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COUNTING CLUSTER-TILTED ALGEBRAS OF TYPE A_n

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Abstract. The purpose of this paper is to give an explicit formula for the number of non-isomorphic cluster-tilted algebras of type A_n, by counting the mutation class of any quiver with underlying graph A_n. It will also follow that if T and T' are cluster-tilting objects in a cluster category $\mathcal{C}$, then $\mathrm{End}_{\mathcal{C}}(T)$ is isomorphic to $\mathrm{End}_{\mathcal{C}}(T')$ if and only if $T = \tau^i T'$.

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1. Cluster-tilted algebras

The cluster category was introduced independently in [7] for type A_n and in [2] for the general case. Let $\mathcal{D}^b(\mathrm{mod}\, H)$ be the bounded derived category of the finitely generated modules over a finite dimensional hereditary algebra H over a field K. In [2] the cluster category was defined as the orbit category $\mathcal{C} = \mathcal{D}^b(\mathrm{mod}\, H)/\tau^{-1}[1]$, where τ is the Auslander-Reiten translation and [1] the suspension functor. The cluster-tilted algebras are the algebras of the form $\Gamma = \mathrm{End}_{\mathcal{C}}(T)^{\mathrm{op}}$, where T is a cluster-tilting object in $\mathcal{C}$ (see [3]).

Let Q be a quiver with no multiple arrows, no loops and no oriented cycles of length two. Mutation of Q at vertex k is a quiver Q' obtained from Q in the following way.

(1) Add a vertex k^*.
(2) If there is a path $i \to k \to j$, then if there is an arrow from j to i, remove this arrow. If there is no arrow from j to i, add an arrow from i to j.
(3) For any vertex i replace all arrows from i to k with arrows from k^* to i, and replace all arrows from k to i with arrows from i to k^*.
(4) Remove the vertex k.

We say that a quiver Q is mutation equivalent to Q', if Q' can be obtained from Q by a finite number of mutations. The mutation class of Q is all quivers mutation equivalent to Q. It is known from [11] that the mutation class of a Dynkin quiver Q is finite.

If Γ is a cluster-tilted algebra, then we say that Γ is of type A_n if it arises from the cluster category of a path algebra of Dynkin type A_n.

Let Q be a quiver of a cluster-tilted algebra Γ. From [4], it is known that if Q' is obtained from Q by a finite number of mutations, then there is a cluster-tilted algebra Γ' with quiver Q'. Moreover, Γ is of finite representation type if and only if Γ' is of finite representation type [3]. We also have that Γ is of type A_n if and only if Γ' is of type A_n. From [5] we know that a cluster-tilted algebra is up to isomorphism uniquely determined by its quiver. See also [8].

It follows from this that to count the number of cluster-tilted algebras of type A_n, it is enough to count the mutation class of any quiver with underlying graph A_n.

2. Category of diagonals of a regular $n+3$ polygon

We recall some results from [7].

Let n be a positive integer and let $\mathcal{P}_{n+3}$ be a regular polygon with $n+3$ vertices. A diagonal is a straight line between two non-adjacent vertices on the border. A triangulation is a maximal set of diagonals which do not cross. If Δ is any triangulation of $\mathcal{P}_{n+3}$, we know that Δ consists of exactly n diagonals.

Let α be a diagonal between vertex v_1 and vertex v_2 on the border of $\mathcal{P}_{n+3}$. In [7] a *pivoting elementary move* $P(v_1)$ is an anticlockwise move of α to another diagonal α' about v_1. The vertices of α' are v_1 and v_2', where v_2 and v_2' are vertices of a border edge and rotation is anticlockwise. A *pivoting path* from α to α' is a sequence of pivoting elementary moves starting at α and ending at α'.

Fix a positive integer n. Categories of diagonals of regular $(n+3)$-polygons were introduced in [7]. Let $\mathcal{C}_n$ be the category with indecomposable objects all diagonals of the polygon, and we take as objects formal direct sums of these diagonals. Morphisms from α to α' are generated by elementary pivoting moves modulo the mesh relations, which are defined as follows. Let α and β be diagonals, with a and b the vertices of α and c and d the vertices of β. Suppose $P(c)P(a)$ takes α to β. Then $P(c)P(a) = P(d)P(b)$. Furthermore, if one of the intermediate edges in a pivoting elementary move is a border edge, this move is zero. It is shown in [7] that this category is equivalent to the cluster category defined in Section 1 in the A_n case.

We have the following from [7].

- The irreducible morphisms in $\mathcal{C}_n$ are the direct sums of pivoting elementary moves.
- The Auslander-Reiten translation of a diagonal is given by clockwise rotation of the polygon.
- $\mathrm{Ext}^1_{\mathcal{C}_n}(\alpha, \alpha') = \mathrm{Ext}^1_{\mathcal{C}}(\alpha, \alpha') = 0$ if and only if α and α' do not cross.

It follows that a tilting object in $\mathcal{C}$ corresponds to a triangulation of $\mathcal{P}_{n+3}$.

For any triangulation Δ of $\mathcal{P}_{n+3}$, it is possible to define a quiver Q_Δ with n vertices in the following way. The vertices of Q_Δ are the midpoints of the diagonals of Δ. There is an arrow between i and j in Q_Δ if the corresponding diagonals bound a common triangle. The orientation is $i \to j$ if the diagonal corresponding to j is obtained from the diagonal corresponding to i by rotating anticlockwise about their common vertex. It is known from [7] that all quivers obtained in this way are quivers of cluster-tilted algebras of type A_n.

We defined the mutation of a quiver of a cluster-tilted algebra above. We also define mutation of a triangulation at a given diagonal, by replacing this diagonal with another one. This can be done in one and only one way. Let Q_Δ be a quiver corresponding to a triangulation Δ. Then mutation of Q_Δ at the vertex i corresponds to mutation of Δ at the diagonal corresponding to i.

It follows that any triangulation gives rise to a quiver of a cluster-tilted algebra, and that a quiver of a cluster-tilted algebra can be associated to at least one triangulation.

Let $\mathcal{M}_n$ be the mutation class of A_n, i.e. all quivers obtained by repeated mutation from A_n, up to isomorphisms of quivers. Let $\mathcal{T}_n$ be the set of all triangulations of $\mathcal{P}_{n+3}$. We can define a function $\gamma : \mathcal{T}_n \to \mathcal{M}_n$, where we set $\gamma(\Delta) = Q_\Delta$ for any triangulation Δ in $\mathcal{T}_n$. Note that γ is surjective.

3. Counting cluster-tilted algebras of type A_n

If a and b are vertices on the border of a regular polygon, we say that the *distance* between a and b is the smallest number of border edges between them. Let us say that a diagonal from a to b is *close to the border* if the distance between a and b is exactly 2. For a quiver Q_Δ corresponding to a triangulation Δ, let us always write v_α for the vertex of Q_Δ corresponding to the diagonal α.

If Q is a quiver of a cluster-tilted algebra of type A_n, we we have the following facts [6,7,12].

- All cycles are oriented.
- All cycles are of length 3.

- There does not exist two cycles that share one arrow.

Lemma 3.1. *If a diagonal α of a triangulation Δ is close to the border, then the corresponding vertex v_α in $\gamma(\Delta) = Q_\Delta$ is either a source, a sink or lies on a cycle (oriented of length 3).*

Proof. All cycles are oriented and of length 3 in the A_n case. Suppose that α is a diagonal in Δ which is close to the border. There are only three cases to consider, shown in Figure 1.

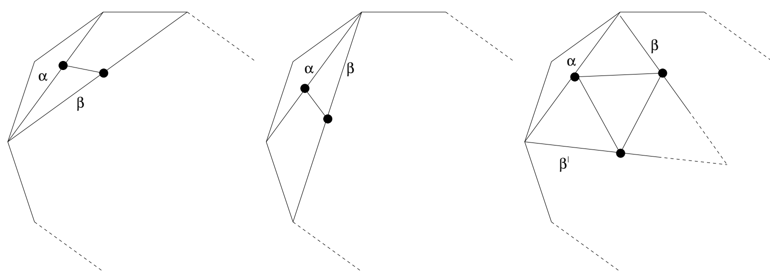

FIGURE 1. See the proof of Lemma 3.1. Sink, source and cycle.

In the first case α corresponds to a sink. There is no other vertex adjacent to v_α but v_β, or else the corresponding diagonal of this vertex would cross β. We have the same for the second case where α is a source. In the third case v_α lies on a cycle. □

Note that if v_α is a sink (or source) then v_α has only one adjacent vertex if and only if α is close to the border.

Lemma 3.2. *Let Δ be a triangulation and let $\gamma(\Delta) = Q_\Delta$ be the corresponding quiver. A quiver Q' obtained from Q_Δ by factoring out a vertex v_α is connected if and only if the corresponding diagonal α is close to the border.*

Proof. Suppose α is close to the border. By Lemma 3.1, α corresponds to a sink, a source or a vertex on a cycle. If v_α is a sink or a source then v_α has only one adjacent vertex, so factoring out v_α does not disconnect the quiver. Suppose v_α lies on a cycle. Then we are in the case shown in the third picture in Figure 1. We see that there can be no other vertex adjacent to v_α except v_β and $v_{\beta'}$, since else the corresponding diagonal would cross β or β'. Hence factoring out v_α does not disconnect the quiver.

Next, suppose that factoring out v_α does not disconnect the quiver. If v_α is a source or a sink with only one adjacent vertex, then v_α is close to the border. If not, first suppose v_α does not lie on a cycle. Then it is clear that factoring out v_α disconnects the quiver, so we may assume that v_α lies on a cycle. Then α is an edge of a triangle consisting of only diagonals (i.e. no border edges), say β and β'. Suppose there is a vertex v_δ adjacent to v_α, with $v_\delta \neq v_\beta$ and $v_\delta \neq v_{\beta'}$. Then v_δ can not be adjacent to v_β or $v_{\beta'}$, since then we would have two cycles sharing one arrow. We also see that v_δ can not be adjacent to any vertex v_γ from which there exists a path to v_β or $v_{\beta'}$ not containing v_α, or else there would be a cycle of length greater than 3. Therefore factoring out v_α would disconnect the quiver, and this is a contradiction, thus there can be no other vertices adjacent to v_α. It follows that α can not be adjacent to any other diagonal but β and β', hence α is close to the border. □

Let Δ be a triangulation of $\mathcal{P}_{n+3}$ and let α be a diagonal close to the border. The triangulation Δ' of $\mathcal{P}_{n+3-1}$ obtained from Δ by factoring out α is defined as the triangulation of $\mathcal{P}_{n+3-1}$ by letting α be a border edge and leaving all the other diagonals unchanged. We write Δ/α for the new triangulation obtained. See Figure 2.

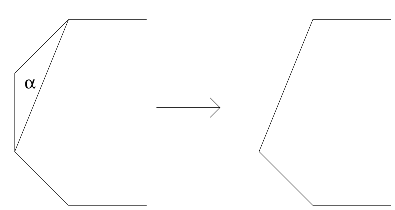

FIGURE 2. Factoring out a diagonal close to the border

Lemma 3.3. *Let Δ be a triangulation and $\gamma(\Delta) = Q_\Delta$. Factoring out a vertex in Q_Δ such that the resulting quiver is connected, corresponds to factoring out a diagonal of Δ close to the border.*

Proof. Factoring out a vertex v_α in Q such that the resulting quiver is connected, implies that α is close to the border by Lemma 3.2. Then consider all cases shown in Figure 1. □

Note that this means that $\gamma(\Delta/\alpha) = Q_\Delta/v_\alpha$. We have the following easy fact.

Proposition 3.4. *Let Q be a quiver of a cluster-tilted algebra of type A_n, with $n \geq 3$. Let Q' be obtained from Q by factoring out a vertex such that Q' is connected. Then Q' is the quiver of some cluster-tilted algebra of type A_{n-1}.*

Proof. It is already known from [4] that Q' is the quiver of a cluster-tilted algebra. Suppose Δ is a triangulation of $\mathcal{P}_{n+3}$ such that $\gamma(\Delta) = Q$. Such a Δ exists since γ is surjective. It is enough, by Lemma 3.2, to consider vertices corresponding to a diagonal close to the border. By Lemma 3.3, factoring out a vertex corresponding to a diagonal α close to the border, corresponds to factoring out α. Then the resulting triangulation of $\mathcal{P}_{(n-1)+3}$ corresponds to a quiver of a cluster-tilted algebra of type A_{n-1}, since it is a triangulation. □

Now we want to do the opposite of factoring out a vertex close to the border. If Δ is a triangulation of $\mathcal{P}_{n+3}$, we want to add a diagonal α such that α is a diagonal close to the border and such that $\Delta \cup \alpha$ is a triangulation of $\mathcal{P}_{(n+1)+3}$. Consider any border edge m on $\mathcal{P}_{n+3}$. Then we have one of the cases shown in Figure 3.

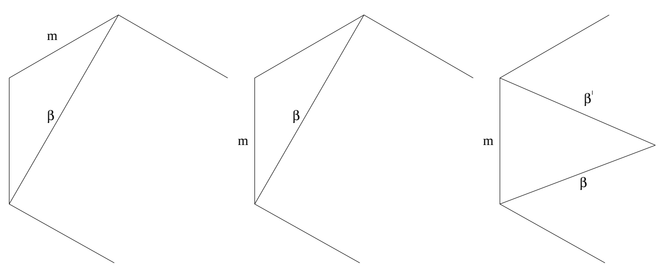

FIGURE 3.

We can extend the polygon at m for each case in Figure 3, and add a diagonal α to the extension. See Figure 4 for the corresponding extensions at m.

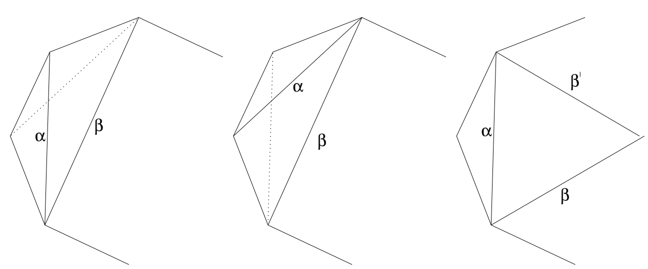

FIGURE 4.

It follows that for a given diagonal β, there are at most three ways to extend the polygon with a diagonal α such that α is adjacent to β, and it is easy to see that these extensions gives non-isomorphic quivers.

For a triangulation Δ of $\mathcal{P}_{n+3}$, let us denote by $\Delta(i)$ the triangulation obtained from Δ by rotating Δ i steps in the clockwise direction. We define an equivalence relation on $\mathcal{T}_n$, where we let $\Delta \sim \Delta(i)$ for all i. We define a new function $\tilde{\gamma}$: $(\mathcal{T}_n/\sim) \to \mathcal{M}_n$ induced from γ. This is well defined, for if $\Delta = \Delta'(i)$ for an i, then obviously $Q_\Delta = Q_{\Delta'}$ in $\mathcal{M}_n$. And hence since γ is a surjection, we also have that $\tilde{\gamma}$ is a surjection. We actually have the following.

Theorem 3.5. *The function $\tilde{\gamma} : (\mathcal{T}_n/\sim) \to \mathcal{M}_n$ is bijective for all $n \geq 2$.*

Proof. We already know that $\tilde{\gamma}$ is surjective.

Suppose $\tilde{\gamma}(\Delta) = \tilde{\gamma}(\Delta')$ in $\mathcal{M}_n$. We want to show that $\Delta = \Delta'$ in $(\mathcal{T}_n/\sim)$ using induction.

It is easy to check that $(\mathcal{T}_3/\sim) \to \mathcal{M}_3$ is injective. Suppose $(\mathcal{T}_{n-1}/\sim) \to \mathcal{M}_{n-1}$ is injective. Let α be a diagonal close to the border in Δ, with image v_α in Q, where Q is a representative for $\tilde{\gamma}(\Delta)$. Then the diagonal α' in Δ' corresponding to v_α in Q is also close to the border. We have $\tilde{\gamma}(\Delta/\alpha) = \tilde{\gamma}(\Delta'/\alpha') = Q/v_\alpha$ by Lemma 3.3, and hence, by hypothesis, $\Delta/\alpha = \Delta'/\alpha'$ in $(\mathcal{T}_n/\sim)$.

We can obtain Δ and Δ' from $\Delta/\alpha = \Delta'/\alpha'$ by extending the polygon at some border edge. Fix a diagonal β in Δ such that v_α and v_β are adjacent. This can be done since Q is connected. Let β' be the diagonal in Δ' corresponding to v_β. By the above there are at most three ways to extend Δ/α such that the new diagonal is adjacent to β. It is clear that these extensions will be mapped by $\tilde{\gamma}$ to non-isomorphic quivers. Also there are at most three ways to extend Δ'/α' such that the new diagonal is adjacent to β', and all these extensions are mapped to non-isomorphic quivers, thus $\Delta = \Delta'$ in $(\mathcal{T}_n/\sim)$. $\square$

Note that this also means that $\Delta = \Delta'(i)$ for an i if and only if $Q_\Delta \simeq Q_{\Delta'}$ as quivers.

Now, let T be a cluster-tilting object of the cluster category $\mathcal{C}$. This object corresponds to a triangulation Δ of $\mathcal{P}_{n+3}$, and all tilting objects obtained from rotation of Δ gives the same cluster-tilted algebra. No other triangulation gives rise to the same cluster-tilted algebra.

The Catalan number $C(i)$ can be defined as the number of triangulations of an i-polygon with $i-3$ diagonals. The number is given by the following formula.

n	$a(n)$	n	$a(n)$
2	1	7	150
3	4	8	442
4	6	9	1424
5	19	10	4522
6	49	11	14924

TABLE 1. Some values of $a(n)$.

$$C(i) = \frac{(2i)!}{(i+1)!i!}$$

We now have the following.

Corollary 3.6. *The number $a(n)$ of non-isomorphic basic cluster-tilted algebras of type A_n is the number of triangulations of the disk with n diagonals, i.e.*

$$a(n) = C(n+1)/(n+3) + C((n+1)/2)/2 + (2/3)C(n/3),$$

where $C(i)$ is the i'th Catalan number and the second term is omitted if $(n+1)/2$ is not an integer and the third term is omitted if $n/3$ is not an integer.

These numbers appeared in a paper by W. G. Brown in 1964 [1]. See Table 1 for some values of $a(n)$.

We have that if T is a cluster-tilting object in $\mathcal{C}$, then the cluster-tilted algebras $\operatorname{End}_{\mathcal{C}}(T)$ and $\operatorname{End}_{\mathcal{C}}(\tau T)$ are isomorphic. In the A_n case we also have the following.

Theorem 3.7. *Let T and T' be tilting objects in $\mathcal{C}$, then the cluster-tilted algebras $\operatorname{End}_{\mathcal{C}}(T)$ and $\operatorname{End}_{\mathcal{C}}(T')$ are isomorphic if and only if $T' = \tau^i T$ for an $i \in \mathbb{Z}$.*

Proof. Let Δ be the triangulation of $\mathcal{P}_{n+3}$ corresponding to T and let Δ' be the triangulation corresponding to T'. If $T' \not\simeq \tau^i T$ for any i, then Δ' is not obtained from Δ by a rotation, and hence $\operatorname{End}_{\mathcal{C}}(T)$ is not isomorphic to $\operatorname{End}_{\mathcal{C}}(T')$ by Theorem 3.5. □

Proposition 3.8. *Let Γ be a cluster-tilted algebra of type A_n. The number of non-isomorphic cluster-tilting objects T such that $\Gamma \simeq \operatorname{End}_{\mathcal{C}}(T)$ has to divide $n+3$.*

Proof. Let T be a tilting object in $\mathcal{C}$ corresponding to the triangulation Δ. Denote by $\Delta(i)$ the rotation of Δ i steps in the clockwise direction. Let $0 < s \leq n$ be the smallest number of rotations needed to obtain the same triangulation Δ, i.e. the smallest s such that $\Delta = \Delta(s)$. It is clear from the above that $T \not\simeq T'$, where T'

corresponds to $\Delta(t)$ with $0 < t < s$, hence s is the number of non-isomorphic tilting objects giving the same cluster-tilted algebra. Now we only need to show that s divides $n+3$, but this is clear. □

The proof of the following is easy and is left to the reader. First recall from [10, Proposition 3.8] that there are exactly $C(n)$ non-isomorphic tilting objects in the cluster category for type A_n, where $C(n)$ denotes the n'th Catalan number.

Proposition 3.9. *Consider the A_n case.*

- *There are always at least 2 non-isomorphic cluster-tilting objects giving the same cluster-tilted algebra.*
- *There are at most $n+3$ non-isomorphic cluster-tilting objects giving the same cluster-tilted algebra.*
- *Let Γ be a cluster-tilted algebra of type A_n. If $n+3$ is prime, there are exactly $n+3$ non-isomorphic cluster-tilting objects giving Γ. In this case there are $C(n)/n+3$ non-isomorphic cluster-tilted algebras, where $C(n)$ denotes the n'th Catalan number.*

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