

THE FURTHER RESULTS ON *SEP* ELEMENTS IN A RING WITH INVOLUTION

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ABSTRACT. In this paper, we further study many new characterizations of *SEP* elements in a ring with involution. Firstly, combining Moore-Penrose invertible element, group invertible element, we find some *PE* elements to characterize *SEP* elements and then further discover some equivalent conditions for *SEP* elements especially around the element aa^*a^+a . Mainly, by constructing some equations in a given set including $a^+, a^*, (a^\#)^*, a^+a, aa^+$, we obtain a lot of new characterizations of *SEP* elements. Next, we study the expression forms of related bivariate equations to depict *SEP* elements. Finally, we use nil-cleanliness of the element aa^*a^+a to link *SEP* elements with *PE* elements.

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1. Introduction

The study of generalized inverse problems has expanded from linear operators to operator algebras, C^* -algebras, and semigroups and rings, the produced conceptions like *EP* elements, *PI* elements, *PE* elements and so on play key roles in the development of generalized inverse in a ring. Numerous challenges in systems and control theory necessitate the addressing of associated equations. The paper aims to discuss the solution of some related equations to further characterize *SEP* elements.

An involution $a \mapsto a^*$ in a ring R is an anti-isomorphism of degree 2, that is, $(a^*)^* = a$; $(a + b)^* = a^* + b^*$; and $(ab)^* = b^*a^*$. In this case, R is called a $*$ -ring.

We know that $a \in R$ satisfying $a^2 = a$ is called an idempotent element. The set of all idempotent elements will be denoted by $E(R)$.

An element $a \in R$ is called Hermitian if $a^* = a$ [14], and a is called a projection if $a^2 = a = a^*$. We denote the set of all projections of R by $PE(R)$.

An element $a \in R$ is called Moore-Penrose invertible if there exists $x \in R$ satisfying the following equations:

$$axa = a, \quad xax = x, \quad (ax)^* = ax, \quad (xa)^* = xa;$$

such an x is the uniquely determined Moore-Penrose inverse (or MP-inverse) of a [3,4], denoted by $x = a^+$. The set of all Moore-Penrose invertible elements of R will be denoted by R^+ .

Let $a \in R$. Then a is called group invertible if there exists $x \in R$ satisfying

$$axa = a, \quad xax = x, \quad ax = xa;$$

such an x is uniquely determined group inverse of a (see [5,8,9]), written $x = a^\#$. Denote by $R^\#$ the set of all group invertible elements of R .

An element $a \in R$ satisfying $a = aa^*a$ is called partial isometry of R [3,4]. Let R^{PI} denote the set of all partial isometries of R . Obviously, we have that $a \in R^+$ is a partial isometry if and only if $a^* = a^+$.

Let $a \in R^\# \cap R^+$. If $a^\# = a^+$, then a is called an *EP* element. We denote the set of all *EP* elements in R by R^{EP} . On the studies of *EP*, the readers can refer to [1,4,6,7,10,14,15,16,17].

If $a \in R^{EP} \cap R^{PI}$, then a is said to be a strong *EP* element of R [3,4,14,19,20]. Let R^{SEP} denote the set of all *SEP* elements of R .

In [14,17], Mosic and Djordjevic give many characterizations of *SEP* elements, we have learned some equivalent conditions for *SEP* elements. In [18], many new characterizations of strongly *EP* elements have been presented. Then many researchers characterize *SEP* elements by constructing related equations. In [19], it is known that $a \in R^\# \cap R^+$ is *SEP* if and only if the equation $yxa^* = yxa^\#$ has at least one solution in a given set $\chi_a^2 =: \{(x, y) | x, y \in \chi_a\}$. In [11], it is shown the basic solution formula of the bivariate equation $xa^+(a^\#)^* = aa^+y$. In [2], Guan uses core invertible elements to characterize and discuss the solution of $xa^* = a \oplus x$. In [12,13], the forms of solutions of parametric equations in a certain given set is proved.

Motivated by these results, this paper mainly study the further characterizations of *SEP* elements by *PE* elements.

2. Characterizing *SEP* elements by projections

Theorem 2.1. *Let $a \in R^\# \cap R^+$. Then $a \in R^{SEP}$ if and only if $aa^*a^+a \in PE(R)$.*

Proof. “ \implies ” If $a \in R^{SEP}$, then $a^+ = a^* = a^\#$, this gives that

$$aa^*a^+a = aa^\#a^+a = a^\#a = a^+a \in PE(R).$$

“ \impliedby ” From the assumption, we have

$$aa^*a^+a = aa^*a^+a^2a^*a^+a,$$

and

$$aa^*a^+a = (aa^*a^+a)^* = a^+a^2a^* = a^+aa^+a^2a^* = a^+a^2a^*a^+a.$$

Multiplying the last equality on the right by $(a^\#)^*a^+a$, one has $a = a^+a^2$. Hence $a \in R^{EP}$ and

$$a = aa^+a = a(a^*a^+a(a^\#)^*a^+)a = aa^*a^+a^2a^*a^+a(a^\#)^*a^+a = aa^*a^+a^2 = aa^*a.$$

It follows that $a \in R^{PI}$. Thus $a \in R^{SEP}$. \square

Noting that $a \in PE(R)$ if and only if $a^* \in PE(R)$. Hence, Theorem 2.1 implies the following corollary.

Corollary 2.2. *Let $a \in R^\# \cap R^+$. Then $a \in R^{SEP}$ if and only if $a^+a^2a^* \in PE(R)$.*

Since $e \in R$ is a projection if and only if $e = ee^*$, this induces

Corollary 2.3. *Let $a \in R^\# \cap R^+$. Then $a \in R^{SEP}$ if and only if $a^+a^2a^* = a^+a^2a^*aa^*a^+a$.*

Lemma 2.4. *Let $e \in R$. Then $e \in R^{Her}$ if and only if $e - ee^* \in R^{Her}$.*

Proof. “ \implies ” It is evident because $e = e^*$.

“ \impliedby ” Assume that $e - ee^* \in R^{Her}$. Then

$$e - ee^* = (e - ee^*)^* = e^* - ee^*.$$

This gives $e = e^*$ and so $e \in R^{Her}$. \square

Lemma 2.4 and Corollary 2.2 imply the following theorem.

Theorem 2.5. *Let $a \in R^\# \cap R^+$. Then $a \in R^{SEP}$ if and only if $a^+a^2a^* \in E(R)$ and $a^+a^2a^* - a^+a^2a^*aa^*a^+a \in R^{Her}$.*

Theorem 2.6. *Let $a \in R^\# \cap R^+$. Then $a \in R^{SEP}$ if and only if $aa^*a^\#a \in PE(R)$.*

Proof. “ \implies ” Since $a \in R^{SEP}$, $a^+ = a^\#$ and $aa^*a^+a \in PE(R)$ by Theorem 2.1. Hence, $aa^*a^\#a \in PE(R)$.

“ \impliedby ” Using the hypothesis, one gets

$$aa^*a^\#a = (aa^*a^\#a)^* = (aa^\#)^*aa^* = ((aa^\#)^*aa^*)aa^+ = (aa^*a^\#a)aa^+.$$

Multiplying the equality on the left by $a^\#(a^+)^*a^+$, one yields $a^\# = a^\#aa^+$. Hence $a \in R^{EP}$ by [14, Theorem 1.2.1], this infers $aa^*a^+a = aa^*a^\#a \in PE(R)$. By Theorem 2.1, $a \in R^{SEP}$. \square

Since $aa^*a^+a = aa^*a^*(a^+)^*$, it follows that for $a \in R^{PI}$, one gets $aa^*a^+a = aa^+a^*(a^+)^*$. Thus, Theorem 2.1 implies the following result.

Theorem 2.7. *Let $a \in R^\# \cap R^+$. Then $a \in R^{SEP}$ if and only if $aa^+a^*(a^+)^* \in PE(R)$.*

Proof. We only show “ \Leftarrow ”. From the hypothesis, we have

$$aa^+a^*(a^+)^* = aa^+a^*(a^+)^*(aa^+a^*(a^+)^*)^*.$$

Multiplying the equality on the left by $aa^*a(aa^\#)^*$, we get

$$a = (a^+)^*aa^+.$$

Hence, $a^* = aa^+a^+$ by [14, Theorem 1.5.3], $a \in R^{SEP}$. \square

According to [14, Theorem 2.1.1], $a \in R^{EP}$ if and only if $aa^+a^+ = a^\#$. Thus we have

Corollary 2.8. *Let $a \in R^\# \cap R^+$. Then $a \in R^{SEP}$ if and only if $a^\#(a^+)^* \in PE(R)$.*

Theorem 2.9. *Let $a \in R^\# \cap R^+$. Then $a \in R^{SEP}$ if and only if $a^\#(a^+)^* \in E(R)$ and $a^\#(a^+)^* - a^\#(a^+)^*a^+(a^\#)^* \in R^{Her}$.*

Proof. “ \Rightarrow ” Assume that $a \in R^{SEP}$. Then $a^\#(a^+)^* \in PE(R)$ by Corollary 2.8, this implies

$$a^\#(a^+)^* \in E(R) \cap R^{Her}.$$

By Lemma 2.4, one gets $a^\#(a^+)^* - a^\#(a^+)^*a^+(a^\#)^* \in R^{Her}$.

“ \Leftarrow ” The condition $a^\#(a^+)^* - a^\#(a^+)^*a^+(a^\#)^* \in R^{Her}$ gives

$$a^\#(a^+)^* - a^\#(a^+)^*a^+(a^\#)^* = (a^\#(a^+)^* - a^\#(a^+)^*a^+(a^\#)^*)^*,$$

this induces

$$a^\#(a^+)^* = (a^\#(a^+)^*)^*.$$

Noting that $a^\#(a^+)^* \in E(R)$. Then $a^\#(a^+)^* \in PE(R)$ and so $a \in R^{SEP}$ by Corollary 2.8. \square

3. Some equivalent conditions for SEP elements

Lemma 3.1. *Let $a \in R^\# \cap R^+$. Then*

- (1) $(aa^*a^+a)^\# = aa^\#(a^\#)^*a^+aa^\#;$
- (2) $(aa^*a^+a)^+ = (a^\#)^*a^+;$
- (3) $aa^*a^+a \in R^{EP}$ if and only if $a \in R^{EP}$;
- (4) $aa^*a^+a \in R^{SEP}$ if and only if $aa^* = (a^\#)^*a^+$.

Proof. (1) and (2) can be verified routinely.

(3) “ \implies ” Since $aa^*a^+a \in R^{EP}$, $(aa^*a^+a)^\# = (aa^*a^+a)^+$. By (1) and (2), one obtains

$$aa^\#(a^\#)^*a^+aa^\# = (a^\#)^*a^+.$$

Multiplying the equality on the left by a^*a^+a , one gets $a^+aa^\# = a^+$. Hence, $a \in R^{EP}$ by [14, Theorem 1.2.1].

“ \Leftarrow ” Assume that $a \in R^{EP}$. Then $a^+ = a^\#$, it follows that

$$aa^\#(a^\#)^*a^+aa^\# = aa^\#(a^+)^*a^+aa^+ = (a^+)^*a^+ = (a^\#)^*a^+.$$

By (1) and (2), $aa^*a^+a \in R^{EP}$.

(4) “ \implies ” Since $aa^*a^+a \in R^{SEP}$ if and only if $aa^*a^+a \in R^{EP}$ and $(aa^*a^+a)^* = (aa^*a^+a)^+$, it follows that

$$aa^* = (aa^*)^* = (aa^*a^+a)^* = (aa^*a^+a)^+ = (a^\#)^*a^+.$$

“ \Leftarrow ” The condition $aa^* = (a^\#)^*a^+$ leads to

$$a^+a^2a^* = a^+a(a^\#)^*a^+ = (a^\#)^*a^+ = aa^*$$

and

$$a = aa^*(a^+)^* = a^+a^2a^*(a^+)^* = a^+a^2.$$

Hence, $a \in R^{EP}$, one has

$$(aa^*a^+a)^* = (aa^*)^* = aa^* = (a^\#)^*a^+ = aa^\#(a^\#)^*a^+aa^\# = (aa^*a^+a)^\#.$$

Thus $aa^*a^+a \in R^{SEP}$. \square

Corollary 3.2. *Let $a \in R^\# \cap R^+$. Then $aa^*a^+a \in R^{SEP}$ if and only if $aa^*a = (a^\#)^*$.*

Proof. “ \implies ” Since $aa^*a^+a \in R^{SEP}$, $a \in R^{EP}$ and $aa^* = (a^\#)^*a^+$ by Lemma 3.1. Hence,

$$aa^*a = (a^\#)^*a^+a = (a^+)^*a^+a = (a^+)^* = (a^\#)^*.$$

“ \Leftarrow ” Using the assumption, one gets

$$(a^\#)^* = aa^*a = (aa^*a)a^+a = (a^\#)^*a^+a,$$

and

$$a^\# = a^+aa^\#.$$

Hence, $a \in R^{EP}$, it follows that $aa^* = aa^*aa^+ = (a^\#)^*a^+$. By Lemma 3.1, $aa^*a^+a \in R^{SEP}$. \square

Corollary 3.3. *Let $a \in R^\# \cap R^+$. Then $a \in R^{SEP}$ if and only if $a \in R^{PI}$ and $aa^*a^+a \in R^{SEP}$.*

Proof. “ \Rightarrow ” Since $a \in R^{SEP}$, $a \in R^{PI}$ and $a^+ = a^\# = a^*$, this gives

$$(a^\#)^* = a = aa^*a.$$

By Corollary 3.2, $aa^*a^+a \in R^{SEP}$.

“ \Leftarrow ” Since $a \in R^{PI}$, $aa^*a^+a \in R^{SEP}$, by Corollary 3.2, one gets

$$a = aa^+a = aa^*a = (a^\#)^*.$$

Hence, $a \in R^{SEP}$. \square

Theorem 3.4. *Let $a \in R^\# \cap R^+$. Then the following are equivalent:*

- (1) $a \in R^{SEP}$;
- (2) $aa^\#(a^\#)^*a^+aa^\# = aa^+$;
- (3) $aa^\#(a^\#)^*a^+aa^\# = a^+a$;
- (4) $a^+aa^\#(a^\#)^*a^+aa^\# = a^+$;
- (5) $(a^\#)^*a^+aa^\# = aa^+$;
- (6) $a^+aa^\# = a^*$.

Proof. (1) \Rightarrow (2) Since $a \in R^{SEP}$, $aa^*a^+a \in R^{SEP}$ by Corollary 3.3. Using Lemma 3.1, one gets

$$aa^\#(a^\#)^*a^+aa^\# = (aa^*a^+a)^\# = (aa^*a^+a)^+ = (a^\#)^*a^+ = aa^+.$$

(2) \Rightarrow (3) From $aa^\#(a^\#)^*a^+aa^\# = aa^+$, one obtains

$$aa^+ = (aa^\#(a^\#)^*a^+aa^\#)a^+a = aa^+a^+a.$$

Hence, $a \in R^{EP}$, this gives

$$aa^\#(a^\#)^*a^+aa^\# = aa^+ = a^+a.$$

(3) \Rightarrow (4) The condition $aa^\#(a^\#)^*a^+aa^\# = a^+a$ gives

$$a^+a = aa^\#a^+a = aa^\#.$$

Hence, $a \in R^{EP}$, which infers $a^+ = a^+a^+a$. Thus,

$$a^+aa^\#(a^\#)^*a^+aa^\# = a^+a^+a = a^+.$$

(4) \Rightarrow (5) From $a^+ = a^+aa^\#(a^\#)^*a^+aa^\#$, one yields

$$a^+ = a^+aa^\#,$$

so $a \in R^{EP}$ and

$$a^+ = a^+aa^\#(a^\#)^*a^+aa^\# = a^+(a^\#)^*a^+aa^\#,$$

$$aa^+ = aa^+(a^\#)^*a^+aa^\# = a^+a(a^\#)^*a^+aa^\# = (a^\#)^*a^+aa^\#.$$

(5) \Rightarrow (6) Since $aa^+ = (a^\#)^*a^+aa^\#$, we get

$$a^* = a^*aa^+ = a^*(a^\#)^*a^+aa^\# = a^+aa^\#.$$

(6) \Rightarrow (1) From the assumption $a^+aa^\# = a^*$, one has

$$aa^* = aa^+aaa^\# = aa^\#.$$

Hence, $a \in R^{SEP}$ by [14, Theorem 1.5.3]. \square

Theorem 3.5. *Let $a \in R^\# \cap R^+$. Then $a \in R^{SEP}$ if and only if $(a^\#a)^*aa^\# = aa^*$.*

Proof. “ \Rightarrow ” Assume that $a \in R^{SEP}$. Then $(a^\#)^*a^+aa^\# = aa^+$ by Theorem 3.4. Noting that $a^+ = a^*$. Then

$$(aa^\#)^*aa^\# = (a^\#)^*a^*aa^\# = (a^\#)^*a^+a^\# = aa^+ = aa^*.$$

“ \Leftarrow ” From the equality $(aa^\#)^*aa^\# = aa^*$, one has

$$a^* = a^+aa^* = a^+(aa^\#)^*aa^\# = a^+aa^\#.$$

Hence, $a \in R^{SEP}$ by Theorem 3.4. \square

Lemma 3.6. *Let $a \in R^\# \cap R^+$. Then $(a^\#a)^*aa^\# \in R^{EP}$ with $((a^\#a)^*aa^\#)^+ = a^+a^2a^+a^+$.*

Proof. It is a routine to verify. \square

Theorem 3.7. *Let $a \in R^\# \cap R^+$. Then $a \in R^{SEP}$ if and only if $((a^\#a)^*aa^\#)^+ = aa^*a^+$.*

Proof. “ \implies ” Since $a \in R^{SEP}$, $a \in R^{EP}$ and $aa^\# = aa^+ = aa^*$. Hence,

$$((a^\# a)^* aa^\#)^+ = (a^\# aaa^\#)^+ = (aa^\#)^+ = (aa^+)^+ = aa^+ = aa^* = aa^* a^+ a.$$

“ \Leftarrow ” From the equality $((a^\# a)^* aa^\#)^+ = aa^* a^+ a$, we get

$$(a^\# a)^* aa^\# = (aa^* a^+ a)^+ = (a^\#)^* a^+.$$

Multiplying the equality on the left by a^* , we obtain

$$a^* aa^\# = a^+.$$

Hence, $a \in R^{SEP}$ by [14, Theorem 1.5.3]. \square

4. Characterizing SEP elements by the solution of equations in a given set

Lemma 4.1. *Let $a \in R^\# \cap R^+$. Then $a \in R^{SEP}$ if and only if $(a^+)^* a^+ a^+ = a^+$.*

Proof. “ \implies ” Since $a \in R^{SEP}$, $(a^+)^* = a$, $a^+ = a^\#$. Hence

$$(a^+)^* a^+ a^+ = aa^+ a^\# = a^\# = a^+.$$

“ \Leftarrow ” From $(a^+)^* a^+ a^+ = a^+$, we have

$$a^* = a^+ aa^* = (a^+)^* a^+ a^+ aa^* = (a^+)^* a^+ a^*,$$

and

$$(aa^\#)^* = (a^+)^* a^+ a^* (a^\#)^* = (a^+)^* a^+.$$

Hence, $a \in R^{EP}$ by [14, Theorem 1.1.3] and $a^* = a^* (aa^\#)^* = a^* (a^+)^* a^+ = a^+$. Thus $a \in R^{SEP}$. \square

Observing Theorem 3.4, we can construct the following equation:

$$aa^\# (a^\#)^* x a^\# = x. \quad (4.1)$$

Theorem 4.2. *Let $a \in R^\# \cap R^+$. Then $a \in R^{SEP}$ if and only if Eq.(4.1) has at least one solution in $\Omega_a = \{a^+, a^*, (a^\#)^*, a^+ a, aa^+\}$.*

Proof. “ \implies ” Since $a \in R^{SEP}$, $a^+ = a^* = a^\#$. Obviously, Eq.(4.1) has at least one solution in $\Omega_a = \{a^+, a^*, (a^\#)^*, a^+ a, aa^+\}$.

“ \Leftarrow ” (1) If $x = a^+$ is the solution of Eq.(4.1), then

$$aa^\# (a^\#)^* a^+ a^\# = a^+.$$

Multiplying the equality on the right by $a^+ a$, we obtain

$$aa^\# (a^\#)^* a^+ a^\# a^+ a = a^+ a^+ a.$$

Noting that $a^\# a^+ a = a^\#$, so

$$a^+ a^+ a = a^+.$$

Then, $a = a^+ a a$, multiplying the equality on the right by $a^\# a^\#$, we obtain

$$a^\# = a^+ a a^\#.$$

Hence, $a \in R^{EP}$ by [14, Theorem 1.2.1]. It follows that $aa^\#(a^\#)^* = aa^\#(a^+)^* = (a^+)^*$. From $aa^\#(a^\#)^* a^+ a^\# = a^+$, we get

$$(a^+)^* a^+ a^+ = a^+.$$

Hence, $a \in R^{SEP}$ by Lemma 4.1.

(2) If $x = a^*$ is the solution of Eq.(4.1), then

$$aa^\#(a^\#)^* a^* a^\# = a^*.$$

Multiplying the equality on the right by $a^+ a$, one obtains $a^* = a^* a^+ a$. We apply the involution to the equality, and then we conclude $a = a^+ a^2$. Hence, $a \in R^{EP}$ by the proof of Theorem 4.2(1). This gives $a^\# = a^t$ and $a^* = aa^\#(a^\#)^* a^* a^\#$ imply

$$a^* = aa^\#(a^\#)^* a^* a^\# = (aa^t)(aa^t)^* a^\# = (aa^t)(aa^t)^* a^\# = (aa^\#)(aa^\#)a^\#.$$

It yields $a^* = a^\#$. Thus $a \in R^{SEP}$.

(3) If $x = (a^\#)^*$ is the solution of Eq.(4.1), then

$$aa^\#(a^\#)^* (a^\#)^* a^\# = (a^\#)^*.$$

Multiplying the equality on the right by $a^+ a$, we obtain $(a^\#)^* = (a^\#)^* a^+ a$. Applying the involution on the equality, one yields $a^\# = a^+ a a^\#$. Hence, $a \in R^{EP}$ by [14, Theorem 1.2.1]. From $aa^\#(a^\#)^* (a^\#)^* a^\# = (a^\#)^*$, we get $(a^\#)^* (a^\#)^* a^+ = (a^\#)^*$ and

$$a^* = a^* a^* (a^\#)^* = a^* a^* (a^\#)^* (a^\#)^* a^+ = a^+.$$

Hence, $a \in R^{SEP}$.

(4) If $x = a^+ a$ is the solution of Eq.(4.1), then $a \in R^{SEP}$ by Theorem 3.4.

(5) If $x = aa^+$ is the solution of Eq.(4.1), then $a \in R^{SEP}$ from Theorem 3.4. \square

Revising Eq.(4.1) as follows

$$aa^\#(a^\#)^* x a^+ = x. \quad (4.2)$$

Theorem 4.3. *Let $a \in R^\# \cap R^+$. Then $a \in R^{SEP}$ if and only if Eq.(4.2) has at least one solution in $\chi_a \cup \{a^+ a\} = \{a, a^\#, a^+, a^*, (a^+)^*, (a^\#)^*, a^+ a\}$.*

Proof. “ \implies ” Since $a \in R^{SEP}$, $a^+ = a^* = a^\#$. Clearly, $x = a^*$ is a solution to Eq.(4.2).

“ \impliedby ” (1) If $x = a$, then

$$aa^\#(a^\#)^*aa^+ = a.$$

Multiplying the equality on the right by aa^+ , we obtain $a = a^2a^+$. Hence, $a \in R^{EP}$ by [14, Theorem 1.2.1]. From $aa^\#(a^\#)^*aa^+ = a$, we get

$$a = aa^\#(a^\#)^*aa^+ = aa^\#(a^+)^*aa^+ = (a^+)^*aa^+ = (a^\#)^*aa^+ = (a^\#)^*.$$

Hence, $a \in R^{SEP}$.

(2) If $x = a^\#$, then

$$aa^\#(a^\#)^*a^\#a^+ = a^\#.$$

Multiplying the equality on the right by aa^+ , we obtain $a^\# = a^\#aa^+$. Hence, $a \in R^{EP}$ by [14, Theorem 1.2.1]. From $aa^\#(a^\#)^*a^\#a^+ = a^\#$, we get $a^+ = (a^+)^*a^+a^+$. Hence, $a \in R^{SEP}$ by Lemma 4.1.

(3) If $x = a^+$, then

$$aa^\#(a^\#)^*a^+a^+ = a^+.$$

Multiplying the equality on the left by $aa^\#$, we obtain $aa^\#a^+ = a^+$. Hence, $a \in R^{EP}$ by [14, Theorem 1.2.1]. Hence, $x = a^+ = a^\#$, by (2), $a \in R^{SEP}$.

(4) If $x = a^*$, then $aa^\#(a^\#)^*a^*a^+ = a^*$, e.g.,

$$aa^\#a^+ = a^*.$$

Hence, $a \in R^{EP}$ by [14, Theorem 1.5.3].

(5) If $x = (a^+)^*$, then

$$aa^\#(a^\#)^*(a^+)^*a^+ = (a^+)^*.$$

Multiplying the equality on the right by aa^+ , we obtain $(a^+)^* = (a^+)^*aa^+$. We take $*$ to the last equality, one yields $a^+ = aa^+a^+$. Hence, $a \in R^{EP}$ by [14, Theorem 1.2.1]. From $aa^\#(a^\#)^*(a^+)^*a^+ = (a^+)^*$, we get $(a^+)^*(a^+)^*a^+ = (a^+)^*$. Applying the involution and Lemma 4.1, one yields $a \in R^{SEP}$.

(6) If $x = (a^\#)^*$, then

$$aa^\#(a^\#)^*(a^\#)^*a^+ = (a^\#)^*.$$

Multiplying the equality on the left by $aa^\#$, we obtain

$$aa^\#(a^\#)^* = (a^\#)^*.$$

By [14, Theorem 1.1.3], $a \in R^{EP}$. Hence, $x = (a^\#)^* = (a^+)^*$, by (5), $a \in R^{SEP}$.

(7) If $x = a^+a$, then $aa^\#(a^\#)^*a^+aa^+ = a^+a$, e.g.,

$$aa^\#(a^\#)^*a^+ = a^+a.$$

Multiplying the equality on the right by aa^+ , we obtain $aa^+ = a^+a^2a^+$. Hence, $a \in R^{EP}$ by [14, Theorem 1.2.1]. From $aa^\#(a^\#)^*a^+aa^+ = a^+a$, we get $(a^+)^*a^+ = a^+a$. This gives

$$aa^* = ((a^+)^*a^+)^+ = (a^+a)^+ = a^+a.$$

Hence, $a \in R^{SEP}$ by [14, Theorem 1.5.3]. \square

Multiplying Eq.(4.1) on the left by $a^\#$, we get

$$a^\#(a^\#)^*xa^\# = a^\#x. \quad (4.3)$$

Theorem 4.4. *Let $a \in R^\# \cap R^+$. Then $a \in R^{SEP}$ if and only if Eq.(4.3) has at least one solution in $\Omega_a \setminus \{a^+a\}$.*

Proof. “ \implies ” Since $a \in R^{SEP}$, $a^+ = a^* = a^\#$. Obviously, $x = a^+$ is a solution to Eq.(4.3).

“ \impliedby ” (1) If $x = a^+$, then

$$a^\#(a^\#)^*a^+a^\# = a^\#a^+.$$

Multiplying the equality on the right by a^+a , we obtain

$$a^\#a^+a^+a = a^\#a^+.$$

Multiplying the last equality on the left by a^+a^2 , we get $a^+a^+a = a^+$. Hence, $a \in R^{EP}$. From $a^\#(a^\#)^*a^+a^\# = a^\#a^+$, we get

$$a^+(a^+)^*a^+a^+ = a^+a^+.$$

This gives

$$(a^+)^*a^+a^+ = aa^+(a^+)^*a^+a^+ = aa^+a^+ = a^+.$$

Hence, $a \in R^{SEP}$ by Lemma 4.1.

(2) If $x = a^*$, then

$$a^\#(a^\#)^*a^*a^\# = a^\#a^*.$$

Multiplying the equality on the right by a^+a , we obtain

$$a^\#a^*a^+a = a^\#a^*.$$

Multiplying the equality on the left by a^+a^2 , we obtain

$$a^*a^+a = a^+a^2(a^\#a^*a^+a) = a^+a^2a^\#a^* = a^*.$$

This infers $a = a^+a^2$. Hence, $a \in R^{EP}$ by [14, Theorem 1.2.1]. From $a^\#(a^\#)^*a^*a^\# = a^\#a^*$, we get

$$a^+(a^+)^*a^*a^+ = a^+a^*,$$

e.g., $a^+a^+ = a^+a^* = a^\#a^*$. Hence, $a \in R^{SEP}$ by [14, Theorem 1.5.3].

(3) If $x = (a^\#)^*$, then

$$a^\#(a^\#)^*(a^\#)^*a^\# = a^\#(a^\#)^*.$$

Multiplying the equality on the right by a^+a , we obtain

$$a^\#(a^\#)^* = a^\#(a^\#)^*a^+a.$$

Multiplying the last equality on the left by a^+a^2 , we obtain

$$(a^\#)^* = (a^\#)^*a^+a.$$

This implies that $a^\# = a^+aa^\#$. Hence, $a \in R^{EP}$ by [14, Theorem 1.2.1]. From $a^\#(a^\#)^*(a^\#)^*a^\# = a^\#(a^\#)^*$, we get

$$a^+(a^+)^*(a^+)^*a^+ = a^+(a^+)^*,$$

and

$$(a^+)^*(a^+)^*a^+ = aa^+(a^+)^*(a^+)^*a^+ = aa^+(a^+)^* = (a^+)^*.$$

Applying the involution, one gets $(a^+)^*a^+a^+ = a^+$. Hence, $a \in R^{SEP}$ by Lemma 4.1.

(4) If $x = aa^+$, then

$$a^\#(a^\#)^*aa^+a^\# = a^\#aa^+.$$

Multiplying the equality on the right by a^+a , we get

$$a^\#aa^+ = a^\#aa^+a^+a.$$

Multiplying the last equality on the left by a^+a , we obtain $a^+a^+a = a^+$. Hence, $a \in R^{EP}$. From $a^\#(a^\#)^*aa^+a^\# = a^\#aa^+$, we get

$$a^+(a^+)^*a^+ = a^+.$$

This gives

$$(a^+)^* = aa^+(a^+)^*a^+a = aa^+a = a.$$

Hence, $a \in R^{SEP}$. □

$$a^\#(a^\#)^*x = x. \tag{4.4}$$

Theorem 4.5. *Let $a \in R^\# \cap R^+$. Then $a \in R^{SEP}$ if and only if Eq.(4.4) has at least one solution in $\Omega_a \setminus \{aa^+\}$.*

Proof. “ \implies ” Since $a \in R^{SEP}$, $x = a^+ = a^* = a^\#$ is a solution to Eq.(4.4).

“ \Leftarrow ” (1) If $x = a^+$, then

$$a^\#(a^\#)^*a^+ = a^+.$$

Multiplying the equality on the right by aa^*a^+a , we get $a^\#a^+ = a^*a^+$. Hence, $a \in R^{SEP}$ by [14, Theorem 1.5.3].

(2) If $x = a^*$, then

$$a^\#(a^\#)^*a^* = a^*.$$

This gives

$$a^\#a^+ = a^\#(a^\#)^*a^*a^+ = a^*a^+.$$

Hence, $a \in R^{SEP}$.

(3) If $x = (a^\#)^*$, then

$$a^\#(a^\#)^*(a^\#)^* = (a^\#)^*.$$

It follows that

$$a^\#(a^\#)^*a^* = a^\#(a^\#)^*(a^\#)^*a^*a^* = (a^\#)^*a^*a^* = a^*.$$

Hence, $a \in R^{SEP}$ by (2).

(4) If $x = a^+a$, then

$$a^\#(a^\#)^*a^+a = a^+a.$$

One gets

$$a^\#(a^\#)^*a^* = a^\#(a^\#)^*a^+aa^* = a^+aa^* = a^*.$$

Hence, $a \in R^{SEP}$ by (2). \square

$$(a^\#)^*xa^\# = x. \quad (4.5)$$

Theorem 4.6. Let $a \in R^\# \cap R^+$. Then $a \in R^{SEP}$ if and only if Eq.(4.5) has at least one solution in $\chi_a \cup \{aa^+\} = \{a, a^\#, a^+, a^*, (a^+)^*, (a^\#)^*, aa^+\}$.

Proof. “ \implies ” Since $a \in R^{SEP}$, $a = (a^\#)^*$. Obviously, Eq.(4.5) has at least one solution $x = a$ in χ_a .

“ \Leftarrow ” (1) If $x = a$, then

$$(a^\#)^*aa^\# = a.$$

It follows that

$$a^* = (aa^\#)^*a^\#.$$

By multiplying the equality on the right by a^+a , we arrive at $a^*a^+a = a^*$. Applying the involution to both sides again leads to $a^+aa = a$. Hence, $a \in R^{EP}$ by [14,

Theorem 1.2.1]. From $(a^\#)^*aa^\# = a$, we get $(a^+)^* = (a^+)^*aa^\# = (a^\#)^*aa^\# = a$, then $a \in R^{PI}$ by [14, Theorem 1.5.1]. Hence, $a \in R^{SEP}$ by [14, Theorem 1.5.3].

(2) If $x = a^\#$, then

$$(a^\#)^*a^\#a^\# = a^\#.$$

This gives

$$a = a^\#a^2 = (a^\#)^*a^\#a^\#a^2 = (a^\#)^*aa^\#.$$

By (1), $a \in R^{SEP}$.

(3) If $x = a^+$, then

$$(a^\#)^*a^+a^\# = a^+.$$

Multiplying the equality on the right by a^+a , we get $a^+a^+a = a^+$. Hence, $a \in R^{EP}$ by [14, Theorem 1.2.1]. Hence $x = a^+ = a^\#$, by (2), $a \in R^{SEP}$.

(4) If $x = a^*$, then

$$(a^\#)^*a^*a^\# = a^*.$$

By multiplying the equality on the right by a^+a , we obtain

$$a^*a^+a = a^*.$$

Therefore, we conclude that $a \in R^{EP}$. By [14, Theorem 1.1.3],

$$a^* = (a^\#)^*a^*a^\# = aa^\#a^\# = a^\#.$$

Hence, $a \in R^{SEP}$.

(5) If $x = (a^+)^*$, then

$$(a^\#)^*(a^+)^*a^\# = (a^+)^*.$$

Applying the involution, one gets

$$(a^\#)^*a^+a^\# = a^+.$$

By (3), $a \in R^{SEP}$.

(6) If $x = (a^\#)^*$, then

$$(a^\#)^*(a^\#)^*a^\# = (a^\#)^*.$$

Applying the involution, one obtains

$$(a^\#)^*a^\#a^\# = a^\#.$$

Hence $a \in R^{SEP}$ by (2).

(7) If $x = aa^+$, then $(a^\#)^*aa^+a^\# = aa^+$. Noting that $(a^\#)^*aa^+ = (a^\#)^*$ and $a^\# = aa^+a^\#$. Then

$$a^\# = aa^+a^\# = (a^\#)^*aa^+a^\#a^\# = (a^\#)^*a^\#a^\#.$$

Hence $a \in R^{SEP}$ by (2). \square

5. Characterize *SEP* elements by the solution of bivariate equations

From Eq.(4.5), we construct the following bivariate equations.

$$(a^\#)^* x a^\# = y. \quad (5.1)$$

Theorem 5.1. *Let $a \in R^\# \cap R^+$. Then the general solution to Eq.(5.1) is given by*

$$\begin{cases} x = p + u - a a^+ u a a^+ \\ y = (a^\#)^* p a^\# \end{cases}, \text{ where } p, u \in R. \quad (5.2)$$

Proof. First the formula (5.2) is the solution to Eq.(5.1). In fact,

$$(a^\#)^* x a^\# = (a^\#)^* (p + u - a a^+ u a a^+) a^\# = (a^\#)^* p a^\# = y.$$

Next, let

$$\begin{cases} x = x_0 \\ y = y_0 \end{cases} \quad (5.3)$$

be any solution to Eq.(5.1). Then

$$(a^\#)^* x_0 a^\# = y_0.$$

Choose $p = a a^+ a^* y_0 a^2 a^+$, and $u = x_0$. Then

$$a a^+ u a a^+ = a a^+ x_0 a a^+ = a a^+ a^* ((a^\#)^* x_0 a^\#) a^2 a^+ = a a^+ a^* y_0 a^2 a^+ = p,$$

it follows that

$$x_0 = p + x_0 - a a^+ u a a^+ = p + u - a a^+ u a a^+.$$

Also,

$$(a^\#)^* p a^\# = (a^\#)^* a a^+ a^* y_0 a^2 a^+ a^\# = (a a^\#)^* y_0 a a^\# = (a a^\#)^* ((a^\#)^* x_0 a^\#) a a^\# = (a^\#)^* x_0 a^\# = y_0.$$

Hence, the general solution to Eq.(5.1) is provided by the formula (5.2). \square

Theorem 5.2. *Let $a \in R^\# \cap R^+$. Then $a \in R^{SEP}$ if and only if the general solution to Eq.(5.1) is given by*

$$\begin{cases} x = p + u - a a^+ u a a^+ \\ y = (a^\#)^* p a^* \end{cases}, \text{ where } p, u \in R. \quad (5.4)$$

Proof. “ \implies ” Since $a \in R^{SEP}$, we have $a^* = a^\#$. Obviously, the formula (5.2) and the formula (5.4) are consistent. By Theorem 5.1, we are done.

“ \impliedby ” For the condition, one gets

$$(a^\#)^* (p + u - a a^+ u a a^+) a^\# = (a^\#)^* p a^*,$$

i.e.,

$$(a^\#)^*pa^\# = (a^\#)^*pa^*$$

for any $p \in R$. Choosing $p = a^*$, one gets

$$(aa^\#)^*a^\# = a^*.$$

Applying the involution on the equality, one has

$$(a^\#)^*aa^\# = a.$$

By Theorem 4.6, $a \in R^{SEP}$. □

Theorem 5.3. *Let $a \in R^\# \cap R^+$. Then $a \in R^{SEP}$ if and only if the general solution to Eq.(5.1) is given by*

$$\begin{cases} x = p + u - aa^+ uaa^+ \\ y = apa^\# \end{cases}, \text{ where } p, u \in R. \quad (5.5)$$

Proof. “ \implies ” Given that $a \in R^{SEP}$, we have $a = (a^\#)^*$. Clearly, the formula (5.2) can be expressed as the formula (5.5), as desired by Theorem 5.1.

“ \impliedby ” From the condition, one obtains

$$(a^\#)^*(p + u - aa^+ uaa^+)a^\# = apa^\#,$$

e.g.,

$$(a^\#)^*pa^\# = apa^\#$$

for each $p \in R$. Choosing $p = a$. Then $(a^\#)^*aa^\# = a$. By Theorem 4.6, $a \in R^{SEP}$. □

We change Eq.(5.1) as follows.

$$a(aa^\#)^*xa^\# = y. \quad (5.6)$$

Theorem 5.4. *Let $a \in R^\# \cap R^+$. Then the general solution to Eq.(5.6) is given by*

$$\begin{cases} x = p + u - aa^+ uaa^+ \\ y = apa^\# \end{cases}, \text{ where } p, u \in R \text{ with } a^+p = a^+a^+ap. \quad (5.7)$$

Proof. Since

$$\begin{aligned} a(aa^\#)^*xa^\# &= a(aa^\#)^*(p + u - aa^+ uaa^+)a^\# = a(aa^\#)^*pa^\# \\ &= a(aa^\#)^*aa^+pa^\# = a(aa^\#)^*aa^+a^+apa^\# = a(aa^\#)^*a^+apa^\# = apa^\# = y, \end{aligned}$$

the formula (5.7) is the solution to Eq.(5.6).

Now let

$$\begin{cases} x = x_0 \\ y = y_0 \end{cases} \quad (5.8)$$

be any solution to Eq.(5.6). It follows $a(aa^\#)^*x_0a^\# = y_0$. Choose $p = a^+y_0a$, and $u = x_0 - p$. This gives

$$\begin{aligned} aa^+uuaa^+ &= aa^+(x_0 - p)aa^+ = aa^+x_0aa^+ - aa^+a^+y_0aaa^+ \\ &= aa^+(aa^\#)^*x_0a^\#aaa^+ - aa^+a^+y_0aaa^+ = aa^+a^+a(aa^\#)^*x_0a^\#aaa^+ - aa^+a^+y_0aaa^+ \\ &= aa^+a^+y_0aaa^+ - aa^+a^+y_0aaa^+ = 0. \end{aligned}$$

Hence $x = p + (x_0 - p) = p + u = p + u - aa^+uuaa^+$. At the same time,

$$apa^\# = a(a^+y_0a)a^\# = a(a^+a(aa^\#)^*x_0a^\#)aa^\# = a(aa^\#)^*x_0a^\# = y_0.$$

Consequently, the formula (5.7) offers the general solution to Eq.(5.6). \square

Theorem 5.5. *Let $a \in R^\# \cap R^+$. Then $a \in R^{SEP}$ if and only if Eq.(5.1) has the same solution as Eq.(5.6).*

Proof. It is evident from Theorem 5.3 and Theorem 5.4. \square

6. The solution to non-homogeneous bivariate equations

$$(a^\#)^*xa^\# - y = a^+. \quad (6.1)$$

Theorem 6.1. *Let $a \in R^\# \cap R^+$. Then the general solution to Eq.(6.1) is given by*

$$\begin{cases} x = p + u - aa^+uuaa^+ \\ y = (a^\#)^*pa^\# - a^+ \end{cases}, \text{where } p, u \in R. \quad (6.2)$$

Proof. Clearly, the formula (6.2) is the solution to Eq.(6.1).

Now assuming that

$$\begin{cases} x = x_0 \\ y = y_0 \end{cases} \quad (6.3)$$

represents any solution to Eq.(6.1), then we have

$$(a^\#)^*x_0a^\# - y_0 = a^+.$$

Choose $p = a^*(a^+ + y_0)a$, and $u = x_0 - p$. Then we have

$$aa^+x_0aa^+ = aa^+a^*((a^\#)^*x_0a^\#)a^2a^+ = aa^+a^*(y_0 + a^+)a^2a^+ = aa^+paa^+.$$

It follows that $aa^+uaa^+ = 0$. Hence

$$x_0 = p + (x_0 - p) = p + u = p + u - aa^+uaa^+.$$

Also we have

$$\begin{aligned} (a^\#)^*pa^\# &= (a^\#)^*a^*(a^+ + y_0)aa^\# = a^+aa^\# + (aa^\#)^*y_0aa^\# \\ &= a^+aa^\# + (aa^\#)^*((a^\#)^*x_0a^\# - a^+)aa^\# = (a^\#)^*x_0a^\# = y_0 + a^+. \end{aligned} \quad \square$$

This induces

$$y_0 = (a^\#)^*pa^\# - a^+.$$

Hence the general solution to Eq.(6.1) is given by (6.2).

Theorem 6.2. *Let $a \in R^\# \cap R^+$. Then $a \in R^{SEP}$ if and only if the general solution to Eq.(6.1) is given by*

$$\begin{cases} x = p + u - aa^+uaa^+ \\ y = (a^\#)^*pa^+ - a^* \end{cases}, \text{where } p, u \in R. \quad (6.4)$$

Proof. “ \Rightarrow ” Given that $a \in R^{SEP}$, we have $a^+ = a^* = a^\#$. It’s evident that the formula (6.2) can be expressed as the formula (6.4), as desired by Theorem 6.1.

“ \Leftarrow ” The condition implies

$$(a^\#)^*(p + u - aa^+uaa^+)a^\# - ((a^\#)^*pa^+ - a^*) = a^+$$

for $p \in R$. Choose $p = 0$. Then one gets $a^+ = a^*$ which follows that $a \in R^{PI}$. Choose $p = a^\#$. Then the equation becomes $(a^\#)^*a^\#a^\# = (a^\#)^*a^\#a^+$, and

$$aa^\# = a^3a^+a^*(a^\#)^*a^\#a^\# = a^3a^+a^*(a^\#)^*a^\#a^+ = aa^+.$$

Thus $a \in R^{EP}$ by [14, Theorem 1.2.1]. Therefore we deduce $a \in R^{SEP}$. \square

Revised Eq.(6.1) as follows:

$$(a^\#)^*xa^+ - y = a^*. \quad (6.5)$$

Similar to the proof of Theorem 6.1, we have the following theorem.

Theorem 6.3. *Let $a \in R^\# \cap R^+$. Then the general solution to Eq.(6.5) is given by*

$$\begin{cases} x = p + u - aa^+ua^+a \\ y = (a^\#)^*pa^+ - a^* \end{cases}, \text{where } p, u \in R. \quad (6.6)$$

Theorem 6.4. *Let $a \in R^\# \cap R^+$. Then $a \in R^{SEP}$ if and only if Eq.(6.5) has the same solution as Eq.(6.1).*

Proof. “ \implies ” Suppose that $a \in R^{SEP}$. Then the formula (6.4) and the formula (6.6) are the same, by Theorem 6.2 and Theorem 6.3, we are done.

“ \Leftarrow ” Indeed,

$$\begin{cases} x = a \\ y = (a^\#)^* - a^* \end{cases} \quad (6.7)$$

clearly represents a solution to Eq.(6.5).

By the assumption, it is also a solution to Eq.(6.1). Hence we obtain

$$(a^\#)^* aa^\# - (a^\#)^* + a^* = a^+.$$

Multiplying the equality on the right by aa^+ , one gets

$$(a^\#)^* aa^\# = (a^\#)^*.$$

This gives $a \in R^{EP}$ by [14, Theorem 1.1.3] and

$$a^+ = (a^\#)^* aa^\# - (a^\#)^* + a^* = a^*.$$

Therefore $a \in R^{SEP}$. \square

7. Using nil-cleanliness of aa^*a^+a to characterize SEP elements

Theorem 7.1. *Let $a \in R^\# \cap R^+$. Then $a \in R^{SEP}$ if and only if $aa^*a^+a = u + p$, where $u^2 = 0$, $p \in PE(R)$, and $pu = up = 0$.*

Proof. “ \implies ” Given that $a \in R^{SEP}$, we know that $aa^*a^+a \in PE(R)$ by Theorem 2.1. Let’s choose $u = 0$ and $p = aa^*a^+a$. Then, we have $aa^*a^+a = u + p$, where $u^2 = 0$ and $pu = up = 0$.

“ \Leftarrow ” From the assumption, we have

$$aa^*a^+ap = (u + p)p = up + p^2 = p,$$

$$paa^*a^+a = p(u + p) = pu + p^2 = p.$$

Furthermore, we find

$$0 = u^2 = (aa^*a^+a - p)^2 = (aa^*a^+a)^2 - aa^*a^+ap - paa^*a^+a + p = aa^*a^+a^2a^*a^+a - p.$$

Thus, we conclude that $aa^*a^+a^2a^*a^+a = p = paa^*a^+a$. Applying the involution to the last equality, one gets

$$a^+a^2a^*a^+a^2a^* = p = aa^*a^+a^2a^*a^+a.$$

Then, we have

$$a^+a^2a^*a^+a^2a^* = a^+a^2a^*a^+a^2a^*a^+a.$$

Multiplying $a^\# a$ on the left of both sides, we get

$$aa^*a^+a^2a^* = aa^*a^+a^2a^*a^+a.$$

To proceed, multiply the equality by $(a^\#)^*a^+$, we obtain $a^+a^2a^* = a^+a^2a^*a^+a$. Finally, multiplying $a^+a^\# a$ on the left, one gets $a^* = a^*a^+a$, thus, $a^+aa = a$. Consequently, $a \in R^{EP}$. We have

$$aa^*aa^* = aa^*a^+a^2a^*a^+a = p = aa^*a^+ap = aa^*p,$$

$$aa^*aa^*aa^* = aa^*p = p = aa^*aa^*,$$

$$a^*aa^*aa^* = a^+aa^*aa^*aa^* = a^+aa^*aa^* = a^*aa^*,$$

it follows by applying the involution to the last equality, one yields

$$aa^*aa^*a = aa^*a,$$

$$a^*aa^* = a^+(aa^*aa^*a)a^+ = a^+(aa^*a)a^+ = a^*.$$

Hence $a \in R^{PI}$, and so $a \in R^{SEP}$. \square

Theorem 7.2. *Let $a \in R^\# \cap R^+$. Then $a \in R^{SEP}$ if and only if $aa^*a^+a = u + p$, where $u^2 = 1$, $up = pu = -p$ and $1 - p = aa^*$, $p \in PE(R)$.*

Proof. “ \implies ” Since $a \in R^{SEP}$, we have $aa^*a^+a = aa^+$ by Theorem 2.1. Choose $u = 2aa^+ - 1$, $p = 1 - aa^+ \in PE(R)$. Then

$$u^2 = 1, \quad up = aa^+ - 1 = -p, \quad pu = aa^+ - 1 = -p, \quad 1 - p = aa^+ = aa^*.$$

Clearly, $aa^*a^+a = u + p$.

“ \impliedby ” From the conditions, we have

$$aa^*a^+ap = (u + p)p = up + p = -p + p = 0,$$

$$paa^*a^+a = p(u + p) = p + pu = p - p = 0,$$

and

$$1 = u^2 = (aa^*a^+a - p)^2 = aa^*a^+a^2a^*a^+a - aa^*a^+ap - paa^*a^+a + p = aa^*a^+a^2a^*a^+a + p.$$

After applying the involution to the last equality, we obtain

$$aa^*a^+a^2a^*a^+a = a^+a^2a^*a^+a^2a^*.$$

Similar to the proof of Theorem 7.1, we get $a \in R^{EP}$. This gives

$$aa^* = aa^*a^+a = u + p,$$

and

$$aa^*aa^* = (u + p)^2 = u^2 + up + pu + p = 1 - p - p + p = 1 - p = aa^*.$$

This leads to

$$a = aa^*(a^+)^* = aa^*aa^*(a^+)^* = aa^*a.$$

In summary, $a \in R^{SEP}$. \square

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