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NPP RINGS, REDUCED RINGS AND SNF RINGS

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Abstract. A ring R is called left NPP if for any nilpotent element a of R, $l(a) = Re, e^2 = e \in R$. A right $R-$module M is called $Nflat$ if for each $a \in N(R)$, the $\mathbb{Z}-$module map $1_M \otimes i : M \underset{R}{\otimes} Ra \longrightarrow M \underset{R}{\otimes} R$ is monic, where $i : Ra \hookrightarrow R$ is the inclusion map. A ring R is called right SNF if every simple right $R-$module is $Nflat$. In this paper, we first show that a ring R is left NPP iff every sum of two injective submodules of a left $R-$module is $nil-$injective. And some properties of left NPP rings are given, for example, if R is left NPP, so is eRe for any $e^2 = e \in R$ satisfying $ReR = R$. Next, we study some properties of reduced rings. A ring R is reduced if and only if R is ZC and right SNF if and only if R is left and right NPP and R has no subrings which is isomorphic to the upper triangular matrix $UT\mathbb{Z}_2$ or $UT(\mathbb{Z}_p)_2$ for some prime p. Finally, we give some characterizations of $n-$regular rings, for example, a ring R is $n-$regular if and only if every right $R-$module is $Nflat$.

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1. Introduction

Throughout R denotes an associative ring with identity and all modules are unitary. For a subset X of R, the left (right) annihilator of X in R is denoted by $l(X)$ $(r(X))$. If $X = \{a\}$, we usually abbreviate it to $l(a)$ $(r(a))$. We write $J(R)$, $Z_l(R)(Z_r(R))$, $N(R)$, $Z(R)$ for the Jacobson radical, the left (right) singular ideal, the set of nilpotent elements, the set of central elements of R, respectively.

A left $R-$module M is called $nil-$injective [8] if every left R- homomorphism from a principal left ideal Ra with $a \in N(R)$ to M extends to one from ${}_RR$ to M. The ring R is called left $nil-$injective if ${}_RR$ is $nil-$injective. Note that left principally injective rings are $nil-$injective, but the converse is not true by [8, Example 2.2]. A ring R is called left NPP if for any $a \in N(R)$, $l(a) = Re, e^2 =$

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$e \in R$. Clearly, left pp ring (that is: for each $a \in R, l(a) = Re, e^2 = e \in R$) is left NPP, but the converse is not true by [8, Example 2.8]. A ring R is called left $NC2$ if ${}_RRa$ projective implies $Ra = Re, e^2 = e \in R$ for all $a \in N(R)$. Clearly, left $C2$ ring [7] is left $NC2$ and by [8, Corollary 2.7], left $nil-$injective ring is left $NC2$. But the converse are all not true by [8, Example 2.21 and Example 2.5]. A ring R is called $n-$regular if $a \in aRa$ for all $a \in N(R)$. Clearly, von Neumann regular rings are $n-$regular, But the converse is not true by [8, Remark 2.19]. A ring R is called reduced if $N(R) = 0$, or equivalently, $a^2 = 0$ implies $a = 0$ in R for all $a \in R$. Clearly, a reduced ring is left $nil-$injective, left NPP and left $NC2$. In this paper, we first give some characterizations of left NPP rings and study some properties of left NPP rings. Next, we consider some conditions for a ring R being reduced. Finally, we introduce right $Nflat$ modules and right SNF rings, giving some characterizations of $n-$regular rings and reduced rings in terms of them.

2. Left NPP rings

Theorem 2.1. *The following conditions are equivalent for a ring R.*

(1) R is left NPP.

(2) Every factor module of an injective left $R-$module is $nil-$injective.

(3) Every sum of two injective submodules of a left $R-$module is $nil-$injective.

(4) Every sum of two isomorphic injective submodules of a left $R-$module is $nil-$injective.

Proof. (3) $\Rightarrow$ (4) and (2) $\Rightarrow$ (1) are trivial. (1) $\Rightarrow$ (2) follows from [8, Theorem 2.10(1)].

(2) $\Rightarrow$ (3) Let N_1 and N_2 be two injective submodules of a left $R-$module M. Since $N_1 \oplus N_2$ is injective and there is an epimorphism $N_1 \oplus N_2 \longrightarrow N_1 + N_2$, $N_1 + N_2$ is $nil-$injective.

(4) $\Rightarrow$ (2) Let M be an injective left $R-$module and N a submodule. Let $U = M \oplus M$, $V = \{(n,n) \,|\, n \in N\}$, $\overline{U} = U/V$, $M_1 = \{\overline{(m,0)} \in \overline{U} \,|\, m \in M\}$, and $M_2 = \{\overline{(0,m)} \in \overline{U} \,|\, m \in M\}$. Then $\overline{U} = M_1 + M_2$ and $M_i \cong M (i = 1, 2)$, so $\overline{U}$ is $nil-$injective by (4). Since M_1 is injective, M_1 is a summand of $\overline{U}$ and $\overline{U}/M_1$ is isomorphic to a summand of $\overline{U}$. Hence $\overline{U}/M_1$ is $nil-$injective. Now there is a canonical isomorphism $M/N \cong \overline{U}/M_1$, via $m + N \longmapsto \overline{(0,m)} + M_1$ and so M/N is $nil-$injective. □

We denote by $M_n(R)$ the ring of n by n matrices over R. Since Morita equivalence preserves summands, epimorphisms, and monomorphisms, it must preserve projective modules. Hence we have the following theorem.

Theorem 2.2. *R is left NPP if and only if every principal left ideal of $M_2(R)$ generated by a nilpotent diagonal matrix is projective as an $M_2(R)-$module.*

Proof. ($\Rightarrow$) is trivial.

($\Leftarrow$) Let $r \in N(R)$ and I be the principal left ideal of $M_2(R)$ generated by the diagonal matrix $\begin{pmatrix} r & 0 \\ 0 & 0 \end{pmatrix}$. Then I is a projective left $M_2(R)-$module. By [4, Theorem 3.2], there is a Morita equivalence between $M_2(R)-$modules and $R-$modules via $M \longrightarrow eM$, where M is a left $M_2(R)-$module and $e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Now $eI \cong Rr$ as $R-$modules, so Rr is a projective $R-$module. Hence R is left NPP. □

Call a ring R left NPF if for each $a \in N(R)$, ${}_RRa$ is flat. Clearly, left NPP ring is left NPF. We have the following theorem.

Theorem 2.3. *The following conditions are equivalent for a ring R.*

(1) R is left NPP.

(2) R is left NPF and for each $a \in N(R)$, $l(a)$ is finitely generated as a left $R-$module.

(3) For each non-empty subset X of R, for each $a \in r(X) \cap N(R)$, there exists a $b \in r(X)$ such that $a = ba$.

Proof. (1) $\Rightarrow$ (2) is trivial.

(2) $\Rightarrow$ (3) Let $\phi \neq X \subseteq R$ and $a \in r(X) \cap N(R)$. Then Ra is finitely presented flat left $R-$module by (2), so Ra is projective as left $R-$module. Hence $l(a) = Re, e^2 = e \in R$. Since $(1-e)R = r(e) = r(Re) = rl(a) \subseteq rlr(X) = r(X)$ and $a \in rl(a)$, $a = (1-e)a$. Set $b = 1-e \in r(X)$. Then $a = ba$.

(3) $\Rightarrow$ (1) Let $a \in N(R)$. Then $a \in r(l(a)) \cap N(R)$, so $a = fa$ for some $f \in r(l(a))$ by (3). Since $1-f \in l(a) \subseteq l(f)$, $f = f^2$ and $R(1-f) \subseteq l(a)$. Now let $x \in l(a)$. Then $xf = 0$, so $x = x(1-f) \in R(1-f)$. Hence $l(a) = R(1-f)$, which implies R is left NPP. □

It is well known that for any left ideal K of a ring R, R/K is a flat left $R-$module if and only if for any $x \in K$, there exists $y \in K$ such that $x = xy$. Hence we have the following theorem.

Theorem 2.4. *Let $e^2 = e \in R$ and $S = eRe$. Then*

(1) If R is NPF, so is S.

(2) Let $ReR = R$ and $x \in N(S)$. If $l_R(x)$ is finite generated as a left $R-$module, so is $l_S(x)$ as a left $S-$module.

(3) Let $ReR = R$. If R is left NPP, so is S.

Proof. (1) Let $x \in N(S)$. Then $x \in N(R)$, so $R/l_R(x) \cong Rx$ is flat left $R-$module by hypothesis. Let $y \in l_S(x)$. Then $yx = 0$ in S, so $y \in l_R(x)$. Hence there exists $z \in l_R(x)$ such that $y = yz$. Thus $y = eye = yeze$. Since $ezex = ezx = 0$, $eze \in l_S(x)$. This shows that $Sx \cong S/l_S(x)$ is a flat left $S-$module and so S is left NPF.

(2) Let $l_R(x) = \sum_{i=1}^{m} Ra_i$ where $a_i \in R$. Since $R = ReR$, $1 = \sum_{j=1}^{n} u_j e v_j$ where $u_j, v_j \in R$. Let $z \in l_S(x)$, then $z \in l_R(x)$. Set $z = \sum_{i=1}^{m} c_i a_i$. Then $z = \sum\sum c_i u_j e v_j a_i e$. So, clearly, as a left $S-$module, $l_S(x)$ is generated by $ev_j a_i e, i = 1, 2, \cdots, m; j = 1, 2, \cdots, n$.

(3) follows from (1), (2) and Theorem 2.3. □

By definition, we have the following theorem.

Theorem 2.5. *Let $R = \prod_{i\in I} R_i$ be the direct product of rings $\{R_i | i \in I\}$. Then*

(1) R is left NPF if and only if R_i is left NPF for all $i \in I$.

(2) R is left NPP if and only if R_i is left NPP for all $i \in I$.

Theorem 2.6. *(1) Left NPF rings have no nonzero central nilpotent elements.*

(2) Left NPP rings have no nonzero central nilpotent elements.

(3) If R is left NPF, then $Z(R)$ is reduced.

(4) If R is left NPP, then $Z(R)$ is reduced.

Proof. (1) Let R be left NPF and $x \in Z(R)$ with $x^n = 0$ and $x^{n-1} \neq 0$. Since $R/l(x) \cong Rx$ is flat and $x^{n-1} \in l(x)$, $x^{n-1} = x^{n-1}y$ for some $y \in l(x)$. Since $yx = 0$ and $x \in Z(R)$, $xy = 0$. Hence $x^{n-1} = 0$, which is a contradiction. So left NPF rings have no nonzero central nilpotent elements.

(2), (3) and (4) follow from (1). □

[8, Theorem 2.9] shows that R is reduced if and only if R is abelian left NPP, where a ring R is *abelian* if every idempotent of R is central. [8, Theorem 2.24] shows that R is $n-$regular if and only if R is left NPP left $NC2$. A ring R is called NI if $N(R)$ forms an ideal of R. A ring R is called $2-primal$ if $N(R) = P(R)$, where $P(R)$ is the prime radical of R. A ring R is called ZC if $ab = 0$ implies that $ba = 0$ for all $a, b \in R$. Clearly, (1) ZC rings are abelian, NI and $2-primal$; (2) abelian rings are $NC2$; (3) $2-primal$ rings are NI.

Theorem 2.7. *The following conditions are equivalent for ring R.*

(1) R is reduced.

(2) R is $n-$regular and abelian.

(3) R is $n-$regular and $N(R)$ forms a left ideal of R.

(4) R is n−regular and N(R) forms a right ideal of R.

(5) R is n−regular and NI.

(6) R is n−regular and 2 − primal.

(7) R is left NPF and ZC.

(8) R is left nil−injective left nonsingular and NI.

Proof. (1) $\Leftrightarrow$ (2) and (1) $\Rightarrow$ (6) $\Rightarrow$ (5) $\Rightarrow$ (4) or (5) $\Rightarrow$ (3) are trivial.

We will prove (3) $\Rightarrow$ (4). The (4) $\Rightarrow$ (1) is similar. Let $a \in R$ with $a^2 = 0$. Then $a = aba$ for some $b \in R$ because R is $n-$regular. Let $e = ba$. Then $e^2 = e$ and $a = ae$. Since $N(R)$ is a left ideal of R and $a \in N(R)$, $e = ba \in N(R)$. So $e = 0$ and then $a = ae = 0$. Hence R is reduced.

(1) $\Rightarrow$ (7) follows from [8, Theorem 2.9] and Theorem 2.3.

(7) $\Rightarrow$ (1) Let $x \in R$ with $x^2 = 0$. Then $R/l(x)$ is flat left $R-$module by (7). So $x = xy$ for some $y \in l(x)$ because $x \in l(x)$. Since R is ZC, $xy = 0$ because $yx = 0$. Thus $x = xy = 0$.

(1) $\Leftrightarrow$ (8) follows from [8, Theorem 2.9]. □

Now we consider the $n \times n$ upper triangular matrix ring UTR_n over a ring R.

Theorem 2.8. *Let R be a ring and $n \geq 2$. Then*

(1) If UTR_n is left NPF, so is R.

(2) If UTR_n is left NPP, so is R.

Proof. (1) Let $a \in N(R)$. Then $A = \begin{pmatrix} a & 0 & 0 & \cdots & 0 \\ 0 & a & 0 & \cdots & 0 \\ 0 & 0 & a & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & a \end{pmatrix} \in N(UTR_n)$. Since UTR_n is left NPF, $UTR_n/l_{UTR_n}(A)$ is flat left UTR_n-module. For any $b \in l_R(a)$, $B = \begin{pmatrix} b & 0 & 0 & \cdots & 0 \\ 0 & b & 0 & \cdots & 0 \\ 0 & 0 & b & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & b \end{pmatrix} \in l_{UTR_n}(A)$. So there exists $C = \begin{pmatrix} c_1 & c_{12} & c_{13} & \cdots & c_{1n} \\ 0 & c_2 & c_{23} & \cdots & c_{2n} \\ 0 & 0 & c_3 & \cdots & c_{3n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & c_n \end{pmatrix} \in l_{UTR_n}(A)$ such that $B = BC$. Clearly, $c_1 \in l(a)$ and $b = bc_1$. This shows that R is left NPF.

(2) It is similar to (1). □

Based on the above preceding result, we consider a kind of subring of $n\times n$ upper triangular matrix rings. For a ring R, we consider the ring

$SUTR_n = \{\begin{pmatrix} b & b_{12} & b_{13} & \cdots & b_{1n} \\ 0 & b & a_{23} & \cdots & b_{2n} \\ 0 & 0 & b & \cdots & b_{3n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & b \end{pmatrix} \mid b, b_{ij} \in R\}$. Then by a similar proof proceeding of Theorem 2.8, we have the following:

Theorem 2.9. *Let R be a ring and $n \geq 2$. Then*

(1) If $SUTR_n$ is left NPF, so is R.

(2) If $SUTR_n$ is left NPP, so is R.

Let R be a ring and M a bimodule over R. The trivial extension of R and M is $R \propto M = \{(a, x) \,|\, a \in R, x \in M\}$ with addition defined componentwise and multiplication defined by $(a, x)(b, y) = (ab, ay + xb)$.

In fact, $R \propto M$ is isomorphic to the subring $\{\begin{pmatrix} a & x \\ 0 & a \end{pmatrix} : a \in R, x \in M\}$ of the formal 2×2 matrix ring $\begin{pmatrix} R & M \\ 0 & R \end{pmatrix}$, and $R \propto R \cong R[x]/(x^2)$. If $\sigma : R \longrightarrow R$ is a ring endomorphism, let $R[x; \sigma]$ denote the ring of skew polynomials over R; that is all formal polynomials in x with coefficients from R with multiplication defined by $xr = \sigma(r)x$. Note that if $R(\sigma)$ is the $(R, R)-$bimodule defined by ${}_RR(\sigma) =_R R$ and $m \circ r = m\sigma(r)$, for all $m \in R(\sigma)$ and $r \in R$, then $R[x; \sigma]/(x^2) \cong R \propto R(\sigma)$. Similar to the proof proceeding of Theorem 2.8, we have the following theorem.

Theorem 2.10. *(1) If one of the following rings is left NPF, so is R.*

(1) $R \propto M$. (2) $R \propto R$. (3) $R \propto R(\sigma)$. (4) $R[x]/(x^2)$.

(2) If one of the following rings is left NPP, so is R.

(1) $R \propto M$. (2) $R \propto R$. (3) $R \propto R(\sigma)$. (4) $R[x]/(x^2)$.

It is well known that there exists a reduced ring R which is not left *pp*. We claim that neither UTR_2 nor $SUTR_2$ is left NPP. In fact, since R is not left *pp*, there exists $a \in R$ such that $l_R(a)$ is not a direct summand of ${}_RR$. Then $A = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \in N(UTR_2)$. If UTR_2 is left NPP, then $l_{UTR_2}(A) = UTR_2E$, where $E^2 = E = \begin{pmatrix} e_1 & x \\ 0 & e_2 \end{pmatrix} \in UTR_2$. By computing, we have $e_1^2 = e_1 \in R$ and $l_R(a) = Re_1$, which is a contradiction. Hence UTR_2 is not left NPP. Similarly, we can show that $SUTR_2$ is not left NPP. Hence there exists a left NPP ring R such that neither UTR_2 nor $SUTR_2$ is left NPP.

3. Reduced rings

In this section, we will prove that a NPP ring R is reduced if and only if R contains no subrings which are isomorphic to the matrix rings $UT\mathbb{Z}_2$ or $UT(\mathbb{Z}_p)_2$, where $\mathbb{Z}$ denotes the integer ring and p is a prime number. For a ring R, let $E(R)$ denotes the set of all idempotents of R. A ring R is called NPP if R is left and right NPP. Thus, our results extend the results by Fraser and Nicholson in [2] and Guo and Shum in [3]. We begin with the following theorem.

Theorem 3.1. *Let R be NPP. Then the following conditions are equivalent.*

(1) R is reduced.

(2) $ef = fe$ for all $e, f \in E(R)$.

(3) $E(R)$ is a subsemigroup of the semigroup $(R, \cdot)$.

(4) $ef = 0$ if and only if $fe = 0$ for all $e, f \in E(R)$.

(5) $N(R) \cap Re = N(R) \cap eR$ for all $e \in E(R)$.

(6) R is NI ring and $eN(R) = N(R)e$ for all $e \in E(R)$.

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3) and (1) $\Rightarrow$ (6) $\Rightarrow$ (5) are trivial.

(3) $\Rightarrow$ (4) Let $e, f \in E(R)$ and $ef = 0$. By (3), $fe \in E(R)$. So $fe = fefe = f(ef)e = 0$.

(4) $\Rightarrow$ (5) Let $x \in N(R) \cap Re$. Then $x(1-e) = 0$, so $1 - e \in r(x) = (1-f)R$ for some $f^2 = f \in R$ because R is NPP and $x \in N(R)$. Hence $f(1-e) = 0$, by (4), $(1-e)f = 0$. Clearly, $1 - e + (1-e)x \in E(R)$ and $f(1 - e + (1-e)x) = 0$. By (4), $(1 - e + (1-e)x)f = 0$. Thus $(1-e)xf = 0$. Since $r(x) = (1-f)R$, $(1-e)x(1-f) = 0$. So $(1-e)x = 0$. Hence $x = ex \in N(R) \cap eR$. This shows that $N(R) \cap Re \subseteq N(R) \cap eR$. Similarly, we can show $N(R) \cap eR \subseteq N(R) \cap Re$.

(5) $\Rightarrow$ (1) Let $x \in R$ with $x^2 = 0$. Since R is NPP, $l(x) = Re, e^2 = e \in R$. So $x \in Re \cap N(R)$ and then $x \in eR$ by (5). Hence $x = ex$ and so $x = 0$ because $l(x) = Re$. Thus R is reduced. □

We first denote by $o(r)$ the additive order of $r \in R$, that is, the smallest positive integer n such that $nr = 0$. If r is of infinite order, then we simply write $o(r) = \infty$. The following theorem is a generalization of [3, Lemma 3.1]. For convenience, we give its brief proof.

Theorem 3.2. *Let R be NPP such that $ef = 0$ and $fe \neq 0$ for some $e, f \in E(R)$. Then, $o(e) = o(f) = o(fe)$. And if $o(e) < \infty$, then there exist $u, v \in E(R)$ and a prime p such that $o(u) = o(v) = o(uv) = p$ with $uv = 0$ but $vu \neq 0$.*

Proof. Since R is NPP, by Theorem 3.1, R is not reduced. Since $ef = 0$, $fe \in N(R)$. So $l(fe) = R(1-g)$ and $r(fe) = (1-h)R$ for some $g, h \in E(R)$. These lead to $l(fe) = l(g)$ and $r(fe) = r(h)$. Thus $g = fg$ because $1-f \in l(fe) = l(g)$, so $gf \in E(R)$ and $l(g) = l(gf)$. Hence $fe = gffe = gfe$ because $1 - gf \in l(gf) = l(g) = l(fe)$. Similarly, there exists $h \in E(R)$ such that $h = he, eh \in E(R), r(eh) = r(fe)$ and $fe = feh$. Hence, $fe = gfeh = (gf)(eh)$ and $(eh)(gf) = ehefgf = 0$. Clearly, $o(gf) = o(eh) = o(fe)$. So if $o(gf) = \infty$, there is nothing to prove. If $o(gf) = pk$, where p is a prime number. Then $o(kfe) = p$. By using similar arguments as above, we have $u, v \in E(R)$ such that $o(u) = o(v) = o(kfe)$ with $uv = 0$ but $vu \neq 0$. □

The following theorem also is a generalization of [3, Theorem 3.2].

Theorem 3.3. *Let R be NPP. Then R is reduced if and only if R has no subrings which are isomorphic to $UT\mathbb{Z}_2$ or $UT(\mathbb{Z}_p)_2$, where p is a prime.*

Proof. Since $UT\mathbb{Z}_2$ and $UT(\mathbb{Z}_p)_2$ both contain some non-commutating idempotents, by Theorem 3.1, the necessity part is clear.

To prove the sufficiency part, we suppose that R is not reduced. Then by Theorems 3.1 and 3.2, there exist $e, f \in E(R)$ such that $ef = 0$, $fe \neq 0$ and $o(e) = o(f) = o(fe)$; and $o(e) = o(f) = o(fe) = p$ if $o(e) < \infty$, where p is a prime. Consider the subring of R generated by e and f. Clearly, $0, e, f, fe$ forms a subsemigroup of R under ring multiplication and so $S = \{af + bfe + ce \mid a, b, c \in \mathbb{Z}\}$ forms a subring of R.

Now let $\theta : UT\mathbb{Z}_2 \longrightarrow S$ defined by $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \longmapsto af + (b-c)fe + ce$. Then θ is a surjective homomorphism.

If $o(e) = o(f) = o(fe) = \infty$, then θ is an isomorphism.

If $o(e) = o(f) = o(fe) = p$, then $ker\theta = \{\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid p|a, p|b, p|c\}$. Since $UT\mathbb{Z}_2/ker\theta \cong UT(\mathbb{Z}_p)_2$, $S \cong UT(\mathbb{Z}_p)_2$. This is a contradiction and therefore our proof is completed. □

A ring R is called left $GC2$ [9] if for $a \in R$ and ${}_RRa \cong_R R$, $Ra = Re$ for some $e^2 = e \in R$. A right $GC2$ ring is defined similarly. A ring R is called *strongly regular* if $a \in a^2R$ for all $a \in R$. Since strongly regular rings are left and right $C2$

[7]; and left (resp. right) $C2$ rings are left (resp. right) $GC2$; strongly regular rings are left and right $GC2$.

Theorem 3.4. *The following conditions are equivalent for a ring R.*

(1) R is strongly regular.

(2) R is abelian, left pp and left $GC2$.

(3) R is abelian, left pp and right $GC2$.

(4) R is von Neumann regular and $N(R)$ forms a left ideal of R.

(5) R is von Neumann regular and NI.

(6) R is von Neumann regular and $2-primal$.

Proof. (1) $\Rightarrow$ (2) and (1) $\Rightarrow$ (6) $\Rightarrow$ (5) $\Rightarrow$ (4) are trivial.

(2) $\Rightarrow$ (1) Let $a \in R$. Since R is left pp, $l(a) = Re, e^2 = e \in R$. Set $b = a + e$. Then $l(b) \subseteq l(a) \cap l(e) = 0$ because R is abelian. Clearly, $(1-e)b = (1-e)a = a$. Since ${}_RRb \cong_R R$ and R is left $GC2$, $Rb = Rg, g^2 = g \in R$. Hence $b = bg = gb$, so $g = 1$ because R is abelian and $l(b) = 0$. So $Ra = R(1-e)b = (1-e)Rb = (1-e)Rg = (1-e)R1 = (1-e)R = R(1-e)$, this implies R is von Neumann regular and so R is strongly regular because R is abelian.

Similarly, we can show (3) $\Rightarrow$ (1).

(4) $\Rightarrow$ (1) By (4), R is $n-$regular. Since $N(R)$ is a left ideal of R, R is reduced by Theorem 2.7. So R is strongly regular. □

Recall that an additive subgroup L of a ring R is said to be a *quasi-ideal* if $xrx \in L$ and $rxr \in L$ whence $x \in L$ and $r \in R$. Obviously, every ideal of R is a quasi-ideal. But there exists an example of a (four-dimensional) Banach algebra A whose quasi-ideal Y is not an ideal, since $A = A * Y$ is the exterior (Grassmann) algebra on a two dimensional real vector space Y [5]. A ring R is called left $MC2$ if $l(k) = Re, e^2 = e \in R$ whence Rk is a projective minimal left ideal of R. By [8, Theorem 2.22], left $NC2$ rings are left $MC2$. But the converse is not true by [8, Remark 2.23]. A left $R-$module M is called $Wnil-$injective [8] if for any $0 \neq a \in N(R)$ (if there exists), there exists a positive integer n such that $a^n \neq 0$ and every left $R-$homomorphism from Ra^n to M extends to one from R to M. Clearly, left $nil-$injective modules and left $YJ-$injective modules [6] are all $Wnil-$injective.

Theorem 3.5. *The following conditions are equivalent for a ring R.*

(1) R is reduced.

(2) R is $n-$regular and $N(R)$ is a quasi-ideal of R such that $aN(R) = N(R)a$ for all $a \in N(R)$.

(3) R is left $MC2$ and NI such that every simple singular left $R-$module is $Wnil-$injective.

(4) R is abelian and $N(R)$ forms a right ideal of R whose simple singular left $R-$modules are $Wnil-$injective.

(5) R is ZI and for any $a \in N(R)$, $l(Ra) = Re, e^2 = e \in R$.

Proof. $(1) \Rightarrow (i), i = 2, 3, 4, 5$ are clear.

$(2) \Rightarrow (1)$ By (2), $a = aba$ for all $a \in N(R)$. Since $N(R)$ is a quasi-ideal of R and $a \in N(R)$, $bab \in N(R)$. Thus $ab = abab = a(bab) \in aN(R) = N(R)a$ and so $a = aba \in N(R)a^2$. This implies R is reduced.

$(3) \Rightarrow (1)$ Let $a \in R$ such that $a^2 = 0$. We claim that $RaR + l(a) = R$. If not, there exists a maximal left ideal M containing $RaR + l(a)$. If M is not essential in $_RR$, then $M = l(e), e^2 = e \in R$. Since R is left $MC2$ ring, R is semiprime by [8, Corollary 3.6]. Since $eaR \in RaR \subseteq M = l(e)$, $eaRe = 0$. Hence $eaRea = 0$ and so $ea = 0$ because R is semiprime. Thus $e \in l(a) \subseteq M = l(e)$, which is a contradiction. Hence M is essential in $_RR$ and so R/M is $Wnil-$injective. This implies there exists $b \in R$ such that $1 - ab \in M$ and so $1 \in M$ because $ab \in RaR$, which is a contradiction. So $RaR + l(a) = R$ and then $a = ya$ for some $y \in RaR$. Since R is NI and $a \in N(R)$, $y \in N(R)$. Hence $y^n = 0$ for some positive integer n. So $a = ya = y^2a = \cdots = y^na = 0$.

$(4) \Rightarrow (1)$ Let $a \in R$ with $a^2 = 0$. If $a \neq 0$, then $l(a) \neq R$, so there exists a left ideal L of R such that $l(a) \oplus L$ is essential in $_RR$. If $l(a) \oplus L \neq R$, there exists a maximal left ideal M of R containing $l(a) \oplus L$. Clearly, M is essential left ideal of R, by hypothesis, R/M is $Wnil-$injective. So there exists $c \in R$ such that $1 - ac \in M$. Since $N(R)$ is a right ideal of R, $ac \in N(R)$, so $1 - ac$ is invertible. Hence $M = R$, which is a contradiction. This shows $l(a) \oplus L = R$. Let $l(a) = Re, e^2 = e \in R$. Clearly, $a = ae = ea = 0$, which is a contradiction. So $a = 0$.

$(5) \Rightarrow (1)$ Let $a^2 = 0$. By (5), $l(Ra) = Re$. Let $x \in l(a)$. Then $xa = 0$, so $xRa = 0$ because R is ZI. Hence $x \in l(Ra)$, this shows that $l(a) = l(Ra)$ and so $l(a) = Re$. Since R is ZI, R is abelian. So $a = 0$. $\square$

Theorem 3.6. *The following conditions are equivalent for a ring R.*

(1) R is reduced.

(2) R is ZC and every essential maximal left ideal of R is $Wnil-$injective.

(3) R is semiprime left nonsingular and for any $a \in N(R)$, Ra is an ideal of R.

(4) R is semiprime left nonsingular and for any $a \in N(R)$, Ra is a left annihilator of a left ideal of R.

Proof. (1) $\Rightarrow$ (4) $\Rightarrow$ (3) and (1) $\Rightarrow$ (2) are trivial.

(2) $\Rightarrow$ (1) Let $a \in R$ with $a^2 = 0$ and L a left ideal of R such that $l(a) \oplus L$ is essential left ideal of R. If $l(a) \oplus L \neq R$, then there exists an essential maximal left ideal M of R containing $l(a) \oplus L$. By hypothesis, ${}_RM$ is $Wnil-$injective. So the inclusion map $Ra \hookrightarrow M$ can be extended to $R \longrightarrow M$, this implies $a = am$ for some $m \in M$. Since R is ZC, $a = ma$. So $1 - m \in l(a) \subseteq M$, which is a contradiction. So $l(a) \oplus L = R$. Then, clearly, $a = 0$ because R is abel.

(3) $\Rightarrow$ (1) Let $a^2 = 0$ and L a left ideal of R such that $l(a) \cap L = 0$. Since Ra is an ideal of R, $aL \subseteq Ra$. Hence $aL \subseteq l(a) \cap L = 0$, so $(La)^2 = 0$. Since R is semiprime, $La = 0$. So $L \subseteq l(a)$ and then $L = 0$. Therefore $l(a)$ is an essential left ideal of R. But R is left nonsingular, so $a = 0$. □

4. $n-$regular rings

In [8, Theorem 2.18], we have shown that a ring R is $n-$regular if and only if every left $R-$module is $nil-$injective. Since $nil-$injective modules are $Wnil-$injective, we can generalize this theorem as follows:

Theorem 4.1. *The following conditions are equivalent for a ring R.*

(1) R is $n-$regular.

(2) Every left $R-$module is $Wnil-$injective.

(3) Every cyclic left $R-$module is $Wnil-$injective.

(4) R is left $Wnil-$injective and left NPP.

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3) and (1) $\Rightarrow$ (4) are trivial.

(3) $\Rightarrow$ (1) Let $a \in N(R)$. By (3), ${}_RRa$ is $Wnil-$injective. If $a^2 = 0$, then the identity map $I : Ra \longrightarrow Ra$ can be extended to $R \longrightarrow R$, this implies there exists $b \in R$ such that $a = aba$. If $a^2 \neq 0$, then there exists a positive integer n such that $a^n \neq 0$ and any left $R-$homomorphism $Ra^n \longrightarrow Ra$ can be extended to $R \longrightarrow Ra$. Set $f : Ra^n \longrightarrow Ra$ is the inclusion map, then, clearly, $f = \cdot ca, c \in R$. So $a^n = f(a^n) = a^n ca$. Let $d = a^{n-1} - a^{n-1}ca$. Then $d^2 = 0$. By the above proof, we can obtain that $d = a^{n-1} - a^{n-1}ca$ is regular element of R. By [1, Lemma 2.1], $a^{n-1} = a^{n-1}da$ for some $d \in R$. Repeating the above-mentioned process, we can obtain $v \in R$ such that $a = ava$.

(4) $\Rightarrow$ (1) Let $0 \neq a \in N(R)$. Since R is left $Wnil-$injective, there exists a positive integer n such that $a^n \neq 0$ and $rl(a^n) = a^nR$. Since R is left NPP and $a^n \in N(R)$, $l(a^n) = R(1-e), e^2 = e \in R$. Hence $eR = r(R(1-e)) = rl(a^n) = a^nR$. This implies that a^n is a regular element of R. If $a^2 = 0$, the argument above shows

that a is a regular element. So, by [1, Theorem 2.2], even if $a^2 \neq 0$, a is also a regular element of R. □

It is well known that R is von Neumann regular if and only if every essential left ideal of R is $YJ-$injective. And note that the direct summand of a $nil-$injective (resp. $Wnil-$injective) module is $nil-$injective (resp. $Wnil-$injective). So we can give the following theorem:

Theorem 4.2. *The following conditions are equivalent for a ring R.*

(1) R is $n-$regular.

(2) Every essential left ideal of R is $nil-$injective as left $R-$module.

(3) Every essential left ideal of R is $Wnil-$injective as left $R-$module.

(4) Every direct product (or sum) of cyclic left $R-$modules is $nil-$injective.

(5) Every direct product (or sum) of cyclic left $R-$modules is $Wnil-$injective.

(6) R is left $nil-$injective and cyclic singular left $R-$modules are $nil-$injective.

Proof. (1) ⇒ (2) ⇒ (3) and (6) ⇔ (1) ⇔ (4) ⇔ (5) follows from Theorem 4.1 and [8, Theorem 2.18].

(3) ⇒ (1) Let $a \in R$ with $a^2 = 0$. Clearly there exists a left ideal L of R such that $Ra \oplus L$ is essential left ideal of R. By (3), $Ra \oplus L$ is $Wnil-$injective, so Ra is $Wnil-$injective. Hence, by the proof of (3) → (1) in Theorem 4.1, we have $a = aba$ for some $b \in R$. □

A right $R-$module M is called $Nflat$ if for any $a \in N(R)$, the map $1_M \otimes i : M \otimes_R Ra \longrightarrow M \otimes_R R$ is monic, where $i : Ra \hookrightarrow R$ is the inclusion map. Clearly, flat modules are $Nflat$.

By definition, we know that every module over any reduced ring is $Nflat$. Since there exists a reduced ring R which is not von Neumann regular, there exists a module over R which is not flat. So there exists a $Nflat$ module which is not flat. The following proposition is trivial.

Proposition 4.3. *(1) The direct sum $\underset{i \in I}{\oplus} M_i$ of left $R-$modules $\{M_i \,|\, i \in I\}$ is $Nflat$ if and only if each M_i is $Nflat$.*

(2) If $\{M_i \,|\, i \in I\}$ is a direct system of $Nflat$ modules, then the direct limit of these modules is also $Nflat$.

(3) If every finitely generated submodule of a right $R-$module M is $Nflat$, then M is $Nflat$.

(4) If M_R is a module such that every cyclic submodule of M is contained in a $Nflat$ submodule then M is $Nflat$.

Let R and S be rings and B an $(S,R)-$bimodule. Then for any left $R-$module A and left $S-$module C, we have a left $\mathbb{Z}-$module isomorphism map:

$$\begin{aligned} \tau_{A,C} : Hom_S(B \otimes_R A, C) &\longrightarrow Hom_R(A, Hom_S(B,C)) \\ h &\longmapsto \tau_{A,C}(h) \end{aligned}$$

where $\tau_{A,C}(h) : A \longrightarrow Hom_S(B,C)$

$$a \longmapsto \tau_{A,C}(h)(a)$$

satisfies $\tau_{A,C}(h)(a)(b) = h(b \otimes a)$ for all $b \in B$.

Theorem 4.4. *Let R and S be rings, B an $(S,R)-$bimodule. If B_R is $Nflat$, C is injective left $S-$module, then as a left $R-$module, $Hom_S(B,C)$ is $nil-$injective.*

Proof. Let $a \in N(R)$ and $f : Ra \longrightarrow Hom_S(B,C)$ be any left $R-$homomorphism. Since B_R is $Nflat$, $1_B \otimes i : B \otimes_R Ra \longrightarrow B \otimes_R R$ is monic. Since ${}_SC$ is injective, $(1_B \underset{R}{\otimes} i)^* : Hom_S(B \otimes_R R, C) \longrightarrow Hom_S(B \otimes_R Ra, C)$ is epic. Since we have the following commutating diagram:

$$\begin{array}{ccc} Hom_S(B \underset{R}{\otimes} R, C) & \xrightarrow{\tau_{R,C}} & Hom_R(R, Hom_S(B,C)) \\ {\scriptstyle (1_B \underset{R}{\otimes} i)^*}\downarrow & & \downarrow{\scriptstyle i^*} \\ Hom_S(B \underset{R}{\otimes} Ra, C) & \xrightarrow{\tau_{Ra,C}} & Hom_R(Ra, Hom_S(B,C)) \end{array}$$

$i^* : Hom_R(R, Hom_S(B,C)) \longrightarrow Hom_R(Ra, Hom_S(B,C))$ is epic. Hence there exists a left $R-$homomorphism $h : R \longrightarrow Hom_S(B,C)$ such that $i^*(h) = f$, that is $hi = f$ or equivalently, $h|_{Ra} = f$. This shows that $Hom_S(B,C)$ is $nil-$injective as a left $R-$module. $\square$

Theorem 4.5. *Right $R-$module B is $Nflat$ if and only if $B^* \overset{\text{def}}{=} Hom_{\mathbb{Z}}(B, \mathbb{Q}/\mathbb{Z})$ is $nil-$injective, where $\mathbb{Q}$ is the field of real numbers.*

Proof. Let B be $Nflat$. Since $\mathbb{Q}/\mathbb{Z}$ is an injective left $\mathbb{Z}-$module, B^* is $nil-$injective as a left $R-$module by Theorem 4.4.

Converse, assume that B^* is a $nil-$injective left $R-$module. Let $a \in N(R)$. We show that $1_B \otimes i : B \underset{R}{\otimes} Ra \longrightarrow B \underset{R}{\otimes} R$ is monic.

Since we have the following commutating diagram:

$$\begin{array}{ccc} Hom_{\mathbb{Z}}(B \underset{R}{\otimes} R, \mathbb{Q}/\mathbb{Z}) & \xrightarrow{\tau_{R,\mathbb{Q}/\mathbb{Z}}} & Hom_R(R, Hom_{\mathbb{Z}}(B, \mathbb{Q}/\mathbb{Z})) \\ {\scriptstyle (1_B \underset{R}{\otimes} i)^*}\downarrow & & \downarrow{\scriptstyle i^*} \\ Hom_{\mathbb{Z}}(B \underset{R}{\otimes} Ra, \mathbb{Q}/\mathbb{Z}) & \xrightarrow{\tau_{Ra,\mathbb{Q}/\mathbb{Z}}} & Hom_R(Ra, Hom_{\mathbb{Z}}(B, \mathbb{Q}/\mathbb{Z})) \end{array}$$

where $\tau_{R,\mathbb{Q}/\mathbb{Z}}$ and $\tau_{Ra,\mathbb{Q}/\mathbb{Z}}$ are $\mathbb{Z}-$isomorphism, $(1_B \otimes i)^*$is epic if and only if i^* is epic.

Since $B^* = Hom_{\mathbb{Z}}(B, \mathbb{Q}/\mathbb{Z})$ is $nil-$injective left $R-$module, i^* is epic. Hence $(1_B \otimes i)^*$ is also epic. Since $\mathbb{Q}/\mathbb{Z}$ is a cogenerator in $\mathbb{Z}-$module category, $(1_B \otimes i)$ is a monic. This shows that B_R is $Nflat$. □

Theorem 4.6. *(1) Let B be an $Nflat$ right $R-$module and $a \in N(R)$. Then there exists a unique $\mathbb{Z}-$module isomorphism $\theta : B \underset{R}{\otimes} Ra \longrightarrow Ba$ satisfies $\theta(b \underset{R}{\otimes} ra) = bra$ for all $b \in B$ and $r \in R$.*

(2) Let B be a right $R-$module and there exists a right $R-$short exact sequence:

$$0 \longrightarrow K \overset{\text{j}}{\hookrightarrow} F \overset{\text{g}}{\longrightarrow} B \longrightarrow 0$$

where F is $Nflat$. Then B_R is $Nflat$ if and only if $K \cap Fa = Ka$ for all $a \in N(R)$.

(3) Let M_R be $Nflat$ and U a submodule of M_R. Then M/U is $Nflat$ if and only if $Ua = U \cap Ma$ for all $a \in N(R)$.

(4) Let I be a right ideal of R. Then R/I is $Nflat$ right $R-$module if and only if $Ia = I \cap Ra$ for all $a \in N(R)$.

Proof. (1) Let $f : B \times Ra \longrightarrow Ba$ satisfy $f((b, ra)) = bra, b \in B, r \in R$. Clearly, f is an $R-$tensorial mapping, so there exists a unique $\mathbb{Z}-$homomorphism

$$\theta : B \underset{R}{\otimes} Ra \longrightarrow Ba, \qquad b \otimes ra \longmapsto bra$$

such that the following diagram is commutative:

$$\begin{array}{ccc} B \times Ra & \xrightarrow{h} & B \underset{R}{\otimes} Ra \\ f \downarrow & & \downarrow \theta \\ Ba & \xrightarrow{I} & Ba \end{array}$$

where $I : Ba \longrightarrow Ba$ is the identity mapping and $h : B \times Ra \longrightarrow B \underset{R}{\otimes} Ra$, $b \times ra \longmapsto b \otimes ra$.

Clearly, $\theta(b \otimes ra) = bra$ and θ is epic. Since B_R is $Nflat$, $1_B \otimes i : B \underset{R}{\otimes} Ra \longrightarrow B \underset{R}{\otimes} R$ is monic. Since $\psi : B \underset{R}{\otimes} R \longrightarrow B,\ b \otimes 1 \longmapsto b$ is a $\mathbb{Z}-$isomorphism, $\theta = \psi(1_B \otimes i)$ is monic. Hence θ is an isomorphism.

(2) Since $\underset{R}{\otimes} Ra$ is right exact, there is an exact sequence:

$$K \underset{R}{\otimes} Ra \overset{\text{j}\otimes 1}{\longrightarrow} F \underset{R}{\otimes} Ra \overset{\text{g}\otimes 1}{\longrightarrow} B \underset{R}{\otimes} Ra \longrightarrow 0.$$

Since F_R is $Nflat$, by (1), there exists a unique $\mathbb{Z}-$isomorphism $\rho : F \underset{R}{\otimes} Ra \longrightarrow Fa$ satisfying $\rho(x \otimes ra) = xra$ for all $x \in F$ and $r \in R$. So there is a $\mathbb{Z}-$epic mapping $(g \otimes 1)\rho^{-1} : Fa \longrightarrow B \underset{R}{\otimes} Ra$. Since $Ker((g \otimes 1)\rho^{-1}) = Ka$, there is a

$\mathbb{Z}-$isomorphism $\gamma : Fa/Ka \longrightarrow B \underset{R}{\otimes} Ra$ satisfying $\gamma(xa + Ka) = g(x) \otimes a$ for all $x \in F$.

On the other hand, $\delta : Ba \longrightarrow Fa/(K \cap Fa)$ defined by $\delta(ba) = xa + (K \cap Fa)$, where $g(x) = b, x \in F, b \in B$ is $\mathbb{Z}-$isomorphism. Hence we obtain $\mathbb{Z}-$homomorphism $\sigma = \delta\theta\gamma : Fa/Ka \longrightarrow Fa/K \cap Fa$ satisfying $\sigma(xa + Ka) = xa + (K \cap Fa), x \in F$.

Since $Ka \subseteq K \cap Fa$, σ is a $\mathbb{Z}-$isomorphism mapping if and only if $Ka = K \cap Fa$. Since $\sigma = \delta\theta\gamma$, σ is a $\mathbb{Z}-$isomorphism mapping if and only if θ is a $\mathbb{Z}-$isomorphism mapping. Hence θ is a $\mathbb{Z}-$isomorphism mapping if and only if $Ka = K \cap Fa$.

The if part: Assume that B_R is $Nflat$, By (1), $\theta : B \underset{R}{\otimes} Ra \longrightarrow Ba$ is a $\mathbb{Z}-$isomorphism mapping, so $Ka = K \cap Fa$.

The only if part: Since $Ka = K \cap Fa$ for all $a \in N(R)$, $\theta : B \underset{R}{\otimes} Ra \longrightarrow Ba$ is a $\mathbb{Z}-$isomorphism mapping. By the following commutating diagram:

$$\begin{array}{ccc} B \underset{R}{\otimes} Ra & \xrightarrow{1_B \otimes i} & B \underset{R}{\otimes} R \\ {\scriptstyle\theta}\big\downarrow & & \big\downarrow{\scriptstyle\psi} \\ Ba & \xrightarrow{\iota} & B \end{array}$$

where $\iota : Ba \hookrightarrow B$ is the inclusion mapping, we have that $1_B \otimes i$ is monic. Hence B_R is $Nflat$.

(3) and (4) follow from (2). $\square$

Theorem 4.7. *The following conditions are equivalent for a ring R.*

(1) R is $n-$regular.

(2) Every right $R-$module is $Nflat$.

(3) Every cyclic right $R-$module is $Nflat$.

Proof. (2) $\Rightarrow$ (3) is trivial.

(1) $\Rightarrow$ (2) Let M be any right $R-$module. Then there is a right $R-$short exact sequence $0 \longrightarrow K \overset{\mathrm{j}}{\hookrightarrow} F \overset{\mathrm{g}}{\longrightarrow} M \longrightarrow 0$ where F_R is free. For any $a \in N(R)$, we always have $Ka \subseteq K \cap Fa$. Let $x \in K \cap Fa$. Then $x = za$ for some $z \in F$. Since R is $n-$regular and $a \in N(R)$, $a = aba$ for some $b \in R$. Set $e = ba$, then $a = ae$ and $e = e^2 = ba \in Ra$. Clearly, $x = za = zae = xe \in Ka$. This shows that $Ka = K \cap Fa$ for all $a \in N(R)$ and so M_R is $Nflat$.

(3) $\Rightarrow$ (1) Let $a \in N(R)$. Since R/aR is a cyclic right $R-$module, R/aR is $Nflat$ by (3). In terms of the following right $R-$short exact sequence

$$0 \longrightarrow aR \overset{\mathrm{i}}{\hookrightarrow} R \overset{\pi}{\longrightarrow} R/aR \longrightarrow 0$$

we have $aRa = aR \cap Ra$. So $a \in aR \cap Ra = aRa$. Thus R is $n-$regular. $\square$

Call a ring R left (resp. right) SNF if every simple left (resp. right) $R-$module is $Nflat$. By Theorem 4.7, $n-$regular rings are SNF. Call a ring R is *left (resp. right) quasi-duo* if every maximal left (resp. right) ideal of R is an ideal. A ring R is called $MELT$ (resp. $MERT$) if every essential maximal left (resp. right) ideal of R is an ideal. A ring R is called left SF if every simple left $R-$module is flat. Clearly, a left SF ring is left SNF, but the converse is not true. Because there exists a reduced $MELT$ ring R which is not von Neumann regular, there exists a reduced $MELT$ ring R which is not left SF by [10, Theorem 1]. On the other hand, by Theorem 4.7, reduced rings are left SNF, so there exists a left SNF ring which is not left SF.

Theorem 4.8. *R is $n-$regular if and only if R is right SNF and every maximal submodule of any cyclic right $R-$module is $Nflat$.*

Proof. The necessity follows from Theorem 4.7.

The sufficiency: Let $a \in N(R)$. Then $aR \neq R$, so there exists a maximal right ideal M of R such that $aR \subseteq M$. Since $(R/aR)/(M/aR) \cong R/M$, M/aR is a maximal submodule of cyclic right $R-$module R/aR. So M/aR is $Nflat$ by hypothesis. Since M is a maximal submodule of cyclic right $R-$module R, M is $Nflat$. In terms of Theorem 4.6 and the following right $R-$short exact sequence:

$$0 \longrightarrow aR \overset{j}{\hookrightarrow} M \overset{\pi}{\longrightarrow} M/aR \longrightarrow 0$$

we have $aRa = aR \cap Ma$ because $a \in N(R)$. Since R is right SNF ring and R/M is simple right $R-$module, R/M is $Nflat$. Hence, by Theorem 4.6, $Ma = M \cap Ra$, so $a \in M \cap Ra = Ma$. Thus $a \in Ma \cap aR = aRa$, obtaining that R is $n-$regular. □

Theorem 4.9. *(1) Let R be left quasi-duo. Then R is reduced if and only if R is right SNF.*

(2) Let R be right SNF. Then

(a) *If R is $MELT$, then R is left nonsingular;*

(b) *If R is left $MC2$ and $MELT$, then R is semiprime and right nonsingular.*

(3) Let R be right SNF. If $r(M)$ is essential in R_R for all maximal right ideal M of R, then R is reduced.

(4) R is reduced if and only R is ZC and right SNF.

Proof. (1) The if part is clear by Theorem 4.7.

The only if part: Let $a \in R$ with $a^2 = 0$. If $a \neq 0$, then there exists a maximal left ideal M containing $l(a)$. Since R is left quasi-duo, M is an ideal of R. So there exists a maximal right ideal L of R such that $M \subseteq L$. Since R/L is a simple right

$R-$module, R/M is $Nflat$ because R is right SNF. By Theorem 4.6, $a = ba$ for some $b \in L$, so $1 - b \in l(a) \subseteq M \subseteq L$ and then $1 \in L$, which is a contradiction. Thus $a = 0$ and so R is reduced.

(2) (a) If $Z_l(R) \neq 0$, then there exists a $0 \neq a \in Z_l(R)$ such that $a^2 = 0$. So there exists a maximal left ideal M of R containing $l(a)$. Clearly, M is an essential left ideal. Since R is $MELT$, M is an ideal of R. By the proof of (1), we can obtain a contradiction. Hence $Z_l(R) = 0$.

(b) First, we show that R is semiprime. Let $a \in R$ satisfy $aRa = 0$. if $a \neq 0$, then there exists a maximal left ideal M of R containing $l(a)$. If M is not an essential left ideal of R, then $M = l(e)$ for some $e^2 = e \in R$. Since $RaR \subseteq l(a)$, $aRe = 0$. Since R is left $MC2$, $eRa = 0$. So $e \in l(a) \subseteq M = l(e)$, which is a contradiction. Hence M is an essential left ideal. The rest proof is similar to (1).

Next, we show that $Z_r(R) = 0$. If not, there exists a $0 \neq a \in Z_r(R)$ such that $a^2 = 0$. We claim that $Z_r(R) + l(a) = R$. If not, there exists a maximal left ideal M of R containing $Z_r(R) + l(a)$. Since R is left $MC2$, similar to the proof of (a), we can show that M is an essential left ideal. By the proof proceeding of (1), we shall give a contradiction. Hence $Z_r(R) + l(a) = R$. Let $1 = z + x$, where $z \in Z_r(R)$ and $x \in l(a)$. Then $a = za$ and so $(1 - z)a = 0$. Since $z \in Z_r(R)$ and $r(z) \cap r(1 - z) = 0$, $r(1 - z) = 0$. Hence $a = 0$, which is a contradiction. This shows that R is right nonsingular.

(3) Let $a \in R$ satisfy $a^2 = 0$. If $a \neq 0$, then $r(a) \subseteq M$ where M is a maximal right ideal of R. Since R is right SNF, R/M is an $Nflat$ right $R-$module. By Theorem 4.6, $a = xa$ for some $x \in M$, so $a \in r(1 - x)$. Since $r(M) \subseteq r(x)$ and $r(M)$ is an essential right ideal of R, $x \in Z_r(R)$. Hence $r(1 - x) = 0$, which implies that $a = 0$, which is a contradiction. Thus $a = 0$.

(4) Let $a \in R$ satisfy $a^2 = 0$. If $a \neq 0$, then, similar to the proof of (3), there exists $x \in M$ such that $a = xa$ where M is a maximal right ideal of R containing $r(a)$. Since R is ZC, $a = ax$. Hence $1 - x \in r(a) \subseteq M$ and so $1 \in M$, which is a contradiction. Thus $a = 0$. $\square$

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