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A NOTE ON σ-NILPOTENCY OF FINITE GROUPS

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Abstract. Let $\sigma = \{\sigma_i \mid i \in I\}$ be some partition of the set $\mathbb{P}$ of all primes, and $\sigma(n) = \{\sigma_i \mid i \in I, \sigma_i \cap \pi(n) \neq \emptyset\}$ for any integer n. A group G is called σ-primary if either $G = 1$ or $|\sigma(G)| = 1$. A group G is σ-nilpotent if $(H/K) \rtimes (G/C_G(H/K))$ is σ-primary for every chief factor H/K of G. In this note, we prove that G is σ-nilpotent if and only if G is a σ-full group and $\pi(|xy|) = \pi(|x||y|)$ for any two elements $x, y \in G$ such that $\sigma(|x|) \cap \sigma(|y|) = \emptyset$.

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1. Introduction

Throughout this note all considered groups are finite and G will always denote a finite group. It is well-known that if G is nilpotent, then $|ab| = |a||b|$ whenever $a, b \in G$ have co-prime orders, where $|x|$ denotes the order of x in G. Conversely, Baumslag and Wiegold [3] proved that G is nilpotent if $|ab| = |a||b|$ for any $a, b \in G$ with co-prime orders. Bastos and Shumyatsky [2] got a similar sufficient and necessary condition for the nilpotency of the commutator subgroup G'. Bastos, Monetta and Shumyatsky [1] proved that the kth term of the lower central series of a finite group G is nilpotent if and only if $|ab| = |a||b|$ for any k-commutators $a, b \in G$ of coprime orders. More results can be found in [6,7,8]. In this note, we shall generalizes these results to finite σ-nilpotent groups.

Let the symbol $\pi(n)$ denote the set of all primes dividing n; as usual, $\pi(G) = \pi(|G|)$, the set of all primes dividing the order of G. Let $\sigma = \{\sigma_i \mid i \in I\}$ be some partition of the set $\mathbb{P}$ of all primes, that is,

$$\mathbb{P} = \bigcup_{i \in I} \sigma_i \text{ and } \sigma_i \cap \sigma_j = \emptyset \text{ for all } i \neq j.$$

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We put $\sigma(n) = \{\sigma_i \mid \sigma_i \cap \pi(n) \neq \emptyset, i \in I\}$ and $\sigma(G) = \sigma(|G|)$. Without loss of a generality, we always assume that $\sigma(G) = \{\sigma_1, \ldots, \sigma_t\}$.

A group G is called σ-primary [10] if either $G = 1$ or $|\sigma(G)| = 1$. G is σ-nilpotent [4] if $(H/K) \rtimes (G/C_G(H/K))$ is σ-primary for every chief factor H/K of G.

A set S of Sylow subgroups of G is called a complete set of Sylow subgroups of G if S contains exact one Sylow p-subgroup of G for every prime $p \in \pi(G)$. By analogy with it, we say that a set $\mathcal{H} = \{H_1, \ldots, H_t\}$ of Hall subgroups of G, where H_i is σ-primary $(i = 1, \ldots, t)$, is a complete Hall σ-set of G if $\gcd(|H_i|, |H_j|) = 1$ for all $i \neq j$ and $\pi(G) = \pi(H_1) \cup \cdots \cup \pi(H_t)$ (see [11,12,13]). Following [4], a group G is a σ-full group if it possesses a complete Hall σ-set.

Our main results are as follows.

Theorem 1.1. *Let G be a finite group. Then G is σ-nilpotent if and only if G is a σ-full group and $\pi(|xy|) = \pi(|x||y|)$ for any two elements $x, y \in G$ such that*

$$\sigma(|x|) \cap \sigma(|y|) = \emptyset.$$

Applying Theorem 1.1, the following theorem is immediately as $|xy| = |x||y|$ implies that $\pi(|xy|) = \pi(|x||y|)$.

Theorem 1.2. *Let G be a finite group. Then G is σ-nilpotent if and only if G is a σ-full group and $|xy| = |x||y|$ for any two elements $x, y \in G$ such that*

$$\sigma(|x|) \cap \sigma(|y|) = \emptyset.$$

Remark 1.3. Let $G = A_5$ be the alternating group of degree 5 and

$$\sigma = \{\sigma_1 = \{2, 3\}, \sigma_2 = \{5\}, \ldots\}.$$

Then G is a σ-full group. Let $H_1 \cong A_4$ be a Hall σ_1-subgroup of G and H_2 be a Sylow 5-subgroup of G. Then $\mathcal{H} = \{H_1, H_2\}$ is a complete Hall σ-set of G. However, G is not σ-nilpotent.

All unexplained notations and terminologies are standard and can be found in [5,9,14].

2. Proof of Theorem

Lemma 2.1. [10] *A group G is σ-nilpotent if and only if G has a complete Hall σ-set $\mathcal{H} = \{H_1, \ldots, H_t\}$ such that $G = H_1 \times \cdots \times H_t$.*

Proof of Theorem 1.1. Suppose that G is a σ-nilpotent group, then G has a complete Hall σ-set $\mathcal{H} = \{H_1, \ldots, H_t\}$ such that $G = H_1 \times \cdots \times H_t$ by Lemma 2.1.

Thus, G is a σ-full group and $xy = yx$ for any two elements $x, y \in G$ such that $\sigma(|x|) \cap \sigma(|y|) = \emptyset$. By well-known results, we get that $|xy| = |x||y|$. In particular, $\pi(|xy|) = \pi(|x||y|)$.

Conversely, suppose that G is a σ-full group and satisfies the theorem hypothesis. Let $\mathcal{H} = \{H_1, \dots, H_t\}$ is a complete Hall σ-set of G. Without loss of a generality, we can assume that H_i is a Hall σ_i-subgroup of G, respectively.

First, we claim that $G = H_1 \cdots H_t$. We do this by counting the number of elements on the right-hand side. Suppose that h_i, k_i are elements of H_i for $i = 1, 2, \dots, t$, and suppose that we have an equality

$$h_1 h_2 \cdots h_t = k_1 k_2 \cdots k_t.$$

Write $x = k_1 \cdots k_{t-1}$ and $y = k_t h_t^{-1} \in H_t$, then $\sigma(|x|) \subseteq \{\sigma_1, \dots, \sigma_{t-1}\}$ and $\sigma(|y|) \subseteq \{\sigma_t\}$. Consequently, we see that $\sigma(|x|) \cap \sigma(|y|) = \emptyset$. Thus, if $y \neq 1$, then $\pi(|xy|) = \pi(|x||y|)$ by hypothesis. Let $g = xy$. Then $\pi(|g|) \cap \sigma_t \neq \emptyset$. On the other hand, observe that

$$g = xy = k_1 k_2 \cdots k_t h_t^{-1} = h_1 h_2 \cdots h_{t-1}.$$

Applying theorem hypothesis again, we know that $\pi(|g|) \cap \sigma_t = \emptyset$ which is a contradiction. So we get $y = 1$ and hence $h_t = k_t$. By induction, we get $h_i = k_i$ for any $i = 1, \dots, k$. Thus, a count of the number of elements in the product $H_1 H_2 \cdots H_t$ shows that there are as many of them as there are elements in G. Therefore, we get that $G = H_1 H_2 \cdots H_t$.

In the following, we show that every subgroup H_i is normal in G. For any $h \in H$, $g \in G$, we have $|h^g| = |h|$, in particular, $\pi(|h^g|) = \pi(|h|) \subseteq \sigma_1$. By above arguments, we know that $G = H_1 H_2 \cdots H_t$. So we can let

$$h^g = h_1 h_2 \cdots h_t, \text{ where } h_i \in H_i \text{ for any } i = 1, 2, \dots, t.$$

Since $|h^g|$ is a σ_1-number, the hypothesis implies that

$$\pi(|h_1||h_2| \cdots |h_t|) = \pi(|h^g|) = \pi(|h|) \subseteq \sigma_1.$$

This leads to $\pi(|h_2|) = \ldots = \pi(|h_t|) = \emptyset$ and hence $h_2 = \ldots = h_t = 1$. So we get $h^g = h_1 \in H_1$. Therefore, H_1 is normal in G. Moreover, we can get that every H_i is normal in G. Hence $G = H_1 \times H_2 \times \cdots \times H_t$. Applying Lemma 2.1, we know that G is σ-nilpotent. The proof of theorem is completed. □

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