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ON GROUP CROSSED PRODUCTS

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Abstract. Let π be a group and let $H=\{H_\alpha\}_{\alpha\in\pi}$ be a Hopf π-coalgebra in the sense of Turaev [8]. Let H act weakly on an algebra A and $\sigma: H_1\otimes H_1\to A$ a k-linear map. Then we first introduce the notion of a π-crossed product $A\#^\pi_\sigma H=\{A\#_\sigma H_\alpha\}_{\alpha\in\pi}$ and find some sufficient and necessary conditions under which each $A\#_\sigma H_\alpha$ forms an algebra. Next we define a comultiplication, a counit and an antipode on $A\#^\pi_\sigma H$ making it into a Hopf π-coalgebra. Finally, we obtain the duality theorem of π-crossed product $A\#^\pi_\sigma H$, generalizing Corollary 5.8 in the authors' paper [6].

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Introduction

As a generalization of ordinary Hopf algebras ([7]), Hopf group-colgebras were studied in the work of Turaev [8] related to homotopy quantum field theories. Let us note that there exists a symmetric monoidal category, the so called Turaev category, the Hopf algebras in which are the same as Hopf group-coalgebras ([4]). A purely algebraic study of Hopf group-coalgebras can be found in the references [9, 10, 11, 12].

It is well-known that crossed products of an algebra and a Hopf algebra are important tools in classical Hopf algebra theory (see [1, 3]). It is natural to ask whether or not there exists an analogue of the crossed product for Hopf algebras in the setting of Hopf π-coalgebras. This becomes a motivation of our paper.

This paper is organized as follows.

In Section 1, we recall definitions and basic results related to Hopf group-coalgebras.

In Section 2, we introduce the notion of a π-crossed product and give a sufficient and necessary condition making each $A\#_\sigma H_\alpha$ into an algebra and $A\#^\pi_\sigma H$ become

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a Hopf π-coalgebra under group-crossed product and the usual tensor product coalgebra (see Theorem 2.5) which extends the results of crossed product and group smash product (cf. [6, 10]). We also get a sufficient condition for π-crossed product algebra to be semisimple in Theorem 2.7.

In Section 3, we prove an analogue of the Blattner-Cohen-Montgomery's duality theorem in [2] for π-crossed products with convolution invertible σ, generalizing Corollary 5.8 in the authors' paper [6] (see Theorem 3.4).

1. Preliminaries

Throughout this paper, we let π be a discrete group (with neutral element 1), k will be a fixed field, and the tensor product $\otimes = \otimes_k$ is always assumed to be over k. If U and V are k-vector spaces, $T_{U,V} : U \otimes V \longrightarrow V \otimes U$ will denote the flip map defined by $T_{U,V}(u \otimes v) = v \otimes u$, for all $u \in U$ and $v \in V$.

Definition 1.1. ([8] and [9]) A π-coalgebra is a family of k-spaces $C = \{C_\alpha\}_{\alpha\in\pi}$ together with a family of k-linear maps $\Delta = \{\Delta_{\alpha,\beta} : C_{\alpha\beta} \longrightarrow C_\alpha \otimes C_\beta\}_{\alpha,\beta\in\pi}$ (called a comultiplication) and a k-linear map $\varepsilon : C_1 \longrightarrow k$ (called a counit), such that Δ is coassociative in the sense that,

- $(\Delta_{\alpha,\beta} \otimes id_{C_\gamma})\Delta_{\alpha\beta,\gamma} = (id_{C_\alpha} \otimes \Delta_{\beta,\gamma})\Delta_{\alpha,\beta\gamma},$ for any $\alpha, \beta, \gamma \in \pi$.
- $(id_{C_\alpha} \otimes \varepsilon)\Delta_{\alpha,1} = id_{C_\alpha} = (\varepsilon \otimes id_{C_\alpha})\Delta_{1,\alpha},$ for all $\alpha \in \pi$.

We use the Sweedler's notation (see Virelizier [9]) for a comultiplication in the following way: for any $\alpha, \beta \in \pi$ and $c \in C_{\alpha\beta}$, we write

$$\Delta_{\alpha,\beta}(c) = c_{(1,a)} \otimes c_{(2,\beta)}.$$

Definition 1.2. ([8] and [9]) A Hopf π-coalgebra is a π-coalgebra $H = (\{H_\alpha\}, \Delta, \varepsilon)$ endowed with a family of k-linear maps $S = \{S_\alpha : H_\alpha \longrightarrow H_{\alpha^{-1}}\}_{\alpha\in\pi}$ (called antipode) such that:

(1) each H_α is an algebra with multiplication m_α and unit element $h_\alpha \in H_\alpha$,
(2) $\varepsilon : H_1 \to k$ and $\Delta_{\alpha,\beta} : H_{\alpha\beta} \to H_\alpha \otimes H_\beta$ are algebra maps, for all $\alpha, \beta \in \pi$,
(3) for each $\alpha \in \pi$, $m_\alpha(S_{\alpha^{-1}} \otimes id_{H_\alpha})\Delta_{\alpha^{-1},\,\alpha} = \varepsilon 1_\alpha = m_\alpha(id_{H_\alpha} \otimes S_{\alpha^{-1}})\Delta_{\alpha,\,\alpha^{-1}}$.

If a π-coalgebra H satisfies conditions (1) and (2), we call it a semi-Hopf π-coalgebra.

We remark that the notion of a Hopf π-coalgebra is notself-dual and In particular, $(H_1, m_1, 1_1, \Delta_{1,1}, \varepsilon, S_1)$ is an ordinary Hopf algebra. The antipode $S = \{S_\alpha\}_{\alpha\in\pi}$ of H is said to be bijective if each S_α is bijective. The antipode of a Hopf π-coalgebra is anti-multiplicative and anti-comultiplicative, i.e., for all $\alpha, \beta \in \pi, a, b \in H_\alpha$,

$$S_\alpha(ab) = S_\alpha(b)S_\alpha(a), \quad S_\alpha(1_\alpha) = 1_{\alpha^{-1}},$$

$$\Delta_{\beta^{-1},\alpha^{-1}} S_{\alpha\beta} = T_{H_{\alpha^{-1}},H_{\beta^{-1}}}(S_\alpha \otimes S_\beta)\Delta_{\alpha,\beta}, \quad \varepsilon S_1 = \varepsilon.$$

Definition 1.3. Let H be a Hopf π-coalgebra and A an algebra over k. H acts weakly on A if there exists a family of maps : $H_\alpha \otimes A \longrightarrow A,\ h \otimes a \mapsto h \cdot a, \forall \alpha \in \pi,\ h \in H_\alpha$, such that

$$(1) \qquad 1_\alpha \cdot a = a, \text{ for any } a \in A,\ \alpha \in \pi,$$

$$(2) \qquad h \cdot (ab) = (h_{(1,\alpha)} \cdot a)(h_{(2,\beta)} \cdot b), \text{ for all } h \in H_{\alpha\beta},\ a, b \in A,$$

$$(3) \qquad h \cdot 1_A = \varepsilon(h)1_A, \text{ for every } h \in H_1.$$

Furthermore, if A is an H_α-module for each $\alpha \in \pi$ and satisfies (2) and (3), we call that A is a π-H-module algebra.

2. π-Crossed Products

Definition 2.1. Let H be a Hopf π-coalgebra and A an algebra over k. H act weakly on A. Let $\sigma : H_1 \otimes H_1 \to A$ be a k-linear map. Define $A \otimes H = \{A \otimes H_\alpha\}_{\alpha \in \pi}$. For each $A \otimes H_\alpha$, we define a multiplication by

$$(a \otimes h)(b \otimes g) = a(h_{(1,1)} \cdot b)\sigma(h_{(2,1)}, g_{(1,1)}) \otimes h_{(3,\alpha)}g_{(2,\alpha)}. \tag{1}$$

If each $A \otimes H_\alpha$ is associative with $1_A \otimes 1_\alpha$ as identity element, we call $A \otimes H$ a π-crossed product, denoted by $A \#_\sigma^\pi H$.

We now determine simple necessary and sufficient conditions on σ and the weak action for $A \#_\sigma^\pi H$ to be a π-crossed product.

Proposition 2.2. *$A \#_\sigma^\pi H$ is a π-crossed product if and only if*

$$\sigma(1_1, h) = \varepsilon(h)1_A = \sigma(h, 1_1),\ \forall\ h \in H_1,\ \textit{here } 1_1 \textit{ is the unit of } H_1, \tag{2}$$

$$(h_{(1,1)} \cdot (g_{(1,1)} \cdot a))\sigma(h_{(2,1)}, g_{(2,1)}) = \sigma(h_{(1,1)}, g_{(1,1)})(h_{(2,1)}g_{(2,1)} \cdot a), \tag{3}$$

$$\sigma(h_{(1,1)}, g_{(1,1)})\sigma(h_{(2,1)}g_{(2,1)}, k) = (h_{(1,1)} \cdot \sigma(g_{(1,1)}, k_{(1,1)}))\sigma(h_{(2,1)}, g_{(2,1)}k_{(2,1)}). \tag{4}$$

Proof. It is similar to the proof of crossed product in [1]. □

Example 2.3. (1) If we set $\pi = \{1\}$, then the π-crossed product is the general crossed product.
(2) If we take $\sigma(h, l) = \varepsilon(h)\varepsilon(l)1_A$, then the π-crossed product has the form of π-smash product. From Proposition 2.2, we get each $A \# H_\alpha$ forms an algebra if A is π-H-module algebra.

If $A\#_\sigma^\pi H$ is a π-crossed product, we will consider the conditions making it be a Hopf π-coalgebra.

Proposition 2.4. *Let $A\#_\sigma^\pi H$ be a π-crossed product and A a bialgebra. Define the comultiplication and counit as follows:*

$$\begin{aligned} \Delta_{\alpha,\beta} : A\#_\sigma H_{\alpha\beta} &\rightarrow (A\#_\sigma H_\alpha)\otimes(A\#_\sigma H_\beta), \\ a\#_\sigma h &\mapsto (a_1\#_\sigma h_{(1,\alpha)})\otimes(a_2\#_\sigma h_{(2,\beta)}), \\ \varepsilon : A\#_\sigma H_1 &\rightarrow k, \\ a\#_\sigma h &\mapsto \varepsilon_A(a)\varepsilon(h), \end{aligned}$$

then $A\#_\sigma^\pi H$ is a semi-Hopf π-coalgebra if and only if

$$\Delta(h\cdot b)=h_{(1,1)}\cdot b_1\otimes h_{(2,1)}\cdot b_2,\ \varepsilon_A(h\cdot b)=\varepsilon(h)\varepsilon_A(b),\ \forall\, h\in H_1, b\in A. \tag{5}$$

$$h_{(1,\alpha)}\otimes h_{(2,1)}\cdot b=h_{(2,\alpha)}\otimes h_{(1,1)}\cdot b,\ \forall\, h\in H_\alpha, b\in A. \tag{6}$$

$$\Delta(\sigma(h,l))=\sigma(h_{(1,1)},l_{(1,1)})\otimes\sigma(h_{(2,1)},l_{(2,1)}),\ \varepsilon_A(\sigma(h,l))=\varepsilon(h)\varepsilon(l). \tag{7}$$

$$h_{(1,\alpha)}l_{(1,\alpha)}\otimes\sigma(h_{(2,1)},l_{(2,1)})=h_{(2,\alpha)}l_{(2,\alpha)}\otimes\sigma(h_{(1,1)},l_{(1,1)}),\ \forall\, h,l\in H_\alpha. \tag{8}$$

Proof. If $A\#_\sigma^\pi H$ satisfy Eqs.(5)-(8), then we prove $A\#_\sigma^\pi H$ is a semi-Hopf π-coalgebra. It is easy to see $\Delta=\{\Delta_{\alpha,\beta}\}_{\alpha,\beta\in\pi}$ and ε are comultiplication and counit. We prove them are algebra maps. For all $a,b\in A$ and $h,g\in H_{\alpha\beta}$,

$$\begin{aligned} & \Delta_{\alpha,\beta}((a\#_\sigma h)(b\#_\sigma g)) \\ \overset{(1)}{=}\ & \Delta_{\alpha,\beta}(a(h_{(1,1)}\cdot b)\sigma(h_{(2,1)},g_{(1,1)})\#_\sigma h_{(3,\alpha\beta)}g_{(2,\alpha\beta)}) \\ \overset{(5)(7)}{=}\ & (a_1(h_{(1,1)}\cdot b_1)\sigma(\underline{h_{(3,1)}},g_{(1,1)})\#_\sigma h_{(5,\alpha)}g_{(3,\alpha)})\otimes \\ & (a_2(\underline{h_{(2,1)}\cdot b_2})\sigma(h_{(4,1)},g_{(2,1)})\#_\sigma h_{(6,\beta)}g_{(4,\beta)}) \\ \overset{(6)}{=}\ & (a_1(h_{(1,1)}\cdot b_1)\sigma(h_{(2,1)},g_{(1,1)})\#_\sigma \underline{h_{(5,\alpha)}g_{(3,\alpha)}})\otimes \\ & (a_2(h_{(3,1)}\cdot b_2)\underline{\sigma(h_{(4,1)},g_{(2,1)})}\#_\sigma h_{(6,\beta)}g_{(4,\beta)}) \\ \overset{(6)(8)}{=}\ & (a_1(h_{(1,1)}\cdot b_1)\sigma(h_{(2,1)},g_{(1,1)})\#_\sigma h_{(3,\alpha)}g_{(2,\alpha)})\otimes \\ & (a_2(h_{(4,1)}\cdot b_2)\sigma(h_{(5,1)},g_{(3,1)})\#_\sigma h_{(6,\beta)}g_{(4,\beta)}) \\ =\ & \Delta_{\alpha,\beta}(a\#_\sigma h)\Delta_{\alpha,\beta}(b\#_\sigma g). \end{aligned}$$

and for all $h, g \in H_1$, we compute

$$\begin{aligned}
\varepsilon((a\#_\sigma h)(b\#_\sigma g)) &= \varepsilon(a(h_{(1,1)}\cdot b)\sigma(h_{(2,1)}, g_{(1,1)})\#_\sigma h_{(3,1)}g_{(2,1)}) \\
&= \varepsilon(a)\varepsilon(h_{(1,1)}\cdot b)\varepsilon(\sigma(h_{(2,1)}, g_{(1,1)}))\varepsilon(h_{(3,1)})\varepsilon(g_{(2,1)}) \\
&\overset{(5)(7)}{=} \varepsilon_A(ab)\varepsilon(hg) \\
&= \varepsilon(a\#_\sigma h)\varepsilon(b\#_\sigma g).
\end{aligned}$$

Conversely, if $\varepsilon((a\#_\sigma h)(b\#_\sigma g)) = \varepsilon(a\#_\sigma h)\varepsilon(b\#_\sigma g)$, then we take $a = b = 1_A$, and we get

$$\varepsilon(\sigma(h, g)) = \varepsilon(h)\varepsilon(g).$$

If we take $a = 1_A, g = 1_1$, we prove

$$\varepsilon_A(h\cdot b) = \varepsilon(h)\varepsilon_A(b).$$

If $\Delta_{\alpha,\beta}((a\#_\sigma h)(b\#_\sigma g)) = \Delta_{\alpha,\beta}(a\#_\sigma h)\Delta_{\alpha,\beta}(b\#_\sigma g)$, taking $a = b = 1_A$ and $h, g \in H_1$, we get

$$\Delta(\sigma(h, g)) = \sigma(h_{(1,1)}, g_{(1,1)}) \otimes \sigma(h_{(2,1)}, g_{(2,1)}).$$

Taking $a = b = 1_A$ and $h, g \in H_\alpha$, we have $[(\sigma(h_{(1,1)}, g_{(1,1)})\#_\sigma h_{(3,\alpha)}g_{(3,\alpha)})] \otimes [\sigma(h_{(2,1)}, g_{(2,1)})$
$\#_\sigma h_{(4,1)}g_{(4,1)}] = [(\sigma(h_{(1,1)}, g_{(1,1)})\#_\sigma h_{(2,\alpha)}g_{(2,\alpha)})] \otimes [\sigma(h_{(3,1)}, g_{(3,1)})\#_\sigma h_{(4,1)}g_{(4,1)}]$, applying $\varepsilon_A \otimes H_\alpha \otimes A \otimes \varepsilon$ to both sides, and we obtain

$$h_{(1,\alpha)}l_{(1,\alpha)} \otimes \sigma(h_{(2,1)}, l_{(2,1)}) = h_{(2,\alpha)}l_{(2,\alpha)} \otimes \sigma(h_{(1,1)}, l_{(1,1)}).$$

If we take $a = 1_A, g = 1_1, h \in H_1$, we get

$$\Delta(h\cdot b) = h_{(1,1)}\cdot b_1 \otimes h_{(2,1)}\cdot b_2.$$

and if we take $a = 1_A, g = 1_1, h \in H_\alpha$, we get $(h_{(1,1)}\cdot b_1\#_\sigma h_{(3,\alpha)}) \otimes (h_{(2,1)}\cdot b_1\#_\sigma h_{(4,1)}) = (h_{(1,1)}\cdot b_1\#_\sigma h_{(2,\alpha)}) \otimes (h_{(3,1)}\cdot b_1\#_\sigma h_{(4,1)})$, applying $\varepsilon_A \otimes I_{H_\alpha} \otimes A \otimes \varepsilon_{H_1}$ to both sides, we obtain

$$h_{(1,\alpha)} \otimes h_{(2,1)}\cdot b = h_{(2,\alpha)} \otimes h_{(1,1)}\cdot b.$$

□

Theorem 2.5. *If $A\#_\sigma^\pi H$ is a semi-Hopf π-coalgebra, A is a Hopf algebra, and H is a Hopf π-coalgebra, then $A\#_\sigma^\pi H$ is a Hopf π-coalgebra. The antipode is defined as :*

$$\begin{aligned}
S_\alpha : A\#_\sigma H_\alpha &\rightarrow A\#_\sigma H_{\alpha^{-1}}, \\
a\#_\sigma h &\mapsto (S_A(\sigma(S_1(h_{(2,1)}), h_{(3,1)}))\#_\sigma S_\alpha(h_{(1,\alpha)}))(S(a)\#_\sigma 1_{\alpha^{-1}}).
\end{aligned}$$

Conversely, if $A\#_\sigma^\pi H$ is a Hopf π-coalgebra, then A is a Hopf algebra and H is a Hopf π-coalgebra.

Proof. We prove $\{S_\alpha\}_{\alpha\in\pi}$ is the antipode of $A\#_\sigma^\pi H$. For all $h\in H_1$, we compute

$$\begin{aligned}
& S_{\alpha^{-1}}(a_1\#_\sigma h_{(1,\alpha^{-1})})(a_2\#_\sigma h_{(2,\alpha)})\\
= & (S_A(\sigma(S_1(h_{(2,1)}),h_{(3,1)}))\#_\sigma S_{\alpha^{-1}}(h_{(1,\alpha^{-1})}))(S(a_1)\#_\sigma 1_\alpha)(a_2\#_\sigma h_{(4,\alpha)})\\
= & \varepsilon(a)S_A(\sigma(S_1(h_{(3,1)}),h_{(4,1)}))\sigma(S_1(h_{(2,1)}),h_{(5,1)})\#_\sigma S_{\alpha^{-1}}(h_{(1,\alpha^{-1})})h_{(6,\alpha)}\\
\overset{(7)}{=} & \varepsilon(a)S_A((\sigma(S_1(h_{(2,1)}),h_{(3,1)}))_1)(\sigma(S_1(h_{(2,1)}),h_{(3,1)}))_2\#_\sigma S_{\alpha^{-1}}(h_{(1,\alpha^{-1})})h_{(4,\alpha)}\\
= & \varepsilon(a)1_A\#_\sigma S_{\alpha^{-1}}(h_{(1,\alpha^{-1})})h_{(2,\alpha)}\\
= & \varepsilon(a\#_\sigma h)(1_A\#_\sigma 1_1).
\end{aligned}$$

and define $\sigma^{-1}: H_1\otimes H_1\to A$, $\sigma^{-1}(h,g)=S_A(\sigma(h,g))$. Since Eq.(7) satisfies, σ^{-1} is the convolution inverse of σ. And from Eq.(4), for all $h,g,k\in H_1$, we have

$$h\cdot\sigma^{-1}(g,k)=\sigma(h_{(1,1)},g_{(1,1)}k_{(1,1)})\sigma^{-1}(h_{(2,1)},g_{(2,1)})\sigma^{-1}(h_{(3,1)},g_{(3,1)}). \qquad (9)$$

So

$$\begin{aligned}
& (a_1\#_\sigma h_{(1,\alpha)})S_{\alpha^{-1}}(a_2\#_\sigma h_{(2,\alpha^{-1})})\\
= & [a_1(h_{(1,1)}\cdot\sigma^{-1}(S_1(h_{(6,1)}),h_{(7,1)}))\sigma(h_{(2,1)},S_1(h_{(5,1)}))\#_\sigma h_{(3,\alpha)}S_{\alpha^{-1}}(h_{(4,\alpha^{-1})})](S(a_2)\#_\sigma 1_\alpha)\\
= & [a_1(h_{(1,1)}\cdot\sigma^{-1}(S_1(h_{(4,1)}),h_{(5,1)}))\sigma(h_{(2,1)},S_1(h_{(3,1)}))\#_\sigma 1_\alpha)](S(a_2)\#_\sigma 1_\alpha)\\
\overset{(9)}{=} & [a_1\sigma(h_{(1,1)},1_1)\sigma^{-1}(h_{(2,1)}S_1(h_{(3,1)}),h_{(4,1)})\#_\sigma 1_\alpha](S(a_2)\#_\sigma 1_\alpha)\\
= & \varepsilon(h)(a_1\#_\sigma 1_\alpha)(S(a_2)\#_\sigma 1_\alpha)\\
= & \varepsilon(a\#_\sigma h)(1_A\#_\sigma 1_1).
\end{aligned}$$

Conversely, if $A\#_\sigma^\pi H$ is a Hopf π-coalgebra, and define $i_\alpha: H_\alpha\to A\#_\sigma H_\alpha$, $i_\alpha(h)=1_A\#_\sigma h$, $\forall\, h\in H_\alpha$, then $i=\{i_\alpha\}_{\alpha\in\pi}$ is a π-coalgebra map. Define a family of algebra maps $p_\alpha: A\#_\sigma H_\alpha\to H_\alpha$, $p_\alpha(b\#_\sigma h)=\varepsilon(b)h$. For all $h\in H_\alpha$, setting $S'_\alpha(h)=p_{\alpha^{-1}}\circ S\circ i_\alpha(h)$, we prove $S'=\{S'_\alpha\}_{\alpha\in\pi}$ is the antipode of H.

$$\begin{aligned}
& S'_{\alpha^{-1}}(h_{(1,\alpha^{-1})})h_{(2,\alpha)}\\
= & (p_\alpha\circ S\circ i_{\alpha^{-1}}(h_{(1,\alpha^{-1})}))(p_\alpha\circ i_\alpha(h_{(2,\alpha)}))\\
= & p_\alpha(S(i_{\alpha^{-1}}(h_{(1,\alpha^{-1})}))i_\alpha(h_{(2,\alpha)}))\\
= & \varepsilon(h)p_\alpha(1_A\#_\sigma 1_\alpha)\\
= & \varepsilon(h)1_\alpha,
\end{aligned}$$

and similarly we can prove $h_{(1,\alpha)}S'_{\alpha^{-1}}(h_{(2,\alpha^{-1})}) = \varepsilon(h)1_\alpha$, $\forall h \in H_1, \alpha \in \pi$. So H is a Hopf π-coalgebra.

Next, we will prove A is a Hopf algebra. Define maps

$$\begin{aligned} p_A \quad &: \quad A\#_\sigma H_1 \to A,\ b\#_\sigma h \mapsto \varepsilon(h)b, \\ j_A \quad &: \quad A \to A\#_\sigma H_1,\ b \mapsto b\#_\sigma 1_1. \end{aligned}$$

It is obvious that j_A is a bialgebra map. We set $A = A\#_\sigma 1_1$ and $\varphi = j_A \circ p_A$,

$$\begin{aligned} \varphi((b\#_\sigma 1_1)(a\#_\sigma h)) &= \varphi(ba\#_\sigma h) \\ &= \varepsilon(h)(ba\#_\sigma 1_1) \\ &= \varepsilon(h)(b\#_\sigma 1_1)(a\#_\sigma h) \\ &= (b\#_\sigma 1_1)\varphi(a\#_\sigma h). \end{aligned}$$

So φ is a left $A\#_\sigma 1_1$-module map. Since $(b_1\#_\sigma 1_1)S(b_2\#_\sigma 1_1) = \varepsilon(b)(1_A\#_\sigma 1_1)$, we get $(b_1\#_\sigma 1_1)\varphi \circ S(b_2\#_\sigma 1_1) = \varepsilon(b)(1_A\#_\sigma 1_1)$. This means $\varphi \circ S|_{A\#_\sigma 1_1}$ is the right inverse of $I_A\#_\sigma 1_1$. So $\varphi \circ S = S$ in $A\#_\sigma 1_1$ and we get $S(A\#_\sigma 1_1) \subset A\#_\sigma 1_1$. We prove A is a Hopf algebra. □

Let H be a Hopf π-coalgebra. H is said to be of finite type if, for all $\alpha \in \pi$, H_α is finite-dimensional as a k-vector space. A Hopf π-coalgebra $H = \{H_\alpha\}_{\alpha\in\pi}$ is said to be semisimple if each algebra H_α is semisimple.

Lemma 2.6. ([9]) *Let $H = \{H_\alpha\}_{\alpha\in\pi}$ be a finite type Hopf π-coalgebra. Then H is semisimple if and only if H_1 is semisimple.*

From Propositions 2.3, 2.4, 2.5, Lemma 2.6, and Theorem 2.6 of [3], we get

Theorem 2.7. *Let A be a finite dimensional Hopf algebra and H a finite type Hopf π-coalgebra. Then the π-crossed product $A\#_\sigma^\pi H$ satisfying Eq.(5)-(8) is a finite type Hopf π-coalgebra with σ invertible. If A and H_1 are semisimple, then $A\#_\sigma H_1$ is semisimple and furthermore $A\#_\sigma^\pi H$ is semisimple.*

3. The Duality Theorem for π-Crossed Products

In this section, we will construct the duality theorem for a group-crossed product. We assume throughout this section that H is a finite type Hopf π-coalgebra, and A is an algebra with weak H-action.

Let H be a finite type Hopf π-coalgebra, then H_1 is a finite dimensional Hopf algebra. So the dual vector space H_1^* has a natural structure of a Hopf algebra with the structure operations dual to those of H_1:

$$\begin{aligned}
&\langle \phi\varphi, h\rangle = \langle \phi\otimes\varphi, \Delta(h)\rangle \triangleq \langle \phi, h_{(1,1)}\rangle\langle \varphi, h_{(2,1)}\rangle,\\
&\langle \widetilde{1}, c\rangle = \varepsilon(c), \text{ where } \widetilde{1} \text{ is the unit of } H_1^*,\\
&\langle \Delta(\phi), h\otimes g\rangle = \langle \phi, hg\rangle \triangleq \langle \phi_1, h\rangle\langle \phi_2, g\rangle,\\
&\varepsilon_{H^*}(\phi) = \langle \phi, 1_1\rangle, \text{ where } 1_1 \text{ is the unit of } H_1,\\
&\langle \widetilde{S}(\phi), h\rangle = \langle \phi, S_1(h)\rangle.
\end{aligned}$$

Lemma 3.1. *Let H be a finite type Hopf π-coalgebra. Then for each $\alpha \in \pi$, $A\#_\sigma H_\alpha$ is a left H_1^*-module algebra via*

$$f\cdot(a\#_\sigma h) = a\#_\sigma f \rightharpoonup h = a\#_\sigma h_{(1,\alpha)}\langle f, h_{(2,1)}\rangle,\ f\in H_1^*, h\in H_\alpha, a\in A.$$

Proof. It is easy to see $A\#_\sigma H_\alpha$ is a left H_1^*-module. We compute

$$\begin{aligned}
&(f_1\cdot(a\#_\sigma h))(f_2\cdot(b\#_\sigma g))\\
=\ &\langle f, h_{(2,1)}g_{(2,1)}\rangle(a\#_\sigma h_{(1,\alpha)})(b\#_\sigma g_{(1,\alpha)})\\
=\ &\langle f, h_{(4,1)}g_{(3,1)}\rangle(a(h_{(1,1)}\cdot b)\sigma(h_{(2,1)}, g_{(1,1)})\#_\sigma h_{(3,\alpha)}g_{(2,\alpha)})\\
=\ &f\cdot(a(h_{(1,1)}\cdot b)\sigma(h_{(2,1)}, g_{(1,1)})\#_\sigma h_{(3,\alpha)}g_{(2,\alpha)})\\
=\ &f\cdot((a\#_\sigma h)(b\#_\sigma g)),
\end{aligned}$$

and

$$f\cdot(1_A\#_\sigma 1_\alpha) = \langle f, 1_1\rangle(1_A\#_\sigma 1_\alpha) = \varepsilon_{H^*}(f)(1_A\#_\sigma 1_\alpha).$$

So $A\#_\sigma H_\alpha$ is a left H_1^*-module algebra, as needed. □

Lemma 3.2. *The map $\alpha : (A\#_\sigma H_\alpha)\# H_1^* \longrightarrow End(A\#_\sigma H_\alpha)_A$ (here $\#$ means smash product and $End(A\#_\sigma H_\alpha)_A$ means the ring of right A-module endomorphisms) defined by*

$$\alpha((x\#_\sigma h)\# f)(y\#_\sigma g) = (x\#_\sigma h)(y\#_\sigma f\rightharpoonup g) = (x\#_\sigma h)(y\#_\sigma\langle f, g_{(2,1)}\rangle g_{(1,\alpha)})$$

for all $x, y \in A, h, g \in H_\alpha, f \in H_1^*$ is a homomorphism of algebras where each $A\#_\sigma H_\alpha$ is a right A-module via $(x\#_\sigma h)\cdot w = (x\#_\sigma h)(w\#_\sigma 1_\alpha)$.

Proof. First, we will show that α commutes with the right action of all $w \in A$.

$$\begin{aligned}
& \alpha((a\#_\sigma h)\#f)((b\#_\sigma g)\cdot w) \\
= \;& \alpha((a\#_\sigma h)\#f)(b(g_{(1,1)}\cdot w)\#_\sigma g_{(2,\alpha)}) \\
= \;& (a\#_\sigma h)(b(g_{(1,1)}\cdot w)\#_\sigma \langle f, g_{(3,1)}\rangle g_{(2,\alpha)}) \\
= \;& a(h_{(1,1)}\cdot b)(h_{(2,1)}\cdot(g_{(1,1)}\cdot w))\sigma(h_{(3,1)}, g_{(2,1)})\#_\sigma\langle f, g_{(4,1)}\rangle h_{(4,\alpha)}g_{(3,\alpha)} \\
\overset{(3)}{=} \;& a(h_{(1,1)}\cdot b)\sigma(h_{(2,1)}, g_{(1,1)})(h_{(3,1)}g_{(2,1)}\cdot w)\#_\sigma\langle f, g_{(4,1)}\rangle h_{(4,\alpha)}g_{(3,\alpha)} \\
= \;& (a(h_{(1,1)}\cdot b)\sigma(h_{(2,1)}, g_{(1,1)})\#_\sigma\langle f, g_{(3,1)}\rangle h_{(3,\alpha)}g_{(2,\alpha)})\cdot w \\
= \;& (\alpha((a\#_\sigma h)\#f)(b\#_\sigma g))\cdot w.
\end{aligned}$$

Next, for all $a, b, x \in A, h, l, y \in H_\alpha$ and $f, g \in H_1^*$,

$$\begin{aligned}
& \alpha([(a\#_\sigma h)\#f][(b\#_\sigma l)\#g])(x\#_\sigma y) \\
= \;& \alpha(\langle f_1, l_{(3,1)}\rangle(a(h_{(1,1)}\cdot b)\sigma(h_{(2,1)}, l_{(1,1)})\#_\sigma h_{(3,\alpha)}l_{(2,\alpha)})\#f_2 g)(x\#_\sigma y) \\
= \;& \langle f, l_{(5,1)}y_{(3,1)}\rangle\langle g, y_{(4,1)}\rangle a(h_{(1,1)}\cdot b)\sigma(h_{(2,1)}, l_{(1,1)})(h_{(3,1)}l_{(2,1)}\cdot x) \\
& \sigma(h_{(4,1)}l_{(3,1)}, y_{(1,1)})\#_\sigma h_{(5,\alpha)}l_{(4,\alpha)}y_{(2,\alpha)}
\end{aligned}$$

and

$$\begin{aligned}
& \alpha((a\#_\sigma h)\#f)\circ\alpha((b\#_\sigma l)\#g)(x\#_\sigma y) \\
= \;& \alpha((a\#_\sigma h)\#f)(b(l_{(1,1)}\cdot x)\sigma(l_{(2,1)}, y_{(1,1)})\#_\sigma\langle g, y_{(3,1)}\rangle l_{(3,\alpha)}y_{(2,\alpha)}) \\
= \;& (a\#_\sigma h)(b(l_{(1,1)}\cdot x)\sigma(l_{(2,1)}, y_{(1,1)})\#_\sigma\langle f, l_{(4,1)}y_{(3,1)}\rangle\langle g, y_{(4,1)}\rangle l_{(3,\alpha)}y_{(2,\alpha)}) \\
= \;& a(h_{(1,1)}\cdot b)(h_{(2,1)}\cdot(l_{(1,1)}\cdot x))\underline{(h_{(3,1)}\cdot\sigma(l_{(2,1)}, y_{(1,1)}))\sigma(h_{(4,1)}, l_{(3,1)}y_{(2,1)})} \\
& \#_\sigma\langle f, l_{(5,1)}y_{(4,1)}\rangle\langle g, y_{(5,1)}\rangle h_{(5,\alpha)}l_{(4,\alpha)}y_{(3,\alpha)} \\
\overset{(4)}{=} \;& a(h_{(1,1)}\cdot b)\underline{(h_{(2,1)}\cdot(l_{(1,1)}\cdot x))\sigma(h_{(3,1)}, l_{(2,1)})}\sigma(h_{(4,1)}l_{(3,1)}, y_{(1,1)}) \\
& \#_\sigma\langle f, l_{(5,1)}y_{(3,1)}\rangle\langle g, y_{(4,1)}\rangle h_{(5,\alpha)}l_{(4,\alpha)}y_{(2,\alpha)} \\
\overset{(3)}{=} \;& \langle f, l_{(5,1)}y_{(3,1)}\rangle\langle g, y_{(4,1)}\rangle a(h_{(1,1)}\cdot b)\sigma(h_{(2,1)}, l_{(1,1)})(h_{(3,1)}l_{(2,1)}\cdot x) \\
& \sigma(h_{(4,1)}l_{(3,1)}, y_{(1,1)})\#_\sigma h_{(5,\alpha)}l_{(4,\alpha)}y_{(2,\alpha)}
\end{aligned}$$

Therefore, α is a homomorphism of algebras. □

Let $\{f_i\}$ be a basis of H_1 and $\{\psi_i\}$ be the dual basis of H_1^*, i.e., such that $\langle f_i, \psi_j\rangle = \delta_{ij}$ for all i, j. Then we have identities:

$$\sum_i f_i\langle h, \psi_i\rangle = h,\ \sum_i\langle f_i, \phi\rangle\psi_i = \phi,$$

for all $h \in H_1, \phi \in H_1^*$.

Lemma 3.3. *Let $A\#_\sigma^\pi H$ be a π-crossed product with σ convolution invertible. Define a linear map $\beta\ :\ End(A\#_\sigma H_\alpha)_A \longrightarrow (A\#_\sigma H_\alpha)\#H_1^*$ by*

$$\beta\ :\ T\mapsto \sum_i [T(\sigma^{-1}(f_{i(3,1)},S_1^{-1}(f_{i(2,1)}))\#_\sigma f_{i(4,\alpha)})(1_A\#_\sigma S_\alpha^{-1}(f_{i(1,\alpha^{-1})}))]\#\psi_i.$$

The maps α and β are inverses of each other.

Proof. We need to check that

$$\beta\circ\alpha = id_{(A\#_\sigma H_\alpha)\#H_1^*},\quad \alpha\circ\beta = id_{End(A\#_\sigma H_\alpha)_A}.$$

For all $x\in A, h\in H_\alpha, \phi\in H_1^*$, we have

$$\begin{aligned}
& \beta\circ\alpha((x\#_\sigma h)\#\phi)\\
= & \sum_i [(x\#_\sigma h)(\sigma^{-1}(f_{i(3,1)},S_1^{-1}(f_{i(2,1)}))\#_\sigma \langle\phi, f_{i(5,1)}\rangle f_{i(4,\alpha)})(1_A\#_\sigma S_\alpha^{-1}(f_{i(1,\alpha^{-1})}))]\#\psi_i\\
= & \sum_i [x(h_{(1,1)}\cdot\sigma^{-1}(f_{i(4,1)},S_1^{-1}(f_{i(3,1)})))\underline{\sigma(h_{(2,1)},f_{i(5,1)})\sigma(h_{(3,1)}f_{i(6,1)},S_1^{-1}(f_{i(2,1)}))}\\
& \#_\sigma h_{(4,\alpha)}f_{i(7,\alpha)}S_\alpha^{-1}(f_{i(1,\alpha^{-1})})]\#\psi_i\langle\phi,f_{i(8,1)}\rangle\\
\overset{(4)}{=} & \sum_i [x(h_{(1,1)}\cdot\sigma^{-1}(f_{i(5,1)},S_1^{-1}(f_{i(4,1)})))(h_{(2,1)}\cdot\sigma(f_{i(6,1)},S_1^{-1}(f_{i(3,1)})))\\
& \sigma(h_{(3,1)},f_{i(7,1)}S_1^{-1}(f_{i(2,1)}))\#_\sigma h_{(4,\alpha)}f_{i(8,\alpha)}S_\alpha^{-1}(f_{i(1,\alpha^{-1})})]\#\psi_i\langle\phi,f_{i(9,1)}\rangle\\
= & \sum_i [x(h_{(1,1)}\cdot(\sigma^{-1}(f_{i(5,1)},S_1^{-1}(f_{i(4,1)}))\sigma(f_{i(6,1)},S_1^{-1}(f_{i(3,1)}))))\sigma(h_{(2,1)},f_{i(7,1)}S_1^{-1}(f_{i(2,1)}))\\
& \#_\sigma h_{(3,\alpha)}f_{i(8,\alpha)}S_\alpha^{-1}(f_{i(1,\alpha^{-1})})]\#\psi_i\langle\phi,f_{i(9,1)}\rangle\\
= & \sum_i [x\sigma(h_{(1,1)},f_{i(3,1)}S_1^{-1}(f_{i(2,1)}))\#_\sigma h_{(2,\alpha)}f_{i(4,\alpha)}S_\alpha^{-1}(f_{i(1,\alpha^{-1})})]\#\psi_i\langle\phi,f_{i(5,1)}\rangle\\
= & \sum_i (x\#_\sigma h)\#\psi_i\langle\phi,f_i\rangle = (x\#_\sigma h)\#\phi.
\end{aligned}$$

From Eq.(3) and Eq.(4), we get the following equations.

$$\sigma^{-1}(h_{(1,1)},g_{(1,1)})(h_{(2,1)}\cdot(g_{(2,1)}\cdot a)) = (h_{(1,1)}g_{(1,1)}\cdot a)\sigma^{-1}(h_{(2,1)},g_{(2,1)}), \tag{10}$$

$$\sigma^{-1}(h_{1,1},g_{(1,1)})(h_{(2,1)}\cdot\sigma(g_{(2,1)},k)) = \sigma(h_{(1,1)}g_{(1,1)},k_{(1,1)})\sigma^{-1}(h_{(2,1)},g_{(2,1)}k_{(2,1)}). \tag{11}$$

Also for every $T \in End(A\#_\sigma H_\alpha)_A$, we compute

$$\begin{aligned}
& \alpha \circ \beta(T)(y\#_\sigma g) \\
= \ & \sum_i \alpha([T(\sigma^{-1}(f_{i(3,1)}, S_1^{-1}(f_{i(2,1)}))\#_\sigma f_{i(4,\alpha)})(1_A\#_\sigma S_\alpha^{-1}(f_{i(1,\alpha^{-1})}))]\#\psi_i)(y\#_\sigma g) \\
= \ & \sum_i T(\sigma^{-1}(f_{i(3,1)}, S_1^{-1}(f_{i(2,1)}))\#_\sigma f_{i(4,\alpha)})(1_A\#_\sigma S_\alpha^{-1}(f_{i(1,\alpha^{-1})}))(y\#_\sigma \langle\psi_i, g_{(2,1)}\rangle g_{(1,\alpha)}) \\
= \ & T(\sigma^{-1}(g_{(5,1)}, S_1^{-1}(g_{(4,1)}))\#_\sigma g_{(6,\alpha)})[(S_1^{-1}(g_{(3,1)}) \cdot y)\sigma(S_1^{-1}(g_{(2,1)}), g_{(1,1)})\#_\sigma 1_\alpha] \\
= \ & T[(\sigma^{-1}(g_{(5,1)}, S_1^{-1}(g_{(4,1)}))\#_\sigma g_{(6,\alpha)})((S_1^{-1}(g_{(3,1)}) \cdot y)\sigma(S_1^{-1}(g_{(2,1)}), g_{(1,1)})\#_\sigma 1_\alpha)] \\
= \ & T[\underline{\sigma^{-1}(g_{(5,1)}, S_1^{-1}(g_{(4,1)}))(g_{(6,1)} \cdot (S_1^{-1}(g_{(3,1)}) \cdot y))}(g_{(7,1)} \cdot \sigma(S_1^{-1}(g_{(2,1)}), g_{(1,1)}))\#_\sigma g_{(8,\alpha)}] \\
\overset{(10)}{=} \ & T[(g_{(5,1)}S_1^{-1}(g_{(4,1)}) \cdot y)\sigma^{-1}(g_{(6,1)}, S_1^{-1}(g_{(3,1)}))(g_{(7,1)} \cdot \sigma(S_1^{-1}(g_{(2,1)}), g_{(1,1)}))\#_\sigma g_{(8,\alpha)}] \\
= \ & T[y\underline{\sigma^{-1}(g_{(4,1)}, S_1^{-1}(g_{(3,1)}))(g_{(5,1)} \cdot \sigma(S_1^{-1}(g_{(2,1)}), g_{(1,1)}))}\#_\sigma g_{(6,\alpha)}] \\
\overset{(11)}{=} \ & T[y\sigma(g_{(5,1)}S_1^{-1}(g_{(4,1)}), g_{(1,1)})\sigma^{-1}(g_{(6,1)}, S_1^{-1}(g_{(3,1)})g_{(2,1)})\#_\sigma g_{(7,\alpha)}] \\
= \ & T(y\#_\sigma g).
\end{aligned}$$

So $End(A\#_\sigma H_\alpha)_A \cong (A\#_\sigma H_\alpha)\#H_1^*$. □

Now we have the main result of this section as follows:

Theorem 3.4. *Let H be a finite type Hopf π-coalgebra and $A\#_\sigma^\pi H$ be a π-crossed product with convolution inverse σ, then there is a canonical isomorphism between the algebras $(A\#_\sigma H_\alpha)\#H_1^*$ and $End(A\#_\sigma H_\alpha)_A$.*

From Example 2.3 and Theorem 3.4, we immediately get the following results.

Corollary 3.5. *Let H a finite dimensional Hopf algebra and $A\#_\sigma H$ be a crossed product with convolution inverse σ, then there is a canonical isomorphism between the algebras $(A\#_\sigma H)\#H^*$ and $End(A\#_\sigma H)_A$.*

Corollary 3.6. ([6]) *Let A be a π-H-module algebra and H be a finite type Hopf π-coalgebra, then there is a canonical isomorphism between the algebras $(A\#H_\alpha)\#H_1^*$ and $End(A\#H_\alpha)_A$.*

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