(\Lambda + \mathcal{B}(\mathfrak{g} \otimes \mathcal{S}; V \otimes \mathcal{S}))$

belongs to $\mathcal{H}(\mathfrak{g};V)\otimes\mathcal{S}$, by Proposition 2.1, there are μ_j in $\mathcal{Z}$ such that

$$\alpha(\Lambda+\mathcal{B}(\mathfrak{g}\otimes\mathcal{S};V\otimes\mathcal{S}))=(\mu_1+\mathcal{B})\otimes s_1+\cdots+(\mu_m+\mathcal{B})\otimes s_m. \tag{38}$$

By (34), $(\iota'\otimes\mathcal{S})\circ\alpha=\Psi$, and using (21) and (33) it follows that:

$$\begin{aligned}&(\iota'\otimes\mathcal{S})\circ\alpha(\Lambda+\mathcal{B}(\mathfrak{g}\otimes\mathcal{S};V\otimes\mathcal{S}))=\Psi\left(\Lambda+\mathcal{B}(\mathfrak{g}\otimes\mathcal{S};V\otimes\mathcal{S})\right)\\&=(\mathcal{L}(\Lambda_1)+\mathcal{B})\otimes s_1+\ldots+(\mathcal{L}(\Lambda_m)+\mathcal{B})\otimes s_m.\end{aligned} \tag{39}$$

Applying $\iota'\otimes\mathcal{S}$ to (38) we get

$$\begin{aligned}&(\iota'\otimes\mathcal{S})\circ\alpha(\Lambda+\mathcal{B}(\mathfrak{g}\otimes\mathcal{S};V\otimes\mathcal{S}))\\&=(\iota'\otimes\mathcal{S})\left(\sum_{j=1}^{m}(\mu_j+\mathcal{B})\otimes s_j\right)=\sum_{j=1}^{m}(\mu_j+\mathcal{B})\otimes s_j.\end{aligned} \tag{40}$$

From (39) and (40), it follows $\mu_j+\mathcal{B}=\mathcal{L}(\Lambda_j)+\mathcal{B}$ for all j. Hence, by (38), we obtain

$$\alpha(\Lambda+\mathcal{B}(\mathfrak{g}\otimes\mathcal{S};V\otimes\mathcal{S}))=(\mathcal{L}(\Lambda_1)+\mathcal{B})\otimes s_1+\ldots+(\mathcal{L}(\Lambda_m)+\mathcal{B})\otimes s_m. \tag{41}$$

(ii) We shall prove that if $f:C(\mathfrak{g};V)\to C(\mathfrak{g};W)$ is a map of complexes, then the following diagram is commutative:

$$\begin{array}{ccc}\mathcal{H}(\mathfrak{g}\otimes\mathcal{S};V\otimes\mathcal{S}) & \xrightarrow{\mathcal{H}(\mathcal{T}(f))} & \mathcal{H}(\mathfrak{g}\otimes\mathcal{S};W\otimes\mathcal{S})\\ \Big\downarrow{\scriptstyle\alpha_V} & & \Big\downarrow{\scriptstyle\alpha_W}\\ \mathcal{H}(\mathfrak{g};V)\otimes\mathcal{S} & \xrightarrow{\mathcal{T}(\mathcal{H}(f))} & \mathcal{H}(\mathfrak{g};W)\otimes\mathcal{S}\end{array} \tag{42}$$

where $\mathcal{T}(\mathcal{H}(f))=\mathcal{H}(f)\otimes\mathcal{S}$. Let $f':\mathcal{Z}'(\mathfrak{g};V)\to\mathcal{Z}'(\mathfrak{g};W)$ be the map induced by f, i.e., $f'(\lambda+\mathcal{B})=f(\lambda)+\mathcal{B}$. Then $\pi'\circ f=f'\circ\pi'$. Since diagram (34) is commutative, (33) implies that:

$$\begin{aligned}&(\iota'\otimes\mathcal{S})\circ(\alpha_W\circ\mathcal{H}(\mathcal{T}(f))\\&=((\iota'\otimes\mathcal{S})\circ\alpha_W)\circ\mathcal{H}(\mathcal{T}(f))=\Psi\circ\mathcal{H}(\mathcal{T}(f))=\mathcal{T}(f')\circ\Psi.\end{aligned} \tag{43}$$

By (19), $\mathcal{T}(f')=f'\otimes\mathcal{S}$ and by (34), $\Psi=(\iota'\otimes\mathcal{S})\circ\alpha_V$. In addition, $\mathcal{T}(\iota')=\iota'\otimes\mathcal{S}$ and since f is a map of complexes, it follows that $\iota'\circ\mathcal{H}(f)=f'\circ\iota'$. Hence:

$$\begin{aligned}&\mathcal{T}(f')\circ\Psi=\mathcal{T}(f')\circ(\mathcal{T}(\iota')\circ\alpha_V)=(\mathcal{T}(f')\circ\mathcal{T}(\iota'))\circ\alpha_V\\&=\mathcal{T}\left(f'\circ\iota'\right)\circ\alpha_V=\mathcal{T}\left(\iota'\circ\mathcal{H}(f)\right)\circ\alpha_V\\&=((\iota'\circ\mathcal{H}(f))\otimes\mathcal{S})\circ\alpha_V=(\iota'\otimes\mathcal{S})\circ((\mathcal{H}(f)\otimes\mathcal{S})\circ\alpha_V).\end{aligned} \tag{44}$$

As $\iota'\otimes\mathcal{S}$ is injective, from (43) and (44), we deduce that $\alpha_W\circ\mathcal{H}(\mathcal{T}(f))=(\mathcal{H}(f)\otimes\mathcal{S})\circ\alpha_V$. Whence, the diagram (42) is commutative. □

Let $\mathcal{R}^0 = \{0\}$ and for each $p > 0$, define $\mathcal{R}^p$ as the subspace of $C^p(\mathfrak{g}\otimes\mathcal{S}; V\otimes\mathcal{S})$ generated by all the cochains Θ satisfying $\Theta(x_1\otimes 1,\dots,x_p\otimes 1) = 0$. Let $\mathcal{R} = \oplus_{p\geq 0}\mathcal{R}^p$ and define $\mathcal{Q}$ by

$$\mathcal{Q} = \left(\mathcal{Z}(\mathfrak{g}\otimes\mathcal{S}; V\otimes\mathcal{S})\cap\mathcal{R} + \mathcal{B}(\mathfrak{g}\otimes\mathcal{S}; V\otimes\mathcal{S})\right)/\mathcal{B}(\mathfrak{g}\otimes\mathcal{S}; V\otimes\mathcal{S}). \tag{45}$$

Now we shall state the main result of this work.

Theorem 4.3. *Let $\mathfrak{g}$ be a Lie algebra and let $\mathcal{S}$ be an m-dimensional, associative and commutative algebra with unit, over a field $\mathbb{F}$. Let $\mathfrak{g}\otimes\mathcal{S}$ be the current Lie algebra of $\mathfrak{g}$ by $\mathcal{S}$. Let $\alpha : \mathcal{H}(\mathfrak{g}\otimes\mathcal{S}; V\otimes\mathcal{S}) \to \mathcal{H}(\mathfrak{g}; V)\otimes\mathcal{S}$ be the map of Proposition 4.2. Then the following short sequence is exact*

$$0 \longrightarrow \mathcal{Q} \xrightarrow{\ \iota\ } \mathcal{H}(\mathfrak{g}\otimes\mathcal{S}; V\otimes\mathcal{S}) \xrightarrow{\ \alpha\ } \mathcal{H}(\mathfrak{g}; V)\otimes\mathcal{S} \longrightarrow 0, \tag{46}$$

where ι is the inclusion map and $\mathcal{Q}$ is the subspace defined in (45).

Proof. By (3), observe that a cochain Θ belongs to $\mathcal{R}$ if and only if $\Theta_j(x_1\otimes 1,\dots,x_p\otimes 1) = \mathcal{L}(\Theta_j)(x_1,\dots,x_p) = 0$, for all j. Then Θ is in $\mathcal{R}$ if and only if Θ_j belongs to $\operatorname{Ker}(\mathcal{L})$ for all j.

In the proof of Claim 4, we showed that $\operatorname{Ker}(\alpha) = \operatorname{Ker}(\Psi)$. We claim that $\operatorname{Ker}(\Psi) = \mathcal{Q}$. First we assume that $p > 0$. Let $\Lambda + \mathcal{B}(\mathfrak{g}\otimes\mathcal{S}; V\otimes\mathcal{S})$ be in $\operatorname{Ker}(\Psi)$. We will find a cochain Θ in $\mathcal{R}$ such that $\Lambda + \mathcal{B}(\mathfrak{g}\otimes\mathcal{S}; V\otimes\mathcal{S}) = \Theta + \mathcal{B}(\mathfrak{g}\otimes\mathcal{S}; V\otimes\mathcal{S})$. Indeed, by (33), we have

$$\Psi(\Lambda + \mathcal{B}(\mathfrak{g}\otimes\mathcal{S}; V\otimes\mathcal{S})) = \sum_{j=1}^{m}(\mathcal{L}(\Lambda_j) + \mathcal{B})\otimes s_j = 0. \tag{47}$$

Then $\mathcal{L}(\Lambda_j)$ belongs to $\mathcal{B}$ for all j. Hence, there exists θ_j in $C^{p-1}(\mathfrak{g}; V)$ such that $\mathcal{L}(\Lambda_j) = \mathrm{d}\,\theta_j$. For each j, define Δ_j in $C^{p-1}(\mathfrak{g}\otimes\mathcal{S}; V)$ by

$$\Delta_j(x_1\otimes t_1,\dots,x_{p-1}\otimes t_{p-1}) = \omega_1(t_1\cdots t_{p-1})\,\theta_j(x_1,\dots,x_{p-1}).$$

Then $\mathcal{L}(\Delta_j) = \theta_j$ (see (6)). Let Δ in $C^{p-1}(\mathfrak{g}\otimes\mathcal{S}; V\otimes\mathcal{S})$ be defined by

$$\Delta(X_1,\dots,X_{p-1}) = \Delta_1(X_1,\dots,X_{p-1})\otimes s_1 + \ldots + \Delta_m(X_1,\dots,X_{p-1})\otimes s_m,$$

where $X_1,\dots,X_{p-1}$ are in $\mathfrak{g}\otimes\mathcal{S}$. From (8), we have

$$\mathcal{L}(\Lambda_j) = \mathrm{d}\,\theta_j = \mathrm{d}(\mathcal{L}(\Delta_j)) = \mathcal{L}((\mathrm{D}\,\Delta)_j),\ \text{ for all } 1\leq j\leq m.$$

Then there exists Θ_j in $\operatorname{Ker}(\mathcal{L})$ such that $\Lambda_j = (\mathrm{D}\,\Delta)_j + \Theta_j$. Let Θ in $C^p(\mathfrak{g}\otimes\mathcal{S}; V\otimes\mathcal{S})$ be defined by

$$\Theta(X_1,\dots,X_p) = \Theta_1(X_1,\dots,X_p)\otimes s_1 + \ldots + \Theta_m(X_1,\dots,X_p)\otimes s_m,$$

for all $X_1, \dots, X_p$ in $\mathfrak{g} \otimes \mathcal{S}$. Since $\Lambda_j = (\mathrm{D}\,\Delta)_j + \Theta_j$ for each j, $\Lambda = \mathrm{D}\,\Delta + \Theta$ (see (5)). Since Θ_j belongs to $\mathrm{Ker}(\mathcal{L})$, Θ belongs to $\mathcal{R}$ and $\Lambda + \mathcal{B}(\mathfrak{g} \otimes \mathcal{S}; V \otimes \mathcal{S}) = \Theta + \mathcal{B}(\mathfrak{g} \otimes \mathcal{S}; V \otimes \mathcal{S})$ belongs to $\mathcal{Q}$.

Since $\mathrm{D}\,\Lambda = 0$, $\mathrm{D}\,\Theta = 0$. Therefore $\Lambda + \mathcal{B}(\mathfrak{g} \otimes \mathcal{S}; V \otimes \mathcal{S}) = \Theta + \mathcal{B}(\mathfrak{g} \otimes \mathcal{S}; V \otimes \mathcal{S})$ belongs to $\mathcal{Q}$, which proves that $\mathrm{Ker}(\alpha) \subset \mathcal{Q}$.

Now we affirm $\mathcal{Q} \subset \mathrm{Ker}(\alpha)$. Let $\Theta + \mathcal{B}(\mathfrak{g} \otimes \mathcal{S}; V \otimes \mathcal{S})$ be in $\mathcal{Q}$, where Θ is in $\mathcal{Z}(\mathfrak{g} \otimes \mathcal{S}; V \otimes \mathcal{S}) \cap \mathcal{R}$. As $\Theta(x_1 \otimes 1, \dots, x_p \otimes 1) = 0$, then Θ_j belongs to $\mathrm{Ker}(\mathcal{L})$ for all j. By (47), we have that $\Theta + \mathcal{B}(\mathfrak{g} \otimes \mathcal{S}; V \otimes \mathcal{S})$ belongs to $\mathrm{Ker}(\Psi) = \mathrm{Ker}(\alpha)$. Then $\mathrm{Ker}(\alpha) = \mathcal{Q}$. Since α is surjective, we deduce that the short exact sequence (46) is exact for $p > 0$.

For $p = 0$, we have $\mathcal{Q}^0 = \{0\}$, because by hypothesis, $\mathcal{R}^0 = \{0\}$. In Lemma 4.1, we showed that $\mathcal{H}^0(\mathfrak{g} \otimes \mathcal{S}; V \otimes \mathcal{S}) = \mathcal{H}^0(\mathfrak{g}; V) \otimes \mathcal{S}$ while in Claim 2, we proved that α is the identity. Therefore, $\mathrm{Ker}(\alpha) = \{0\} = \mathcal{Q}^0$ and the sequence (46) is exact for $p = 0$. □

Corollary 4.4. *Let $\mathfrak{g}$ be a Lie algebra and let $\mathcal{S}$ be a finite dimensional, associative and commutative algebra with unit over a field $\mathbb{F}$. Let V be a $\mathfrak{g}$-module. Then $\mathcal{H}(\mathfrak{g} \otimes \mathcal{S}; V \otimes \mathcal{S})$ is isomorphic to $\mathcal{H}(\mathfrak{g}; V) \otimes \mathcal{S}$ if and only if*

$$\mathcal{Z}^p(\mathfrak{g} \otimes \mathcal{S}, V \otimes \mathcal{S}) \cap \mathcal{R}^p \subset \mathcal{B}^p(\mathfrak{g} \otimes \mathcal{S}; V \otimes \mathcal{S}), \quad \text{for all } p > 0. \tag{48}$$

Proof. By Theorem 4.3, we know $\mathrm{Ker}(\alpha) = \mathcal{Q}$. By (45), it is clear that $\mathcal{Q} = \{0\}$ if and only if (48) holds. Observe that for $p = 0$, $Q^0 = 0$ by definition. Moreover, we proved that $\mathcal{H}^0(\mathfrak{g} \otimes \mathcal{S}; V \otimes \mathcal{S}) = \mathcal{H}^0(\mathfrak{g}; V) \otimes \mathcal{S}$ (see Lemma 4.1) and that α is the identity map (see Claim 2). □

4.1. Current Lie algebras over semisimple Lie algebras. In the next result, we will determine the cohomology group $\mathcal{H}(\mathfrak{g} \otimes \mathcal{S}; V \otimes \mathcal{S})$, where $\mathfrak{g}$ is a semisimple Lie algebra and V is an irreducible $\mathfrak{g}$-module. It is a well known result that in this case $\mathcal{H}(\mathfrak{g}; V) = \{0\}$ (see [4, Theorem 24.1]). We shall use this fact in proving the following:

Proposition 4.5. *Let $\mathfrak{g}$ be a semisimple Lie algebra and V an irreducible $\mathfrak{g}$-module. Let $\mathcal{S}$ be a finite dimensional, associative and commutative algebra with unit over a field $\mathbb{F}$ of zero characteristic. Then $\mathcal{H}(\mathfrak{g} \otimes \mathcal{S}; V \otimes \mathcal{S}) = \mathcal{Q}$.*

Proof. As $\mathcal{H}(\mathfrak{g}; V) = \{0\}$, by Theorem 4.3, we get $\mathcal{H}(\mathfrak{g} \otimes \mathcal{S}; V \otimes \mathcal{S}) = \mathcal{Q}$. Now we shall verify this result without using Theorem 4.3.

Let $\Lambda + \mathcal{B}(\mathfrak{g} \otimes \mathcal{S}; V \otimes \mathcal{S})$ be in $\mathcal{H}(\mathfrak{g} \otimes \mathcal{S}; V \otimes \mathcal{S})$. Then $\mathrm{D}\,\Lambda = 0$ and $\widehat{\omega}_j \circ \mathrm{D}\,\Lambda = (\mathrm{D}\,\Lambda)_j = 0$ for all j. By Proposition 2.2, $0 = \mathcal{L}\left((\mathrm{D}\,\Lambda)_j\right) = \mathrm{d}\left(\mathcal{L}(\Lambda_j)\right)$, which implies that $\mathcal{L}(\Lambda_j)$ belongs to $\mathcal{Z} = \mathcal{B}$. Hence, there exists μ_j in $C(\mathfrak{g}; V)$ such that $\mathcal{L}(\Lambda_j) = \mathrm{d}\,\mu_j$. Then by (6), we have

$$\Lambda_j(x_1 \otimes 1, \ldots, x_p \otimes 1) = \mathrm{d}\,\mu_j(x_1, \ldots, x_p), \quad \text{for all } 1 \leq j \leq m. \tag{49}$$

Let Ω in $C(\mathfrak{g} \otimes \mathcal{S}; V \otimes \mathcal{S})$ be defined by

$$\Omega(x_1 \otimes t_1, \ldots, x_p \otimes t_p) = \sum_{j=1}^{m} \mu_j(x_1, \ldots, x_p) \otimes s_j\, \widetilde{t}, \tag{50}$$

where $\widetilde{t} = t_1 \cdots t_p$. We claim that

$$\mathrm{D}\,\Omega(x_1 \otimes t_1, \ldots, x_p \otimes t_p) = \sum_{j=1}^{m} \mathrm{d}\,\mu_j(x_1, \ldots, x_p) \otimes s_j\, \widetilde{t}. \tag{51}$$

Indeed, if we write Ω as in (5), we obtain

$$\Omega(x_1 \otimes t_1, \ldots, x_p \otimes t_p) = \sum_{j=1} \Omega_j(x_1 \otimes t_1, \ldots, x_p \otimes t_p) \otimes s_j. \tag{52}$$

Using (50), (52) and the definition of $\mathcal{L}$, it follows $\mathcal{L}(\Omega_j)(x_1, \ldots, x_p) = \Omega_j(x_1 \otimes 1, \ldots, x_p \otimes 1) = \mu_j(x_1, \ldots, x_p)$, then $\mathcal{L}(\Omega_j) = \mu_j$. From (50), this implies that

$$\Omega(x_1 \otimes t_1, \ldots, x_p \otimes t_p) = \sum_{j=1}^{m} \Omega_j(x_1 \otimes 1, \ldots, x_p \otimes 1) \otimes\ s\widetilde{t}. \tag{53}$$

On the other hand, by (8), we get

$$\mathcal{L}\left((\mathrm{D}\,\Omega)_j\right) = \mathrm{d}\,\mathcal{L}(\Omega_j) = \mathrm{d}\,\mu_j \quad \text{for all } 1 \leq j \leq m. \tag{54}$$

By Remark 3.1 and (53), we deduce that $\mathcal{T}(\mathrm{Id})(\Omega) = \Omega$. Then by Proposition 3.3, $\mathrm{D}\,\Omega = \mathrm{D}\,\mathcal{T}(\mathrm{Id})(\Omega) = \mathcal{T}(\mathrm{Id})(\mathrm{D}\,\Omega)$. Hence by (18) and (54),

$$\begin{aligned} \mathrm{D}\,\Omega(x_1 \otimes t_1, \ldots, x_p \otimes t_p) &= \mathcal{T}(\mathrm{Id})(\mathrm{D}\,\Omega)((x_1 \otimes t_1, \ldots, x_p \otimes t_p)) \\ &= \sum_{j=1}^{m} \mathcal{L}\left((\mathrm{D}\,\Omega)_j\right)(x_1, \ldots, x_p) \otimes s_j\, \widetilde{t} = \sum_{j=1}^{m} \mathrm{d}\,\mu_j(x_1, \ldots, x_p) \otimes s_j\, \widetilde{t}, \end{aligned}$$

which proves (51). Let $\Theta = \Lambda - \mathrm{D}\,\Omega$. By (49) and (51), we obtain

$$\begin{aligned} \Theta(x_1 \otimes 1, \ldots, x_p \otimes 1) &= \Lambda(x_1 \otimes 1, \ldots, x_p \otimes 1) - \mathrm{D}\,\Omega(x_1 \otimes 1, \ldots, x_p \otimes 1) \\ &= \sum_{j=1}^{m} \Lambda_j(x_1 \otimes 1, \ldots, x_p \otimes 1) \otimes s_j - \sum_{j=1}^{m} \mathrm{d}\,\mu_j(x_1, \ldots, x_p) \otimes s_j = 0. \end{aligned}$$

Then $\Theta + \mathcal{B}(\mathfrak{g} \otimes \mathcal{S}; V \otimes \mathcal{S})$ belongs to $\mathcal{Q}$ and $\mathcal{H}(\mathfrak{g} \otimes \mathcal{S}; V \otimes \mathcal{S}) = \mathcal{Q}$. □

Acknowledgement. The author wishes to thank the referees for their comments, criticism and valuable suggestions, as they have provided the opportunity to better present the results, substantially improving the original presentation. The author thanks the support provided by a post-doctoral fellowship CONAHCYT 769309.

Disclosure statement. The author reports there are no competing interests to declare.

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