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FACTOR CATEGORIES AND INFINITE DIRECT SUMS

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Abstract. We give an improved categorical version of the Weak Krull-Schmidt Theorem for serial modules proved by the second author in [10]. The main improvement consists in the fact that it applies not only to serial modules, but also, more generally, to arbitrary direct summands of serial modules. The technique is based on categorical methods, essentially representing the category add(SUsr) of direct summands of serial modules into factors of the category add(SUsr) modulo suitable ideals, one for each uniserial module of type 1 and two for each uniserial module of type 2. Our categorical technique can be applied to further broader settings.

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1. Introduction

Recall that a module is *uniserial* if its lattice of submodules is linearly ordered by inclusion, and is *serial* if it is a direct sum of uniserial submodules. In this paper we prove that direct summands of serial modules are completely described up to isomorphism by a family of cardinal numbers (Theorem 7.4). These cardinal numbers are the dimensions of suitable vector spaces over division rings that are homomorphic images of endomorphism rings of uniserial modules. The technique we use to prove our result is based on factoring the category of all serial modules modulo suitable ideals. Our result is rather surprising, because there exist direct summands of serial modules which are not direct sums of indecomposable submodules [13].

Recall that serial modules decompose as a direct sum of uniserial modules in different ways, and the uniqueness of direct sum decompositions is completely described up to isomorphism by the Weak Krull-Schmidt Theorem proved by the second author in [10, Theorem 2.6]. If V and U are arbitrary modules over a

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ring R, we write $[V]_m = [U]_m$, and say that V and U are in the same *monogeny* class, if there exist a monomorpism $V \to U$ and a monomorpism $U \to V$, and write $[V]_e = [U]_e$, and say that V and U are in the same *epigeny* class, if there exist an epimorphism $V \to U$ and an epimorphism $U \to V$. According to the Weak Krull-Schmidt Theorem, if $\{ U_i \mid i \in I \}$ and $\{ V_j \mid j \in J \}$ are two families of non-zero uniserial modules, I' is the set of all indices $i \in I$ with U_i quasi-small and J' is the set of all $j \in J$ with V_j quasi-small, then $\oplus_{i\in I} U_i \cong \oplus_{j\in J} V_j$ if and only if there exist a bijection $\sigma \colon I \to J$ and a bijection $\tau \colon I' \to J'$ such that $[U_i]_m = [V_{\sigma(i)}]_m$ for every $i \in I$ and $[U_i]_e = [V_{\tau(i)}]_e$ for every $i \in I'$. Equivalently, a serial module $\oplus_{i\in I} U_i$ is completely determined up to isomorphism by a family of cardinal numbers, one for each monogeny class $[U]_m$ of non-zero uniserial modules U (the cardinality of the set of all indices $i \in I$ with $[U_i]_m = [U]_m$) and one for each epigeny class $[U]_e$ of non-zero quasi-small uniserial modules U (the cardinality of the set of all $i \in I$ with $[U_i]_e = [U]_e$). In Theorem 7.4 we extend this Weak Krull-Schmidt Theorem from serial modules to arbitrary direct summands of serial modules.

The technique we make use of to prove our Theorem 7.4 is essentially the following. For a fixed ring R, let add(SUsr) be the full subcategory of Mod-R whose objects are all direct summands of serial modules. Recall that if U_R is a non-zero uniserial module, then $\mathrm{End}_R(U)$ has two important ideals — one given by non-injective endomorphisms, and one given by non-surjective endomorphisms. If these two ideals are comparable with respect to inclusion, $\mathrm{End}_R(U)$ is a local ring and its Jacobson radical is the union of these ideals. In this case we say that U is of *type 1*. If these two ideals are not comparable with respect to inclusion, then they are the only (left, right, two-sided) maximal ideals of $\mathrm{End}_R(U)$ and we say that U is of *type 2* (see [2, Theorem 9.1] for details). Fix a non-zero uniserial module U_R, and fix a maximal ideal I of $\mathrm{End}_R(U)$. Let $\mathcal{I}$ be the ideal of the category add(SUsr) consisting of all the morphisms $f \colon X \to Y$ such that $\beta f \alpha \in I$ for every $\alpha \colon U \to X$ and every $\beta \colon Y \to U$. We call $\mathcal{I}$ the *ideal of* add(SUsr) *associated to* I. Under mild hypotheses (in particular, a property (*) considered in Section 3), the factor category add(SUsr)$/\mathcal{I}$ turns out to be equivalent to the category Mod-$\mathrm{End}_R(U)/I$ of all right vector spaces over the division ring $\mathrm{End}_R(U)/I$ (Lemma 3.1). Let $F_I \colon \mathrm{add(SUsr)} \to \mathrm{add(SUsr)}/\mathcal{I} \cong \text{Mod-}\mathrm{End}_R(U)/I$ denote the canonical functor. Then the family of all cardinal numbers that describe up to isomorphism all direct summands M_R of serial modules consists essentially of all the dimensions of the $\mathrm{End}_R(U)/I$-vector spaces $F_I(M_R)$, where I ranges in the set of all maximal ideals of all the endomorphism rings $\mathrm{End}_R(U)$ of uniserial right R-modules. For further details on the categorical technique employed, which seems to be very general and should find applications in broader settings, see Section 2.

Assume we have a countable direct sum $M = \oplus_{i\in\mathbb{N}} U_i$ of uniserial modules U_i. Let $M^{(\aleph_0)}$ be the direct sum of countably many copies of M, and $V(M^{(\aleph_0)})$ be the additive monoid of isomorphism classes of direct summands A of $M^{(\aleph_0)}$. Here the addition is induced by direct sum. The cardinal invariants $\text{i-dim}_{U_i}, \text{m-dim}_{U_i}, \text{e-dim}_{U_i}$ of Corollary 7.3 are mappings $V(M^{(\aleph_0)}) \to \mathbb{N}_0^*$, where $\mathbb{N}_0^* = \mathbb{N}_0 \cup \{+\infty\}$. Hence, by Corollary 7.3, they induce an injective morphism of monoids $V(M^{(\aleph_0)}) \to (\mathbb{N}_0^*)^{\aleph_0}$. In particular, the cardinality of $V(M^{(\aleph_0)})$ is $\leq 2^{\aleph_0}$, that is, there are at most $2^{\aleph_0}$ direct summands of $M^{(\aleph_0)}$ up to isomorphism.

The remaining problem in classifying direct summands of serial modules is now understanding which sequences of cardinals may occur as cardinal invariants of these modules, and this problem is still open.

In this paper, rings are associative rings with identity and modules are unital right modules.

2. Factoring the category modulo an ideal of the endomorphism ring of an object

In the following, $\mathcal{A}$ is always a full subcategory of Mod-R and $\mathrm{Ob}(\mathcal{A})$ is its class of objects. An *ideal* $\mathcal{I}$ of $\mathcal{A}$ is a subgroup $\mathcal{I}(X,Y)$ of $\mathcal{A}(X,Y)$ for every pair of objects $X, Y \in \mathrm{Ob}(\mathcal{A})$ such that for every morphism $\varphi\colon Z \to X$, $\psi\colon X \to Y$ and $\omega\colon Y \to W$ with $\psi \in \mathcal{I}(X,Y)$ one has that $\omega\psi\varphi \in \mathcal{I}(Z,W)$ (an ideal is a subfunctor of the two variable functor $\mathcal{A}(-,-)$, [8, p. 18]). The *factor category* $\mathcal{A}/\mathcal{I}$ of $\mathcal{A}$ modulo the ideal $\mathcal{I}$ has the same objects as $\mathcal{A}$, and, for objects $X, Y \in \mathrm{Ob}(\mathcal{A}) = \mathrm{Ob}(\mathcal{A}/\mathcal{I})$, the morphisms $X \to Y$ in the factor category $\mathcal{A}/\mathcal{I}$ are the cosets of $\mathcal{A}(X,Y)$ modulo $\mathcal{I}(X,Y)$, that is, they are the elements of the abelian group $\mathcal{A}(X,Y)/\mathcal{I}(X,Y)$.

Let U_R be a non-zero module, and fix an ideal I of $\mathrm{End}_R(U)$. Let $\mathcal{I}$ be the ideal of $\mathcal{A}$ defined as follows: a morphism $f\colon X \to Y$ is in $\mathcal{I}$ if and only if $\beta f \alpha \in I$ for every $\alpha\colon U \to X$ and every $\beta\colon Y \to U$. We will call $\mathcal{I}$ the *ideal of* $\mathcal{A}$ *associated to* I. If U is an object of $\mathcal{A}$, then $\mathcal{I}$ is the greatest among the ideals $\mathcal{I}'$ of $\mathcal{A}$ with $\mathcal{I}'(U,U) \subseteq I$, and in this case, as it is easily seen, $\mathcal{I}(U,U) = I$. Let $F\colon \mathcal{A} \to \mathcal{A}/\mathcal{I}$ denote the canonical functor.

Lemma 2.1. *Suppose that* U *is an object of* $\mathcal{A}$*. Let* I *be a proper ideal of* $\mathrm{End}_R(U)$*,* $\mathcal{I}$ *be the ideal of* $\mathcal{A}$ *associated to* I*, and* $F\colon \mathcal{A} \to \mathcal{A}/\mathcal{I}$ *be the canonical functor. Then* $F(U)$ *is a non-zero object of* $\mathcal{A}/\mathcal{I}$*. Moreover, if* I *is completely prime in* $\mathrm{End}_R(U)$*, then* $F(U)$ *is an indecomposable object of* $\mathcal{A}/\mathcal{I}$*.*

Proof. From $1_U \notin \mathcal{I}$, it follows that $F(1_U) \neq 0$ in $\mathcal{A}/\mathcal{I}$. Assume I completely prime in $\mathrm{End}_R(U)$. Then I is a proper ideal, so that $F(U) \neq 0$. If $F(U) \cong X \oplus Y$ in $\mathcal{A}/\mathcal{I}$ with X, Y non-zero objects, then there are non-zero orthogonal idempotent elements in $\mathrm{End}_{\mathcal{A}/\mathcal{I}}(F(U)) \cong \mathrm{End}_R(U)/I$. This is not possible, because I is completely prime. □

Lemma 2.2. *Let U be an object of $\mathcal{A}$, I an ideal of $\mathrm{End}_R(U)$, $\mathcal{I}$ the ideal of $\mathcal{A}$ associated to I and $F\colon \mathcal{A} \to \mathcal{A}/\mathcal{I}$ the canonical functor. Let V be an object of $\mathcal{A}$ such that $F(U) \cong F(V)$ in $\mathcal{A}/\mathcal{I}$. Let K be the ideal of $\mathrm{End}_R(V)$ given by $K := \mathcal{I}(V, V)$. Let $\mathcal{K}$ be the ideal of $\mathcal{A}$ associated to K. Then $\mathcal{K} = \mathcal{I}$.*

Proof. Clearly, $\mathcal{K} \supseteq \mathcal{I}$.

Conversely, let $f\colon X \to Y$ be in $\mathcal{K}$. As $F(U) \cong F(V)$, there are homomorphisms $\alpha\colon U \to V$ and $\beta\colon V \to U$ such that $1_U - \beta\alpha \in \mathcal{I}$ and $1_V - \alpha\beta \in \mathcal{I}$. In order to prove that f is in $\mathcal{I}$, fix $\gamma\colon U \to X$ and $\delta\colon Y \to U$. We must show that $g := \delta f \gamma \in I$. Now $\alpha g \beta \in K$ and, consequently, $\beta\alpha g \beta\alpha \in I$. Now $\beta\alpha g\beta\alpha - g = \beta\alpha g(\beta\alpha - 1_U) + (\beta\alpha - 1_U)g \in I$. Thus $g \in I$. □

In order to have that the canonical functor F respect infinite direct sums, we add a rather technical condition to U and I. Recall that a family of morphisms $f_\lambda\colon U \to M_\lambda$, $\lambda \in \Lambda$, is *summable* if for every $x \in U$ there is a finite subset Λ_x of Λ such that $f_\lambda(x) = 0$ for every $\lambda \in \Lambda \setminus \Lambda_x$. Equivalently, if the position $x \mapsto (f_\lambda(x))_{\lambda\in\Lambda}$ defines a mapping $U \to \oplus_{\lambda\in\Lambda} M_\lambda$.

Lemma 2.3. *Let U be a right R-module and let I be an ideal in $\mathrm{End}_R(U)$. The following conditions are equivalent:*

- (a) *For every family of modules M_λ, $\lambda \in \Lambda$, and homomorphisms $\alpha\colon U \to \oplus_{\lambda\in\Lambda} M_\lambda$ and $\beta\colon \oplus_{\lambda\in\Lambda} M_\lambda \to U$ with $\beta\alpha \notin I$, there exists $\mu \in \Lambda$ such that $\beta\iota_\mu\pi_\mu\alpha \notin I$.*
- (b) *For every summable family $f_\lambda\colon U \to U$, $\lambda \in \Lambda$, of endomorphisms of U such that $f_\lambda \in I$ for every $\lambda \in \Lambda$, one has that $\sum_{\lambda\in\Lambda} f_\lambda \in I$.*
- (c) *For every set Λ and every homomorphism $F\colon U \to U^{(\Lambda)}$ such that the composite mapping $\Sigma \circ F\colon U \to U$, where $\Sigma\colon U^{(\Lambda)} \to U$ is the homomorphism $(x_\lambda)_{\lambda\in\Lambda} \mapsto \sum_{\lambda\in\Lambda} x_\lambda$, is not in I, there exists $\mu \in \Lambda$ such that $\pi_\mu F \notin I$.*

Proof. The proof is elementary. For (a)⇒(b), it suffices to assume (a) true and take as α the morphism $u \mapsto (f_\lambda(u))_{\lambda\in\Lambda}$ and as β the homomorphism Σ defined in (c). For (b)⇒ (c), it suffices to assume (b) true and take as f_λ's the morphisms $\pi_\lambda F$'s. For (c)⇒(a), it suffices to assume (c) true and take as F the morphism $u \mapsto (\beta\iota_\lambda\pi_\lambda\alpha(u))_{\lambda\in\Lambda}$. □

If U is a right R-module and I is an ideal of $\mathrm{End}_R(U)$, we say that U is *I-small* if it satisfies the conditions of the previous Lemma.

Corollary 2.4. *Let U be a non-zero module, let I be an ideal of $\mathrm{End}_R(U)$ and let $\mathcal{I}$ be the ideal of $\mathcal{A}$ associated to I. Suppose that M_λ, $\lambda \in \Lambda$, is a family of objects of $\mathcal{A}$ such that $\oplus_{\lambda\in\Lambda} M_\lambda$ is an object of $\mathcal{A}$. If U is I-small, then a morphism $f\colon \oplus_{\lambda\in\Lambda} M_\lambda \to N$ in $\mathcal{A}$ is in $\mathcal{I}$ if and only if $f\iota_\mu\colon M_\mu \to N$ is in $\mathcal{I}$ for every embedding $\iota_\mu\colon M_\mu \to \oplus_{\lambda\in\Lambda} M_\lambda$.*

Proof. Suppose that $f\iota_\mu \in \mathcal{I}$ for every $\mu \in \Lambda$. If $f \notin \mathcal{I}$, then there are $\alpha\colon U \to \oplus_{\lambda\in\Lambda} M_\lambda$ and $\beta\colon N \to U$ such that $\beta f \alpha \notin I$. This is not possible by Lemma 2.3(a). □

Notice that coproducts in a full additive subcategory $\mathcal{A}$ of Mod-R can be different in $\mathcal{A}$ and Mod-R. For instance, if $\mathcal{A}$ is the category of all non-singular injective right R-modules, then coproducts in $\mathcal{A}$ are the injective envelopes of the coproducts in Mod-R [6, Proposition 1.12]. The next statement shows that F preserves the coproducts that are equal to the direct sum, that is, equal to the coproduct in Mod-R.

Lemma 2.5. *Suppose that M_λ, $\lambda \in \Lambda$, is a family of objects in $\mathcal{A}$ such that $\oplus_{\lambda\in\Lambda} M_\lambda \in \mathrm{Ob}(\mathcal{A})$. Let U be a non-zero module, I an ideal of $\mathrm{End}_R(U)$. Consider the ideal $\mathcal{I}$ in $\mathcal{A}$ associated to I and the canonical functor $F\colon \mathcal{A} \to \mathcal{A}/\mathcal{I}$. For every $\mu \in \Lambda$, let $\iota_\mu\colon M_\mu \to \oplus_{\lambda\in\Lambda} M_\lambda$ be the embedding. Suppose that, for any morphism $f\colon \oplus_{\lambda\in\Lambda} M_\lambda \to X$ in the category $\mathcal{A}$, the morphism f is in $\mathcal{I}$ if and only if $f\iota_\lambda \in \mathcal{I}$ for every $\lambda \in \Lambda$. Then $F(\oplus_{\lambda\in\Lambda} M_\lambda)$ with the morphisms $F(\iota_\lambda), \lambda \in \Lambda$, is the coproduct of the family of objects $F(M_\lambda), \lambda \in \Lambda$, in the factor category $\mathcal{A}/\mathcal{I}$.*

Proof. Let $F(X)$ be an object of $\mathcal{A}/\mathcal{I}$ and let $F(f_\lambda)\colon F(M_\lambda) \to F(X)$, $\lambda \in \Lambda$, be morphisms in $\mathcal{A}/\mathcal{I}$. Clearly, there exists a morphism $g\colon \oplus_{\lambda\in\Lambda} M_\lambda \to X$ such that $f_\lambda = g\iota_\lambda$ for every $\lambda \in \Lambda$. Therefore $F(f_\lambda) = F(g)F(\iota_\lambda)$ for every $\lambda \in \Lambda$. Now let $g'\colon \oplus_{\lambda\in\Lambda} M_\lambda \to X$ be a morphism such that $F(g')F(\iota_\lambda) = F(f_\lambda)$ for every $\lambda \in \Lambda$. Then $F((g - g')\iota_\lambda) = 0$ for all $\lambda \in \Lambda$. Equivalently, $(g - g')\iota_\lambda \in \mathcal{I}$ for every $\lambda \in \Lambda$. By our hypothesis, $g - g' \in \mathcal{I}$, and consequently $F(g) = F(g')$. □

Remark 2.6. The condition "$f \in \mathcal{I}$ if $f\iota_\lambda \in \mathcal{I}$ for every $\lambda \in \Lambda$" is necessary in the statement of Lemma 2.5. To see this, suppose that there exists an object X of $\mathcal{A}$ and $f\colon \oplus_{\lambda\in\Lambda} M_\lambda \to X$ such that $f \notin \mathcal{I}$ but $f\iota_\lambda \in \mathcal{I}$ for every $\lambda \in \Lambda$. Then $F(f)$ and $0\colon F(\oplus_{\lambda\in\Lambda} M_\lambda) \to F(X)$ are two different morphisms, and $F(f)F(\iota_\lambda) = 0 = 0F(\iota_\lambda)$ for every $\lambda \in \Lambda$.

The next Corollary follows immediately from Corollary 2.4 and Lemma 2.5.

Corollary 2.7. *Let U be a non-zero module, I an ideal of $\mathrm{End}_R(U)$ such that U is I-small. If $\mathrm{Ob}(\mathcal{A})$ is closed under arbitrary direct sums, and $\mathcal{I}$ is the ideal of $\mathcal{A}$ associated to I, then the canonical functor $F\colon \mathcal{A} \to \mathcal{A}/\mathcal{I}$ preserves coproducts.*

By Lemma 2.5, we get:

Proposition 2.8. *Let U be an I-small module, $\mathcal{I}$ the ideal of $\mathcal{A}$ associated to I, and $F\colon \mathcal{A} \to \mathcal{A}/\mathcal{I}$ the canonical functor. If M_λ, $\lambda \in \Lambda$, is a family of objects of $\mathcal{A}$ such that $\oplus_{\lambda\in\Lambda} M_\lambda \in \mathrm{Ob}(\mathcal{A})$, then $F(\oplus_{\lambda\in\Lambda} M_\lambda) = 0$ if $F(M_\lambda) = 0$ for every $\lambda \in \Lambda$.*

We will deal with infinite direct sums and factor categories of module categories, and it is convenient to fix the notation we will use in this setting. Let U be an I-small module, and $\mathcal{A}$ a full subcategory of Mod-R. Let A_λ, $\lambda \in \Lambda$, and B_μ, $\mu \in M$, be families of objects in $\mathcal{A}$ such that $A = \oplus_{\lambda\in\Lambda} A_\lambda$ and $B = \oplus_{\mu\in M} B_\mu$ are objects of $\mathcal{A}$. Let $\mathcal{I}$ be the ideal of $\mathcal{A}$ associated to I and let $F\colon \mathcal{A} \to \mathcal{A}/\mathcal{I}$ be the canonical functor. Any morphism $f\colon A \to B$ of right R-modules can be represented by an $M \times \Lambda$ matrix $(f_{\mu,\lambda})_{\mu\in M,\lambda\in\Lambda}$, where $f_{\mu,\lambda} = \pi_{B_\mu} f \iota_{A_\lambda}$ and, for every fixed $\overline{\lambda} \in \Lambda$, the family of morphisms $f_{\mu,\overline{\lambda}}\colon A_{\overline{\lambda}} \to B_\mu$, $\mu \in M$, is summable. Let us prove that the morphism $F(f)$ is uniquely determined by the morphisms $F(f_{\mu,\lambda}) = F(\pi_{B_\mu})F(f)F(\iota_{A_\lambda})$. Suppose that $f\colon \oplus_{\lambda\in\Lambda} A_\lambda \to \oplus_{\mu\in M} B_\mu$ and $f'\colon \oplus_{\lambda\in\Lambda} A_\lambda \to \oplus_{\mu\in M} B_\mu$ are such that $F(\pi_{B_\mu})F(f)F(\iota_{A_\lambda}) = F(\pi_{B_\mu})F(f')F(\iota_{A_\lambda})$. Let us prove that $F(f) = F(f')$. By Lemma 2.5, the $F(\iota_{A_\lambda})$'s are the coproduct morphisms, so that it is enough to show that $F(f)F(\iota_{A_\lambda}) = F(f')F(\iota_{A_\lambda})$ for every $\lambda \in \Lambda$. Assume that $(f - f')\iota_{A_{\overline{\lambda}}} \notin \mathcal{I}$ for some $\overline{\lambda}$. That is, there are $\alpha\colon U \to A_{\overline{\lambda}}$ and $\beta\colon \oplus_{\mu\in M} B_\mu \to U$ such that $\beta(f - f')\iota_{A_{\overline{\lambda}}}\alpha \notin I$. As U is I-small, there exists $\overline{\mu} \in M$ such that $\beta\iota_{B_{\overline{\mu}}}\pi_{B_{\overline{\mu}}}(f - f')\iota_{A_{\overline{\lambda}}}\alpha \notin I$. This is contrary to our assumption that $\pi_{B_\mu}(f - f')\iota_{A_\lambda} \in I$. In the following remark, we consider when a morphism $f = (f_{\mu,\lambda})_{\mu\in M,\lambda\in\Lambda}\colon \oplus_{\lambda\in\Lambda} A_\lambda \to \oplus_{\mu\in M} B_\mu$ is in the ideal $\mathcal{I}$ of $\mathcal{A}$ associated to I.

Remark 2.9. Let U be a non-zero module, I an ideal in $\mathrm{End}_R(U)$, and $\mathcal{I}$ the ideal of $\mathcal{A}$ associated to I. Suppose that U is I-small and that A_λ, $\lambda \in \Lambda$, and B_μ, $\mu \in M$, are objects of $\mathcal{A}$ such that $\oplus_{\lambda\in\Lambda} A_\lambda$ and $\oplus_{\mu\in M} B_\mu$ are also objects of $\mathcal{A}$. Then $f\colon \oplus_{\lambda\in\Lambda} A_\lambda \to \oplus_{\mu\in M} B_\mu$ is in $\mathcal{I}$ if and only if $\pi_\mu f \iota_\lambda \in \mathcal{I}$ for every $\lambda \in \Lambda$ and $\mu \in M$. That is, the homomorphism f is in $\mathcal{I}$ if and only if all entries of the corresponding matrix are in $\mathcal{I}$.

Lemma 2.10. *Let U be a non-zero right R-module and let I be an ideal of* $\mathrm{End}_R(U)$ *such that* $\mathrm{End}_R(U)/I$ *is a division ring and U is I-small. Then every summable family of morphisms f_λ, $\lambda \in \Lambda$, belonging to* $\mathrm{End}_R(U) \setminus I$ *is finite.*

Proof. Let f_i, $i \in \mathbb{N}$, be a summable family of morphisms belonging to $\mathrm{End}_R(U) \setminus I$. Since I is a maximal left ideal, for every $i \in \mathbb{N}$ there exists $g_i \in \mathrm{End}_R(U)$ with $h_i := 1_U - g_i f_i \in I$. Consider the family $h_1, h_2 - h_1, h_3 - h_2, \dots$, which is easily seen to be a summable family. All the homomorphisms in this family belong to I, but the sum of the family is 1_U, so that Property (b) of Lemma 2.3 does not hold for this family. Hence U is not I-small. □

Corollary 2.11. *Let κ be a cardinal, U a non-zero right R-module and I an ideal of* $\mathrm{End}_R(U)$ *with U I-small and* $\mathrm{End}_R(U)/I$ *a division ring. If U and $U^{(\kappa)}$ are objects of the category $\mathcal{A}$, then, for any homomorphism $f \colon U \to U^{(\kappa)}$, $F(\pi_j)F(f) \neq 0$ only for finitely many $j < \kappa$.*

Proof. The homomorphisms $\pi_j f$, $j < \kappa$, form a summable family of $\mathrm{End}_R(U)$. By Lemma 2.10 only finitely many of the $\pi_j f$'s are not elements of I. □

3. Property (*)

In all this section, we suppose that *$\mathcal{A}$ is a full subcategory of* Mod-R *closed under arbitrary direct sums, U is a non-zero object of $\mathcal{A}$ and I is an ideal of* $\mathrm{End}_R(U)$ *such that* $\mathrm{End}_R(U)/I$ *is a division ring and U is I-small. Let $\mathcal{I}$ be the ideal of $\mathcal{A}$ associated to I and let $F \colon \mathcal{A} \to \mathcal{A}/\mathcal{I}$ be the canonical functor.*

Consider the following property on $\mathcal{A}$:

(*) *Every object M of $\mathcal{A}$ is a direct sum of modules $M = \oplus_{\lambda \in \Lambda} M_\lambda$, where, for every $\lambda \in \Lambda$, the module M_λ is an object of $\mathcal{A}$ with either $F(M_\lambda) = 0$ or $F(M_\lambda) \cong F(U)$.*

Recall that $F(U)$ is indecomposable in $\mathcal{A}/\mathcal{I}$ (Lemma 2.1). For a family M_λ, $\lambda \in \Lambda$, of objects in $\mathcal{A}$, $F(\oplus_{\lambda \in \Lambda} M_\lambda)$ is the coproduct in $\mathcal{A}/\mathcal{I}$ of the family of objects $F(M_\lambda)$, $\lambda \in \Lambda$. Thus if $F(M_\lambda)$ is either 0 or isomorphic to $F(U)$, every object $F(M)$ of $\mathcal{A}/\mathcal{I}$ is necessarily isomorphic to the coproduct $F(U^{(\kappa)})$, where κ is the cardinality of the set of all $\lambda \in \Lambda$ with $F(M_\lambda) \neq 0$.

Lemma 3.1. *If $\mathcal{A}$ satisfies Property (*), then $G := \mathrm{Hom}_{\mathcal{A}/\mathcal{I}}(F(U), -)$ is a category equivalence of the category $\mathcal{A}/\mathcal{I}$ into the category of all right vector spaces over the division ring* $\mathrm{End}_R(U)/I$*:*

$$G \colon \mathcal{A}/\mathcal{I} \to \text{Mod-}(\mathrm{End}_R(U)/I).$$

Proof. We know that every object of $\mathcal{A}/\mathcal{I}$ is isomorphic to an object of the form $F(U^{(\kappa)})$ for some cardinal κ. So we can consider only objects of this form. We must prove that the functor G is full, faithful and dense.

From Corollary 2.11, it follows that $G(F(U^{(\kappa)}))$ is a vector space whose basis is given by the morphisms $F(\iota_j), j < \kappa$, where $\iota_j \colon U \to U^{(\kappa)}$ denotes the embedding. So $G(F(U^{(\kappa)})) \cong G(F(U))^{(\kappa)}$, hence G is dense.

In order to show that G is faithful, fix a morphism $f \colon U^{(\kappa)} \to U^{(\lambda)}$ with $F(f) \neq 0$. Let $(f_{j,i})_{j<\lambda, i<\kappa}$ be the matrix corresponding to f. At least one of the $f_{j,i}$ is not in $\mathcal{I}$ by Remark 2.9. Thus $F(f\iota_i) \neq 0$. But $F(f\iota_i) = F(f)F(\iota_i) = G(F(f))(F(\iota_i))$. Hence $G(F(f)) \neq 0$.

It remains to show that G is full. Fix $g \colon G(F(U^{(\kappa)})) \to G(F(U^{(\lambda)}))$ and consider the canonical bases $F(\iota_i), i < \kappa$, and $F(\nu_j), j < \lambda$, of $G(F(U^{(\kappa)})) = \mathrm{Hom}_{\mathcal{A}/\mathcal{I}}(F(U), F(U^{(\kappa)}))$ and $G(F(U^{(\lambda)}))$, respectively. So

$$g(F(\iota_i)) = \sum_{j<\lambda} F(\nu_j)s_{j,i},$$

where for every fixed $i < \kappa$, only finitely many $s_{j,i}$'s are non-zero elements of $\mathrm{End}_R(U)/I$. Then there is a column finite matrix $(t_{j,i})_{j<\lambda, i<\kappa}$, $t_{j,i} \colon U \to U$, such that $F(t_{j,i}) = s_{j,i}$ for every $i < \kappa$ and $j < \lambda$. This matrix defines a morphism $f \colon U^{(\kappa)} \to U^{(\lambda)}$. Now the matrix corresponding to the vector $G(F(f))(F(\iota_i))$ is the column matrix $(F(\pi_j)F(f)F(\iota_i))_j = (F(t_{j,i}))_j = (s_{j,i})_j$, so that $G(F(f))(F(\iota_i)) = \sum_{j<\lambda} F(\nu_j)s_{j,i} = g(F(\iota_i))$. Hence $G(F(f)) = g$. □

The full subcategory of Mod-R whose objects are all R-modules that are isomorphic to direct summands of modules in $\mathrm{Ob}(\mathcal{A})$ will be denoted by $\mathrm{add}(\mathcal{A})$.

Proposition 3.2. *If $\mathcal{A}$ satisfies Property (*) and $\mathcal{K}$ is the ideal of $\mathrm{add}(\mathcal{A})$ associated to I, then the category $\mathrm{add}(\mathcal{A})/\mathcal{K}$ is equivalent to the category of all right vector spaces over the division ring $\mathrm{End}_R(U)/I$.*

Proof. Let $F \colon \mathcal{A} \to \mathcal{A}/\mathcal{I}$ and $F' \colon \mathrm{add}(\mathcal{A}) \to \mathrm{add}(\mathcal{A})/\mathcal{K}$ be the canonical functors. Let $I \colon \mathcal{A} \to \mathrm{add}(\mathcal{A})$ be the inclusion functor. The definition of factor categories gives a full and faithful functor $E \colon \mathcal{A}/\mathcal{I} \to \mathrm{add}(\mathcal{A})/\mathcal{K}$, and $EF = F'I$. We have to prove that E is dense, that is, if U_λ, $\lambda \in \Lambda$, are objects in $\mathcal{A}$ and $X \oplus Y = \oplus_{\lambda\in\Lambda} U_\lambda$ in Mod-R, then $F'(X) \cong EF(U^{(\kappa)})$ in the category $\mathrm{add}(\mathcal{A})/\mathcal{K}$ for some cardinal κ. By Property (*) we can suppose that, for every $\lambda \in \Lambda$, either $F(U_\lambda) = 0$ or $F(U_\lambda) \cong F(U)$.

If κ' is the cardinality of the set of all $\lambda \in \Lambda$ with $F(U_\lambda) \cong F(U)$, then $F(\oplus_{\lambda\in\Lambda} U_\lambda) \cong F(U^{(\kappa')})$ (Proposition 2.8). Let $c \colon F(\oplus_{\lambda\in\Lambda} U_\lambda) \to F(U^{(\kappa')})$ be an isomorphism. Let $\iota_X\colon X \to \oplus_{\lambda\in\Lambda}U_\lambda$ be the embedding and $\pi_X\colon \oplus_{\lambda\in\Lambda} U_\lambda \to X$ be the canonical projection. Then $E(c)F'(\iota_X)F'(\pi_X)E(c^{-1})$ is an idempotent endomorphism of $F'(U^{(\kappa')})$ in $\mathrm{add}(\mathcal{A})/\mathcal{K}$, hence an idempotent endomorphism of $F(U^{(\kappa')})$ in $\mathcal{A}/\mathcal{I}$. The functor G considered in the statement of Lemma 3.1 is a category equivalence between $\mathcal{A}/\mathcal{I}$ and Mod-$(\mathrm{End}_R(U)/I)$, so that

$$G(E(c)F'(\iota_X)F'(\pi_X)E(c^{-1}))$$

is an idempotent endomorphism of $GF(U^{(\kappa')})$. As idempotents split in the category of vector spaces over $\mathrm{End}_R(U)/I$, there exist a cardinal κ and morphisms $\alpha\colon GF(U^{(\kappa)}) \to GF(U^{(\kappa')})$ and $\beta\colon GF(U^{(\kappa')}) \to GF(U^{(\kappa)})$ with $\beta\alpha = 1_{GF(U^{(\kappa)})}$ and $\alpha\beta = G(E(c)F'(\iota_X)F'(\pi_X)E(c^{-1}))$. Via the equivalence G, there exist morphisms $\alpha'\colon F(U^{(\kappa)}) \to F(U^{(\kappa')})$ and $\beta'\colon F(U^{(\kappa')}) \to F(U^{(\kappa)})$ with $\beta'\alpha' = 1_{F(U^{(\kappa)})}$ and $\alpha'\beta' = E(c)F'(\iota_X)F'(\pi_X)E(c^{-1})$. Thus

$$E(\alpha')\colon F'(U^{(\kappa)}) \to F'(U^{(\kappa')}) \quad \text{and} \quad E(\beta')\colon F'(U^{(\kappa')}) \to F'(U^{(\kappa)})$$

are morphisms in $\mathrm{add}(\mathcal{A})/\mathcal{K}$ with the property that

$$E(\beta')E(\alpha') = 1_{F'(U^{(\kappa)})} \quad \text{and} \quad E(\alpha')E(\beta') = E(c)F'(\iota_X)F'(\pi_X)E(c^{-1}).$$

Then $F'(\pi_X)E(c^{-1})E(\alpha')$ and $E(\beta')E(c)F'(\iota_X)$ are mutually inverse isomorphisms between $F'(U^{(\kappa)})$ and $F'(X)$ in $\mathrm{add}(\mathcal{A})/\mathcal{K}$. Thus $F'(X) \cong F'(U^{(\kappa)}) = EF(U^{(\kappa)})$, as desired. □

Lemma 3.3. *Let $\mathcal{A}$ satisfy Property (*) and $\mathcal{K}$ be the ideal of* $\mathrm{add}(\mathcal{A})$ *associated to I. Let L be an ideal in* $\mathrm{End}_R(U)$ *not contained in I. Let $F'\colon \mathrm{add}(\mathcal{A}) \to \mathrm{add}(\mathcal{A})/\mathcal{K}$ be the canonical functor. Consider a countable family M_i, $i \in \mathbb{N}$, of objects in $\mathcal{A}$ such that, for every $i \in \mathbb{N}$, either $F'(M_i) = 0$ or $F'(M_i) \cong F'(U)$. Assume that $\oplus_{i\in\mathbb{N}}M_i = A_1 \oplus B_1 = A_2 \oplus B_2$. If $F'(A_1) \cong F'(A_2)$, then there exist homomorphisms $f, g\colon \oplus_{i\in\mathbb{N}} M_i \to \oplus_{i\in\mathbb{N}}M_i$ with the following properties:*

(i) *If $i, j \in \mathbb{N}$ and $\pi_j f\iota_i \neq 0$ or $\pi_j g\iota_i \neq 0$, then $F'(M_i) \neq 0$ and $F'(M_j) \neq 0$.*

(ii) *The homomorphisms $1_{A_1} - \pi_{A_1} g\iota_{A_2}\pi_{A_2} f\iota_{A_1}$ and $1_{A_2} - \pi_{A_2} f\iota_{A_1}\pi_{A_1} g\iota_{A_2}$ are in $\mathcal{K}$.*

(iii) *For every $i, j \in \mathbb{N}$, both $\pi_j f\iota_i$ and $\pi_i g\iota_j$ belong to the ideal of $\mathcal{A}$ generated by L.*

Proof. Let $\alpha\colon A_1 \to A_2$ and $\beta\colon A_2 \to A_1$ be homomorphisms with

$$F'(\alpha)\colon F'(A_1) \to F'(A_2) \qquad \text{and} \qquad F'(\beta)\colon F'(A_2) \to F'(A_1)$$

mutually inverse isomorphisms. Set

$$\alpha' := \begin{pmatrix} \alpha & 0 \\ 0 & 0 \end{pmatrix} : A_1 \oplus B_1 = \oplus_{i\in\mathbb{N}} M_i \to A_2 \oplus B_2 = \oplus_{i\in\mathbb{N}} M_i$$

and

$$\beta' := \begin{pmatrix} \beta & 0 \\ 0 & 0 \end{pmatrix} : A_2 \oplus B_2 = \oplus_{i\in\mathbb{N}} M_i \to A_1 \oplus B_1 = \oplus_{i\in\mathbb{N}} M_i.$$

Write α' and β' in matrix form with respect to the decomposition $\oplus_{i\in\mathbb{N}} M_i$ as

$$\alpha' = (\alpha'_{j,i})_{j,i} \quad \text{and} \quad \beta' = (\beta'_{i,j})_{i,j},$$

where $\alpha'_{j,i}\colon M_i \to M_j$ and $\beta'_{i,j}\colon M_j \to M_i$ for every i, j. There exist $\varphi \in I$ and $\psi \in L$ such that $\varphi + \psi = 1_U$. Thus $F'(\psi) = F'(1_U)$. For every $i \in \mathbb{N}$ with $F'(M_i) \cong F'(U)$, fix $\gamma_i\colon M_i \to U$ and $\delta_i\colon U \to M_i$ with $F'(\gamma_i)$ and $F'(\delta_i)$ mutually inverse isomorphisms.

Let $f := (f_{j,i})_{j,i}$, $g := (g_{i,j})_{i,j}$ be defined by $f_{j,i} = \alpha'_{j,i}\delta_i\psi\gamma_i$, $g_{i,j} = \delta_i\psi\gamma_i\beta'_{i,j}$ for every i, j with $F'(M_i) \cong F'(U)$ and $F'(M_j) \cong F'(U)$, and $f_{j,i} = 0, g_{i,j} = 0$ for every i, j with $F'(M_i) = 0$ or $F'(M_j) = 0$. Then $F'(f) = F'(\alpha')$ and $F'(g) = F'(\beta')$ by Remark 2.9. Then (i) holds trivially by the way f and g have been defined, and (ii) follows from the fact that $F'(\alpha)$ and $F'(\beta)$ are mutually inverse isomorphisms. Finally, (iii) follows from the fact that $\psi \in L$, so that, for every $i, j \in \mathbb{N}$, both $\pi_j f \iota_i = f_{j,i}$ and $\pi_i g \iota_j = g_{i,j}$ belong to the ideal of $\mathcal{A}$ generated by L. $\square$

4. Uniform modules

Recall that a module U is *uniform* if it is non-zero and the intersection of any two non-zero submodules of U is non-zero.

Fix a uniform right R-module U. For any R-module A, we defined in [4] the invariant $\text{m-dim}_U(A) := \sup\{\, k \in \mathbb{N}_0 \mid \text{there exist morphisms } f\colon U^k \to A \text{ and } g\colon A \to U^k \text{ with } gf \text{ a monomorphism}\,\}$. We now define the invariant m-dim_U on right R-module homomorphisms as well. If $\varphi\colon A \to B$ is a right R-module homomorphism, set $\text{m-dim}_U(\varphi) := \sup\{\, k \in \mathbb{N}_0 \mid \text{there exist morphisms } f\colon U^k \to A \text{ and } g\colon B \to U^k \text{ with } g\varphi f \text{ a monomorphism}\,\}$. It is either a non-negative integer or ∞. The following lemma collects some basic properties of m-dim_U. The proof is easy.

Lemma 4.1. *Let U be a uniform module and let $\varphi\colon A \to B$ be a homomorphism. Then*

(i) *For every $\alpha\colon X \to A$, $\text{m-dim}_U(\varphi\alpha) \le \text{m-dim}_U(\varphi)$. If α is a split epimorphism, then the equality holds.*

(ii) *For every* $\alpha\colon B \to X$, $\text{m-dim}_U(\alpha\varphi) \le \text{m-dim}_U(\varphi)$. *If* α *is a split monomorphism, then the equality holds.*

(iii) $\text{m-dim}_U(1_X) = \text{m-dim}_U(X)$ *for every module* X.

(iv) $\text{m-dim}_U(\varphi) = 0$ *if and only if* $g\varphi f$ *is not injective for every* $f\colon U \to A$ *and* $g\colon B \to U$.

(v) *If* φ *has an essential kernel, then* $\text{m-dim}_U(\varphi) = 0$.

(vi) *If* $\varphi'\colon A \to B$ *and* $\text{m-dim}_U(\varphi) = \text{m-dim}_U(\varphi') = 0$, *then also* $\text{m-dim}_U(\varphi + \varphi') = 0$.

Now let $\mathcal{A}$ be any additive full subcategory of Mod-R, let U be a uniform module, and consider the ideal $\mathcal{M}_U$ in the category $\mathcal{A}$ consisting of all morphisms φ in $\mathcal{A}$ with $\text{m-dim}_U(\varphi) = 0$. More precisely, for all objects A, B in $\mathcal{A}$, define $\mathcal{M}_U(A, B) := \{ \varphi \in \text{Hom}_R(A, B) \mid \text{m-dim}_U(\varphi) = 0 \}$. If I_U denotes the completely prime ideal of $\text{End}_R(U)$ consisting of all endomorphisms of U_R that are not injective, then $\mathcal{M}_U$ is the ideal of $\mathcal{A}$ associated to I_U (Lemma 4.1(iv)). Hence we can apply the results of the previous sections.

Construct the factor category $\mathcal{A}/\mathcal{M}_U$. Notice that in our previous paper [5], we had defined another ideal $\mathcal{M}'_U$ consisting of all morphisms φ in $\mathcal{A}$ that can be factored through some object C of $\mathcal{A}$ with $\text{m-dim}_U(C) = 0$. In this notation, we have that $\mathcal{M}'_U(A, B) \subseteq \mathcal{M}_U(A, B)$ for every A, B, so that there is a canonical full functor $\mathcal{A}/\mathcal{M}'_U \to \mathcal{A}/\mathcal{M}_U$.

Throughout this section, U is a fixed uniform module and the symbol F will always stand for the canonical functor $F\colon \mathcal{A} \to \mathcal{A}/\mathcal{M}_U$. Observe that the ideal $\mathcal{M}_U$ depends on the category $\mathcal{A}$, but this will cause no confusion.

Lemma 4.2. *Let* U_R *be a uniform module and let* I_U *be the ideal of* $\text{End}_R(U)$ *consisting of all the endomorphisms that are not injective. Then* U_R *is* I_U*-small.*

Proof. We will show that Condition (a) of Lemma 2.3 is satisfied. Let M_λ, $\lambda \in \Lambda$, be a family of modules and $\alpha\colon U \to \oplus_{\lambda\in\Lambda} M_\lambda$, $\beta\colon \oplus_{\lambda\in\Lambda} M_\lambda \to U$ be homomorphisms with $\beta\alpha \notin I_U$. That is, with $\beta\alpha$ injective. Fix a non-zero element $u \in U$. Let $\Lambda_0 := \{ \lambda \in \Lambda \mid \pi_\lambda\alpha(u) \neq 0\}$. The set Λ_0 is finite. Since the restriction of $\beta\alpha$ to uR is a monomorphism $\beta\alpha|_{uR}\colon uR \to U_R$, we have that $0 = \ker(\beta\alpha|_{uR}) \supseteq uR \cap \big(\bigcap_{\lambda\in\Lambda_0} \ker(\beta\iota_\lambda\pi_\lambda\alpha)\big)$. As U_R is uniform, there is a $\mu \in \Lambda_0$ with $\ker(\beta\iota_\mu\pi_\mu\alpha) = 0$. Hence $\beta\iota_\mu\pi_\mu\alpha \notin I_U$. □

From Lemma 2.1 applied to the ideal I_U, we know that $F(U)$ is a non-zero indecomposable object of $\mathcal{A}/\mathcal{M}_U$. The next Lemma and the following Proposition are more precise in this sense.

Lemma 4.3. *Let U be a uniform module and let A be an object of an additive full subcategory $\mathcal{A}$ of* Mod-R. *Then $F(A) = 0$ in $\mathcal{A}/\mathcal{M}_U$ if and only if* $\text{m-dim}_U(A) = 0$.

Proof. An object A is zero if and only if the identity 1_A is the zero morphism. It follows that $F(A) = 0$ in $\mathcal{A}/\mathcal{M}_U$ if and only if $1_A \in \mathcal{M}_U(A, A)$, that is, if and only if $\text{m-dim}_U(A) = 0$. □

Recall that if U, V are arbitrary modules over a ring R, we write $[U]_m = [V]_m$, and say that U and V are in the same *monogeny* class, if there exist a monomorpism $U \to V$ and a monomorpism $V \to U$.

Proposition 4.4. *Let $\mathcal{A}$ be an additive full subcategory of* Mod-R *and let V be a uniform right R-module in* $\text{Ob}(\mathcal{A})$. *Then:*

(a) *$F(V)$ is indecomposable in $\mathcal{A}/\mathcal{M}_U$ if $[U]_m = [V]_m$.*
(b) *$F(V) = 0$ in $\mathcal{A}/\mathcal{M}_U$ if $[U]_m \neq [V]_m$.*
(c) *Suppose that I_U is a maximal right ideal of* $\text{End}_R(U)$. *If $[U]_m = [V]_m$, then I_V is a maximal right ideal of* $\text{End}_R(V)$.

Proof. (a) Since $[U]_m = [V]_m$, $F(V)$ is not zero by Lemma 4.3. The endomorphism ring of $F(V)$ is isomorphic to $\text{End}_R(V)/I_V$. If $F(V) = A \oplus B$ with A and B non-zero, then there are non-zero orthogonal idempotents in $\text{End}(F(V))$, which is not possible as I_V is completely prime.

(b) follows from Lemma 4.3.

(c) Suppose that $[U]_m = [V]_m$ and that I_U is a maximal right ideal. Fix monomorphisms $\alpha\colon U \to V$ and $\beta\colon V \to U$. We will show that, for any monomorphism $f\colon V \to V$, the element $f + I_V$ has a right inverse in $\text{End}_R(V)/I_V$. Observe that $\beta f \alpha$ is a monomorphism. Therefore, by our assumption, there exists $g\colon U \to U$ such that $1_U - \beta f \alpha g$ is not a monomorphism. Then also $\alpha(1_U - \beta f \alpha g)\beta = \alpha\beta(1_V - f\alpha g\beta)$ is not a monomorphism, so $1_V - f\alpha g\beta$ is not a monomorphism. In other words, $\alpha g \beta + I_V$ is a right inverse for $f + I_V$ in $\text{End}_R(V)/I_V$. □

Corollary 4.5. *Let $\mathcal{A}$ be an additive full subcategory of* Mod-R *and let U, V, W be uniform right R-modules in* $\text{Ob}(\mathcal{A})$. *Suppose that I_U is a maximal right ideal of* $\text{End}_R(U)$. *If $f\colon V \to W$, then $F(f)$ is an isomorphism if and only if either $[V]_m \neq [U]_m$ and $[W]_m \neq [U]_m$, or $[V]_m = [U]_m = [W]_m$ and f is a monomorphism.*

Proof. Suppose $F(f)$ is an isomorphism. Then either $F(f) = 0$ and $F(V) = 0$, $F(W) = 0$ (i.e., $[U]_m \neq [V]_m$, $[U]_m \neq [W]_m$), or $F(f) \neq 0$, in which case f has to be a monomorphism and $[V]_m = [U]_m = [W]_m$.

Conversely, if $[V]_m \neq [U]_m$ and $[W]_m \neq [U]_m$, then $F(V) = 0$, $F(W) = 0$ and $F(f)$ is isomorphism. If $[U]_m = [V]_m = [W]_m$ and $f\colon V \to W$ is a monomorphism, then there exists a monomorphism $g\colon W \to V$. By Proposition 4.4(c), $F(g)F(f)$ is an automorphism of $F(V)$ and $F(f)F(g)$ is an automorphism of $F(W)$. Therefore $F(f)$ is an isomorphism. □

Let SUfm be the full subcategory of Mod-R whose objects are all right R-modules that are direct sums of (possibly infinitely many) uniform submodules. (Notice that in our previous paper [5] the symbol SUfm denoted the full subcategory of Mod-R whose objects are all *finite* direct sums of uniform modules.)

Proposition 4.6. *Let U be a uniform module such that I_U is a maximal right ideal. Let $\mathcal{M}_U$ be the ideal of* SUfm *consisting of all homomorphisms* SUfm *whose* m-dim$_U$ *is* 0 *and let $\mathcal{M}'_U$ be the ideal of* add(SUfm) *consisting of all homomorphisms in* add(SUfm) *whose* m-dim$_U$ *is* 0. *Then the categories* SUfm$/\mathcal{M}_U$ *and* add(SUfm)$/\mathcal{M}'_U$ *are both equivalent to* Mod-$(\mathrm{End}_R(U)/I_U)$.

Proof. Both categories contain U, are closed under arbitrary direct sums, the module U is I_U-small, the ideal $\mathcal{M}_U$ is the ideal in the category SUfm associated to I_U and the ideal $\mathcal{M}'_U$ is the ideal of add(SUfm) associated to I_U. It remains to check that if $F\colon \mathrm{SUfm} \to \mathrm{SUfm}/\mathcal{M}_U$ is the canonical functor, then every object M of SUfm has a decomposition $M = \oplus_{\lambda\in\Lambda} U_\lambda$, where for every $\lambda \in \Lambda$ the module U_λ is an object of $\mathcal{A}$ with either $F(U_\lambda) = 0$ or $F(U_\lambda) \cong F(U)$. Every object M of $\mathcal{A}$ is a direct sum of uniform modules, say $M = \oplus_{\lambda\in\Lambda} U_\lambda$. By Proposition 4.4(b), $F(U_\lambda) = 0$ if $[U_\lambda]_m \neq [U]_m$ and, by Corollary 4.5, $F(U_\lambda) \cong F(U)$ if $[U]_m = [U_\lambda]_m$. Hence it is possible to apply Lemma 3.1 and Proposition 3.2. □

Lemma 4.7. *Let U be a uniform module and assume that I_U is a maximal right ideal of* $\mathrm{End}_R(U)$. *Let k be a nonnegative integer. If $f\colon U^k \to U^k$ is a monomorphism, then $F(f)\colon F(U^k) \to F(U^k)$ is an isomorphism.*

Proof. Let $G\colon \mathrm{SUfm}/\mathcal{M}_U \to \text{Mod-}(\mathrm{End}_R(U)/I_U)$ be the category equivalence defined in Lemma 3.1. In order to prove that $F(f)$ is an automorphism, it suffices to show that $GF(f)$ is an automorphism of $GF(U^k)$. Since $GF(U^k)$ is a finite dimensional vector space, it suffices to show that $GF(f)\colon GF(U^k) = \mathrm{Hom}_{\mathrm{SUfm}/\mathcal{M}_U}(F(U), F(U^k)) \to GF(U^k) = \mathrm{Hom}_{\mathrm{SUfm}/\mathcal{M}_U}(F(U), F(U^k))$ is a monomorphism. Let $\alpha\colon U \to U^k$ be an R-module morphism and suppose that $F(\alpha)$ is in the kernel of $GF(f)$. That is, suppose $GF(f)(F(\alpha)) = 0$, equivalently $F(f\alpha) = 0$. Then $f\alpha \in \mathcal{M}_U(U, U^k)$. If $\pi_i\colon U^k \to U$, $i = 1, \dots, k$, is the canonical projection, it follows that $\pi_i f\alpha \in \mathcal{M}_U(U,U)$, i.e., the $\pi_i f\alpha$ are not monomorphisms. As U is uniform, it follows that $f\alpha$ is not a monomorphism, so that α is not a monomorphism. Hence $\ker\alpha$ is an essential submodule of U, therefore $\text{m-dim}_U(\alpha) = 0$ by Lemma 4.1(v). Thus $F(\alpha) = 0$ and $GF(f)$ is a monomorphism. □

Lemma 4.8. *Let U be a uniform module and assume that I_U is a maximal right ideal of* $\text{End}_R(U)$. *Let U_i, $i \in \mathbb{N}$, be a countable family of uniform modules and A a direct summand of $\oplus_{i\in\mathbb{N}} U_i$. Then $F(A) \cong F(U^{(\text{m-dim}_U(A))})$ in the category* $\text{add(SUfm)}/\mathcal{M}_U$.

In the statement of this Lemma, for $\text{m-dim}_U(A) = \infty$ we mean that $F(A) \cong F(U^{(\aleph_0)})$.

Proof. We have shown that every object of $\text{add(SUfm)}/\mathcal{M}_U$ is isomorphic to $F(U^{(\kappa)})$ for some cardinal κ (Proposition 4.6), so that $F(A) \cong F(U^{(\kappa)})$ for some $\kappa \leq \aleph_0$. Let $f\colon U^{(\kappa)} \to A$ and $g\colon A \to U^{(\kappa)}$ be such that $1_{U^{(\kappa)}} - gf \in \mathcal{M}_U$. Represent this homomorphism $1_{U^{(\kappa)}} - gf$ as a matrix. Then all its entries are not monomorphisms, therefore gf is a monomorphism. Consequently, $\kappa \leq \text{m-dim}_U(A)$. Suppose that $\kappa < \text{m-dim}_U(A)$, in particular, κ is finite. Then there exist $f'\colon U^{\kappa+1} \to A$ and $g'\colon A \to U^{\kappa+1}$ such that $g'f'$ is a monomorphism. Therefore $F(g'f')$ is an isomorphism according to Lemma 4.7. But then $G(F(U^{\kappa+1}))$ is a direct summand of $G(F(U^{\kappa}))$, which is not possible, because the first vector space has dimension $\kappa+1$ and the second one has dimension κ. □

Lemma 4.9. *Let U be a uniform module such that I_U is a maximal right ideal. Consider a countable family of uniform modules U_i, $i \in \mathbb{N}$, and assume $\oplus_{i\in\mathbb{N}} U_i = A_1 \oplus B_1 = A_2 \oplus B_2$. Suppose that* $\text{m-dim}_U(A_1) = \text{m-dim}_U(A_2)$. *Then there are homomorphisms $f, g\colon \oplus_{i\in\mathbb{N}} U_i \to \oplus_{i\in\mathbb{N}} U_i$ such that*

(i) *For every $i, j \in \mathbb{N}$ with either $\pi_j f\iota_i \neq 0$ or $\pi_j g\iota_i \neq 0$, one has $[U_i]_m = [U_j]_m = [U]_m$.*

(ii) $\text{m-dim}_U(1_{A_1} - \pi_{A_1} g\iota_{A_2}\pi_{A_2} f\iota_{A_1}) = 0$ *and* $\text{m-dim}_U(1_{A_2} - \pi_{A_2} f\iota_{A_1}\pi_{A_1} g\iota_{A_2}) = 0$.

If, moreover, U is a uniserial module and there exists a monomorphism $U \to U$ that is not an epimorphism, then f and g can be chosen in such a way to satisfy the following property (iii) *also:*

(iii) *For every $i, j \in \mathbb{N}$, the morphisms $\pi_j f\iota_i$ and $\pi_j g\iota_i$ are not epimorphisms.*

Proof. Let $F\colon \mathrm{add}(\mathrm{SUfm}) \to \mathrm{add}(\mathrm{SUfm})/\mathcal{M}_U$ be the canonical functor. Then $\text{m-dim}_U(A_1) = \text{m-dim}_U(A_2)$ implies $F(A_1) \cong F(A_2)$ by Lemma 4.8. We have already seen in the proof of Proposition 4.6 that Lemma 3.3 can be applied to $\mathcal{A} = \mathrm{SUfm}$ and the ideal $\mathcal{M}_U$ of SUfm associated to I_U, that is, given by the morphisms of m-dim_U zero. Then (i) and (ii) follow directly from Lemma 3.3. In order to prove (iii), observe that if K_U denotes the ideal of $\mathrm{End}_R(U)$ consisting of all the endomorphisms of U that are not onto, then $K_U \not\subseteq I_U$. Then, applying Lemma 3.3(iii), f and g can be chosen such that $\pi_j f \iota_i$ and $\pi_j g \iota_i$ are in the ideal of SUfm generated by K_U. Recall that the set of morphisms $U_i \to U_j$ in the ideal generated by K_U is $\mathcal{A}(U, U_j) K_U \mathcal{A}(U_i, U)$. Now U is uniserial and U_j is non-zero, so that pq is not an epimorphism for every $q \in K_U$ and every homomorphism $p\colon U \to U_j$. □

5. Couniform modules

Recall that a module U is *couniform* if it is non-zero and the sum of any two proper submodules of U is a proper submodule of U.

If $\varphi\colon A \to B$ is a homomorphism, and U is a couniform module, define $\text{e-dim}_U(\varphi)$ to be the supremum of the set $\{\, k \in \mathbb{N}_0 \mid$ there are homomorphisms $f\colon U^k \to A$ and $g\colon B \to U^k$ with $g\varphi f$ an epimorphism $\}$. For a module M, $\text{e-dim}_U(M)$ as defined in [4] is exactly $\text{e-dim}_U(1_M)$. The dual of Lemma 4.1 also holds:

Lemma 5.1. *Let $\varphi\colon A \to B$ be a homomorphism and let U be a couniform module. Then*

(i) *For every $\alpha\colon X \to A$, $\text{e-dim}_U(\varphi\alpha) \leq \text{e-dim}_U(\varphi)$. If α is a split epimorphism, then the equality holds.*

(ii) *For every $\alpha\colon B \to X$, $\text{e-dim}_U(\alpha\varphi) \leq \text{e-dim}_U(\varphi)$. If α is a split monomorphism, then the equality holds.*

(iii) $\text{e-dim}_U(\varphi) = 0$ *if and only if $g\varphi f$ is not an epimorphism for every $f\colon U \to A$ and $g\colon B \to U$.*

(iv) *If φ has a superfluous image, then* $\text{e-dim}_U(\varphi) = 0$.

(v) *If $\varphi'\colon A \to B$ and* $\text{e-dim}_U(\varphi) = \text{e-dim}_U(\varphi') = 0$*, then also* $\text{e-dim}_U(\varphi+\varphi') = 0$.

Let $\mathcal{A}$ be any additive subcategory of Mod-R, let U be a couniform module and consider the ideal $\mathcal{E}_U$ of the category $\mathcal{A}$ consisting of all morphisms φ with $\text{e-dim}_U(\varphi) = 0$. That is, for all objects $A, B \in \mathcal{A}$ define $\mathcal{E}_U(A,B) := \{\varphi \in \mathrm{Hom}_R(A,B) \mid \text{e-dim}_U(\varphi) = 0\}$. Construct the factor category $\mathcal{A}/\mathcal{E}_U$. In our previous paper [5] we defined another ideal $\mathcal{E}'_U$ consisting of all morphisms φ in $\mathcal{A}$

that can be factored through some object C of $\mathcal{A}$ with e-$\dim_U(C) = 0$. In this notation, we have $\mathcal{E}'_U \subseteq \mathcal{E}_U$, so that there is a canonical functor $\mathcal{A}/\mathcal{E}'_U \to \mathcal{A}/\mathcal{E}_U$.

Throughout this section, the symbol F always stands for the canonical functor $F\colon \mathcal{A} \to \mathcal{A}/\mathcal{E}_U$.

If U is a couniform module, the ring $\mathrm{End}_R(U)$ has a completely prime ideal consisting of all nonepimorphisms of $\mathrm{End}_R(U)$. We will denote this ideal by K_U. If U is a couniform module and $\mathcal{A}$ is a full subcategory of Mod-R, then $\mathcal{E}_U$ coincides with the ideal of $\mathcal{A}$ associated to K_U (Lemma 5.1(iii)).

Recall that an R-module N_R is said to be *quasi-small* [1, Definition 4.1] if for every family $\{ M_i \mid i \in I \}$ of R-modules such that N_R is isomorphic to a direct summand of $\oplus_{i \in I} M_i$, there is a finite subset $F \subseteq I$ such that N_R is isomorphic to a direct summand of $\oplus_{i \in F} M_i$. A uniserial module U is quasi-small if and only if for every set Λ and every homomorphism $F\colon U \to U^{(\Lambda)}$ such that the composite mapping $\Sigma \circ F\colon U \to U$ is the identity morphism $1_U\colon U \to U$, there exists $\mu \in \Lambda$ with $\pi_\mu F$ an epimorphism [1, Lemma 4.4].

Recall that if U, V are arbitrary modules, we write $[U]_e = [V]_e$, and say that U and V are in the same *epigeny* class, if there exist an epimorphism $U \to V$ and an epimorpism $V \to U$.

Remark 5.2. *In this case, it is not necessarily true that if X is a direct sum of couniform modules having their epigeny classes different from $[U]_e$, then* e-$\dim_U(X) = 0$. *However, this is true if U is a quasi-small uniserial module of type 2.* To see this, recall that, over a suitable ring R, there exists a uniserial module U that is not quasi-small and a uniserial module V non-isomorphic to U such that $V^{(\aleph_0)} \cong U \oplus V^{(\aleph_0)}$ [12, Proposition 8.1]. Then necessarily $[U]_m = [V]_m$ [2, Theorem 9.12], so that $[U]_e \neq [V]_e$. Thus $X = V^{(\aleph_0)}$ is the required example with e-$\dim_U(X) \neq 0$. In fact, we have that $V^{(\aleph_0)} \cong U^k \oplus V^{(\aleph_0)}$ for every k, so that e-$\dim_U(X) = \infty$. The second part of this remark, that is, the part concerning quasi-small uniserial modules of type 2, will follow from Lemma 5.3(ii). Hence we cannot apply Proposition 2.8. The reason is that U is not K_U-small in general.

Let U be a couniform module. We say that U is *epi-small* if U is K_U-small. That is, a couniform module U is epi-small, if for every family M_λ, $\lambda \in \Lambda$, of modules and homomorphisms $f\colon U \to \oplus_{\lambda \in \Lambda} M_\lambda$ and $g\colon \oplus_{\lambda \in \Lambda} M_\lambda \to U$ such that gf is an epimorphism, there exists $\mu \in \Lambda$ with $g\iota_\mu \pi_\mu f$ is an epimorphism.

For example, local modules, that is, the modules with a greatest proper submodule, are epi-small couniform modules.

Lemma 5.3. *Let U be a uniserial module.*

(i) *If U is epi-small, then U is quasi-small.*

(ii) *If U is quasi-small and there exists a monomorphism $U \to U$ which is not an epimorphism, then U is epi-small.*

Proof. (i) Let U be a uniserial module that is not quasi-small. Then, by [2, Proposition 9.30(a)], there exists a countable family $A_n, n \geq 1$, of uniserial R-modules such that $U \oplus (\oplus_{n\geq 1} A_n) \cong \oplus_{n\geq 1} A_n$ and $[A_n]_e \neq [U]_e$ for every $n \geq 1$. It follows that there exists morphisms $f\colon U \to \oplus_{n\geq 1} A_n$ and $g\colon \oplus_{n\geq 1} A_n \to U$ with $gf = 1_U$. But all composed morphisms $U \to A_n \to U$ are not epimorphisms because $[A_n]_e \neq [U]_e$. Hence U is not epi-small.

(ii) Let U be a uniserial module that is not epi-small and has a monomorphism $h\colon U \to U$ which is not an epimorphism. Then there exists a family M_λ, $\lambda \in \Lambda$, of modules and two homomorphisms $f\colon U \to \oplus_{\lambda\in\Lambda} M_\lambda$ and $g\colon \oplus_{\lambda\in\Lambda} M_\lambda \to U$ with gf an epimorphism, but $g\iota_\lambda\pi_\lambda f$ not an epimorphism for every $\lambda \in \Lambda$. We shall distinguish two cases.

First case: gf is a monomorphism. In this case, gf is an automorphism of U. The existence of the family of endomorphisms $(gf)^{-1} g\iota_\lambda\pi_\lambda f\colon U \to U$, $\lambda \in \Lambda$, which are not epimorphisms, but whose sum is 1_U, shows that U is not quasi-small [1, Lemma 4.4].

Second case: gf is not a monomorphism. Then $gf + h$ is an automorphism of U. The existence of the family consisting of the endomorphism $(gf+h)^{-1}h$ and all the endomorphisms $(gf+h)^{-1} g\iota_\lambda\pi_\lambda f$ of U, $\lambda \in \Lambda$, (they are not epimorphisms, but their sum is 1_U) shows that U is not quasi-small by [1, Lemma 4.4] again. □

We do not know what happens for the uniserial modules U for which every monomorphism $U \to U$ is an epimorphism, that is, the case in which $\mathrm{End}_R(U)$ is a local ring in which the maximal ideal consists of all morphisms $U \to U$ that are not monomorphisms. Such a module U is necessarily quasi-small [2, Example 9.29], but we do not know if it must be epi-small.

The same proofs of Lemma 4.3, Proposition 4.4 and Corollary 4.5 give:

Lemma 5.4. *Let U be a couniform module and let A be an object of an additive full subcategory $\mathcal{A}$ of* Mod-R. *Then $F(A) = 0$ in $\mathcal{A}/\mathcal{E}_U$ if and only if* $\text{e-dim}_U(A) = 0$.

Proposition 5.5. *Let $\mathcal{A}$ be an additive full subcategory of* Mod-R *and let V be a couniform right R-module in* $\mathrm{Ob}(\mathcal{A})$. *Then:*

(a) *$F(V)$ is indecomposable in $\mathcal{A}/\mathcal{E}_U$ if $[V]_e = [U]_e$;*

(b) *$F(V) = 0$ in $\mathcal{A}/\mathcal{M}_U$ if $[V]_e \neq [U]_e$.*

(c) *Suppose that* K_U *is a maximal right ideal of* $\mathrm{End}_R(U)$. *If* $[V]_e = [U]_e$, *then* K_V *is a maximal right ideal of* $\mathrm{End}_R(V)$.

Corollary 5.6. *Let* $\mathcal{A}$ *be an additive full subcategory of* Mod-R *and let* U, V, W *be couniform right* R*-modules in* $\mathrm{Ob}(\mathcal{A})$. *Suppose that* K_U *is a maximal right ideal of* $\mathrm{End}_R(U)$. *If* $f\colon V \to W$, *then* $F(f)$ *is an isomorphism if and only if either* $[V]_e \neq [U]_e$ *and* $[W]_e \neq [U]_e$, *or* $[V]_e = [U]_e = [W]_e$ *and* f *is an epimorphism.*

Let SCfm be the full subcategory of Mod-R whose objects are all right R-modules that are direct sums of (possibly infinitely many) couniform submodules.

Proposition 5.7. *Let* U *be an epi-small couniform module such that* K_U *is a maximal right ideal. Let* $\mathcal{E}_U$ *be the ideal of* SCfm *consisting of all homomorphisms in* SCfm *whose* e-$\dim_U$ *is* 0 *and let* $\mathcal{E}'_U$ *be the ideal of* add(SCfm) *consisting of all homomorphisms in* add(SCfm) *whose* e-$\dim_U$ *is* 0. *Then the categories* $\mathrm{SCfm}/\mathcal{E}_U$ *and* $\mathrm{add(SCfm)}/\mathcal{E}'_U$ *are both equivalent to* Mod-$(\mathrm{End}_R(U)/K_U)$.

Proof. Both categories contain U, are closed under arbitrary direct sums and the module U is K_U-small. By Lemma 5.1(iii), $\mathcal{E}_U$ is the ideal in the category SCfm associated to K_U and $\mathcal{E}'_U$ is the ideal of add(SCfm) associated to K_U. It remains to prove that if $F\colon \mathrm{SCfm} \to \mathrm{SCfm}/\mathcal{E}_U$ is the canonical functor, then any object M of SCfm has a decomposition $M = \oplus_{\lambda\in\Lambda} U_\lambda$ such that the U_λ's are objects of $\mathcal{A}$ and $F(U_\lambda) = 0$ or $F(U_\lambda) \cong F(U)$ for every $\lambda \in \Lambda$. Any object M of $\mathcal{A}$ is a direct sum of couniform modules, say $M = \oplus_{\lambda\in\Lambda} U_\lambda$, where U_λ is couniform and hence an object of SCfm for any $\lambda \in \Lambda$. By Proposition 5.5, $F(U_\lambda) = 0$ if $[U_\lambda]_e \neq [U]_e$ and, by Corollary 5.6, $F(U_\lambda) \cong F(U)$ if $[U]_e = [U_\lambda]_e$. Now it is possible to apply Lemma 3.1 and Proposition 3.2. □

Lemma 5.8. *Let* U *be a couniform module and assume that* K_U *is a maximal right ideal of* $\mathrm{End}_R(U)$. *Let* k *be a nonnegative integer. If* $f\colon U^k \to U^k$ *is an epimorphism, then* $F(f)\colon F(U^k) \to F(U^k)$ *is an isomorphism in* $\mathrm{SCfm}/\mathcal{E}_U$.

Proof. We argue as in the proof of Lemma 4.7. Let G be the category equivalence defined in Lemma 3.1 with $\mathcal{A} = \mathrm{SCfm}$ and $I = K_U$. In order to prove that $F(f)$ is an automorphism, it suffices to show that $GF(f)$ is an automorphism of $GF(U^k)$. Since $GF(U^k)$ is a finite dimensional vector space, it suffices to show that $GF(f)\colon GF(U^k) = \mathrm{Hom}_{\mathrm{SCfm}/\mathcal{E}_U}(F(U), F(U^k)) \to GF(U^k) = \mathrm{Hom}_{\mathrm{SCfm}/\mathcal{E}_U}(F(U), F(U^k))$ is a monomorphism. Let $\alpha\colon U \to U^k$ be an R-module morphism and suppose that $F(\alpha)$ is in the kernel of $GF(f)$. That is, suppose $GF(f)(F(\alpha)) = 0$, i.e., $F(f\alpha) = 0$. Then $f\alpha \in \mathcal{E}_U(U, U^k)$. If $\pi_i\colon U^k \to U$,

$i = 1, \dots, k$, is the canonical projection, it follows that $\pi_i f\alpha \in \mathcal{E}_U(U,U)$, i.e., the $\pi_i f\alpha$ are not epimorphisms. Thus the images of the $\pi_i f\alpha$'s are superfluous submodules of U, so that the image of $f\alpha$ is a superfluous submodule of U^k. We claim that the image of α is a superfluous submodule of U^k. To prove the claim, take a submodule A of U^k with $A+\alpha(U) = U^k$. Applying the epimorphism f, we get that $f(A) + f\alpha(U) = U^k$. Thus $f(A) = U^k$, from which $A + \ker f = U^k$. Applying [2, Proposition 2.42(d)], we know that $\ker f$ is a superfluous submodule of U^k, so that $A = U^k$. This proves the claim. From Lemma 5.1, we have that $\text{e-dim}_U(\alpha) = 0$. Thus $F(\alpha) = 0$ and $GF(f)$ is a monomorphism. □

Lemma 5.9. *Let U be an epi-small couniform module and assume that K_U is a maximal right ideal of $\text{End}_R(U)$. Let U_i, $i \in \mathbb{N}$, be a countable family of couniform modules. For any direct summand A of $\oplus_{i\in\mathbb{N}} U_i$, we have $F(A) \cong F(U^{(\text{e-dim}_U(A))})$ in the category $\text{add(SCfm)}/\mathcal{E}_U$.*

Again, when $\text{e-dim}_U(A) = \infty$ we mean that $F(A) \cong F(U^{(\aleph_0)})$. The proof of Lemma 5.9 is the same as the proof of Lemma 4.8.

Lemma 5.10. *Let U be an epi-small couniform module such that K_U is a maximal right ideal. Consider a countable family of couniform modules U_i, $i \in \mathbb{N}$, and assume $\oplus_{i\in\mathbb{N}} U_i = A_1 \oplus B_1 = A_2 \oplus B_2$. Suppose that $\text{e-dim}_U(A_1) = \text{e-dim}_U(A_2)$. Then there are homomorphisms $f, g\colon \oplus_{i\in\mathbb{N}} U_i \to \oplus_{i\in\mathbb{N}} U_i$ such that:*

(i) *For every $i, j \in \mathbb{N}$ with either $\pi_j f \iota_i \neq 0$ or $\pi_j g \iota_i \neq 0$, one has $[U_i]_e = [U_j]_e = [U]_e$.*

(ii) $\text{e-dim}_U(1_{A_1} - \pi_{A_1} g \iota_{A_2} \pi_{A_2} f \iota_{A_1}) = 0$ *and* $\text{e-dim}_U(1_{A_2} - \pi_{A_2} f \iota_{A_1} \pi_{A_1} g \iota_{A_2}) = 0$.

If, moreover, U is a uniserial module and there exists an epimorphism $U \to U$ that is not a monomorphism, then f and g can be chosen in such a way to satisfy the following property (iii) *also:*

(iii) *For every $i, j \in \mathbb{N}$, the morphisms $\pi_j f \iota_i$ and $\pi_j g \iota_i$ are not monomorphisms.*

The proof is the dual of the proof of Lemma 4.9.

6. Local endomorphism ring

Throughout this section, U is a module with a local endomorphism ring and J_U is the unique maximal (right) ideal of $\text{End}_R(U)$, which consists of all nonisomorphisms of $\text{End}_R(U)$. Observe that U is J_U-small because whenever we have homomorphisms $f\colon U \to \oplus_{\lambda\in\Lambda} M_\lambda$ and $g\colon \oplus_{\lambda\in\Lambda} M_\lambda \to U$ with gf an automorphism of U, then there exists $\mu \in \Lambda$ with $g\iota_\mu \pi_\mu f$ an isomorphism. (In order to see this, fix any non-zero $u \in U$ and consider the finite set $\Lambda_0 := \{ \lambda \in \Lambda \mid \pi_\lambda f(u) \neq 0\}$. If $\iota_{\Lambda_0} \colon \oplus_{\lambda\in\Lambda_0} M_\lambda \to \oplus_{\lambda\in\Lambda} M_\lambda$ is the embedding and $\pi_{\Lambda_0} \colon \oplus_{\lambda\in\Lambda} M_\lambda \to \oplus_{\lambda\in\Lambda_0} M_\lambda$ is the canonical projection, then $g\iota_{\Lambda_0}\pi_{\Lambda_0} f$ is an isomorphism, so there exists $\mu \in \Lambda_0$ with $g\iota_\mu\pi_\mu f$ an isomorphism.)

We can define a dimension i-dim_U as follows. For $\alpha \colon A \to B$, define $\text{i-dim}_U(\alpha)$ to be the supremum of $\{ k \in \mathbb{N}_0 \mid \text{there are } f \colon U^k \to A \text{ and } g \colon B \to U^k \text{ with } g\alpha f \text{ an isomorphism} \}$. Observe that, for $n \in \mathbb{N}$, $\text{i-dim}_U(1_A) \geq n$ if and only if A contains a direct summand isomorphic to U^n. This is not true for $\text{i-dim}_U(1_A) = \infty$, that is, it is not necessarily true that $\text{i-dim}_U(1_A) = \infty$ if and only if A contains a direct summand isomorphic to $U^{(\aleph_0)}$. That is, there exist modules U and A over a suitable ring R with $\text{End}_R(U)$ local, A with a direct summand isomorphic to U^n for every $n \geq 1$, but A without direct summands isomorphic to $U^{(\aleph_0)}$. For instance, let V_k be a vector space of infinite dimension over a commutative field k and $R = \text{End}(V_k)$, so that ${}_RV$ is a simple left R-module. It is easy to see that for every subspace W of V_k, $S_W := \{ \varphi \in R \mid \varphi(W) = 0 \}$ is a left ideal of R. If W is a vector subspace of V_k of finite codimension n, then $S_W \cong {}_RV^n$ as a left R-module. If $V_k = W \oplus W'$, then ${}_RR = S_W \oplus S_{W'}$. It follows that ${}_RR$ has direct summands that are isomorphic to ${}_RV^n$ for every $n \geq 1$. But ${}_RR$ does not have direct summands that are direct sums of infinitely many non-zero modules, because it is finitely generated. Notice that $\text{End}_R(V)$ is local.

Lemma 4.1 can be adapted to i-dim as well:

Lemma 6.1. *Let $\varphi \colon A \to B$ be a homomorphism and let U be a module with a local endomorphism ring. Then*

(i) *For every $\alpha \colon X \to A$,* $\text{i-dim}_U(\varphi\alpha) \leq \text{i-dim}_U(\varphi)$. *If α is a split epimorphism, then the equality holds.*

(ii) *For every $\alpha \colon B \to X$,* $\text{i-dim}_U(\alpha\varphi) \leq \text{i-dim}_U(\varphi)$. *If α is a split monomorphism, then the equality holds.*

(iii) *If φ has an essential kernel or a superfuous image, then* $\text{i-dim}_U(\varphi) = 0$.

(iv) *If $\varphi' \colon A \to B$ and* $\text{i-dim}_U(\varphi) = \text{i-dim}_U(\varphi') = 0$, *then also* $\text{i-dim}_U(\varphi + \varphi') = 0$.

Now let $\mathcal{A}$ be any additive full subcategory of Mod-R, let U be a module with local endomorphism ring, and consider the ideal $\mathcal{I}_U$ in the category $\mathcal{A}$ consisting of all morphisms φ in $\mathcal{A}$ with $\text{i-dim}_U(\varphi) = 0$. Thus, for all objects A, B in $\mathcal{A}$, one has $\mathcal{I}_U(A, B) = \{ \varphi \in \text{Hom}_R(A, B) \mid \text{i-dim}_U(\varphi) = 0 \}$. Construct the factor category $\mathcal{A}/\mathcal{I}_U$.

Throughout this section the symbol F always stands for the canonical functor $F\colon \mathcal{A} \to \mathcal{A}/\mathcal{I}_U$.

Notice that $\mathcal{I}_U$ is the ideal of $\mathcal{A}$ associated to J_U, the Jacobson radical of the local ring $\mathrm{End}_R(U)$.

We can study the category $\mathcal{A}/\mathcal{I}_U$ and give in Corollary 6.5 a proof of the Krull-Schmidt theorem similar to that given in [7].

Proposition 6.2. *Let $\mathcal{A}$ be an additive full subcategory of* Mod-R *and let U, V be right R-modules in* $\mathrm{Ob}(\mathcal{A})$ *with local endomorphism rings. Then $F(U)$ is indecomposable, and $F(V) \neq 0$ if and only if $V \cong U$.*

Proof. If $U \ncong V$, then $\text{i-dim}_U(1_V) = 0$, and $F(V) = 0$. The endomorphism ring of $F(U)$ is a division ring, hence its idempotents are the trivial ones only, so that $F(U)$ is indecomposable as a biproduct in the additive category $\mathcal{A}/\mathcal{I}_U$. □

The next Lemma describes the ideal $\mathcal{I}_U$ in the case in which $\mathcal{A}$ is the full subcategory SLer of Mod-R whose objects are modules that are direct sums of modules with local endomorphism rings. (This is exactly the full subcategory of Mod-R considered in the Krull-Schmidt-Azumaya Theorem.)

Lemma 6.3. *Let U_λ, $\lambda \in \Lambda$, V_μ, $\mu \in M$, and U be modules with local endomorphism rings. Then $f\colon \oplus_{\lambda\in\Lambda} U_\lambda \to \oplus_{\mu\in M} V_\mu$ is in $\mathcal{I}_U$ if and only if $\pi_\mu f \iota_\lambda$ is not an isomorphism for every λ, μ such that $U_\lambda \cong U \cong U_\mu$.*

Proof. Remark 2.9. □

Proposition 6.4. *Let U be a module with local endomorphism ring and let J_U be the maximal right ideal of* $\mathrm{End}_R(U)$. *Let $\mathcal{I}_U$ be the ideal of* SLer *consisting of all homomorphisms in* SLer *whose* i-dim_U *is* 0 *and let $\mathcal{I}'_U$ be the ideal of* add(SLer) *consisting of all homomorphisms in* add(SLer) *whose* i-dim_U *is* 0. *Then the categories* SLer$/\mathcal{I}_U$ *and* add(SLer)$/\mathcal{I}'_U$ *are both equivalent to* Mod-$(\mathrm{End}_R(U)/J_U)$.

Proof. Both categories contain U, are closed under arbitrary direct sums and the module U is J_U-small. We have already noticed that the ideal $\mathcal{I}_U$ is the ideal in the category SLer associated to J_U, and the ideal $\mathcal{I}'_U$ is the ideal of add(SUfm) associated to J_U. It remains to check that if $F\colon \text{SLer} \to \text{SLer}/\mathcal{I}_U$ is the canonical functor, then any object M of SLer has a decomposition $M = \oplus_{\lambda\in\Lambda} U_\lambda$, such that U_λ's are objects of $\mathcal{A}$, and $F(U_\lambda) = 0$ or $F(U_\lambda) \cong F(U)$ for every $\lambda \in \Lambda$. Any object M of SLer is a sum of modules with local endomorphism ring. Apply Proposition 6.2, Lemma 3.1 and Proposition 3.2. □

As a corollary, we get another proof of the Krull-Schmidt-Azumaya Theorem:

Corollary 6.5. *Let U_i, $i \in I$, V_j, $j \in J$ be families of modules with local endomorphism ring. Then $\oplus_{i\in I} U_i \cong \oplus_{j\in J} V_j$ if and only if there exists a bijection $\sigma\colon I \to J$ such that $U_i \cong V_{\sigma(i)}$ for every $i \in I$.*

Proof. Suppose $\oplus_{i\in I} U_i \cong \oplus_{j\in J} V_j$. Obviously, it is enough to prove that for any module U with local endomorphism ring, if $\kappa = |\{i \in I \mid U_i \cong U\}|$ and $\kappa' = |\{j \in J \mid V_j \cong U\}|$, then $\kappa = \kappa'$. Consider the canonical functor $F\colon \mathrm{SLer} \to \mathrm{SLer}/\mathcal{I}_U$. Since $\oplus_{i\in I} U_i \cong \oplus_{i\in J} V_j$, then $F(\oplus_{i\in I} U_i) \cong F(\oplus_{j\in J} V_j)$. Using Lemma 2.5 and Proposition 6.2, we see that $F(\oplus_{i\in I} U_i)$ is a coproduct of κ objects isomorphic to $F(U)$ and $F(\oplus_{j\in J} V_j)$ is the coproduct of κ' objects isomorphic to $F(U)$. Now apply the equivalence G of Lemma 3.1 to see that $GF(U)^{(\kappa)} \cong GF(U)^{(\kappa')}$ in Mod-$(\mathrm{End}_R(U)/J_U)$. But $GF(U)$ is the vector space of dimension 1, so $\kappa = \kappa'$. □

Lemma 6.6. *Let U be a module with a local endomorphism ring. Consider a countable family U_i, $i \in \mathbb{N}$, of modules with local endomorphism rings. For any direct summand A of $\oplus_{i\in\mathbb{N}} U_i$ we have $F(A) \cong F(U^{(\text{i-dim}_U(A))})$ in the category* $\mathrm{add}(\mathrm{SLer})/\mathcal{I}_U$.

Again, we mean that $F(A) \cong F(U^{(\aleph_0)})$ for $\text{i-dim}_U(A) = \infty$.

Proof. This follows easily from the fact that if there are $f\colon U^k \to A$ and $g\colon A \to U^k$ such that gf is an isomorphism, then A contains a direct summand isomorphic to U^k. Moreover, $F(A)$ must be a direct summand of $F(U)^{(\aleph_0)}$. □

Lemma 6.7. *Let U be a module with a local endomorphism ring. Consider a countable family of modules U_i, $i \in \mathbb{N}$, with local endomorphism rings, and assume that $\oplus_{i\in\mathbb{N}} U_i = A_1 \oplus B_1 = A_2 \oplus B_2$. Suppose that $\text{i-dim}_U(A_1) = \text{i-dim}_U(A_2)$. Then there are homomorphisms $f, g\colon \oplus_{i\in\mathbb{N}} U_i \to \oplus_{i\in\mathbb{N}} U_i$ with the following two properties:*

(i) *For every $i, j \in \mathbb{N}$ with either $\pi_j f \iota_i \neq 0$ or $\pi_j g \iota_i \neq 0$, one has $U_i \cong U_j \cong U$.*

(ii) $\text{i-dim}_U(1_{A_1} - \pi_{A_1} g \iota_{A_2} \pi_{A_2} f \iota_{A_1}) = 0$ *and* $\text{i-dim}_U(1_{A_2} - \pi_{A_2} f \iota_{A_1} \pi_{A_1} g \iota_{A_2}) = 0$.

Proof. Observe that $\text{i-dim}_U(A_1) = \text{i-dim}_U(A_2)$ implies $F(A_1) \cong F(A_2)$, where $F\colon \mathrm{add}(\mathrm{SLer}) \to \mathrm{add}(\mathrm{SLer})/\mathcal{I}_U$ (Lemma 6.6). We have already checked in the proof of Proposition 6.4 that Lemma 3.3 can be applied to $\mathcal{A} = \mathrm{SLer}$ and the ideal $\mathcal{I}$ of SLer associated to J_U (that is, given by the morphisms of i-dim_U zero). Hence we can conclude by Lemma 3.3. □

Propositions 6.9 and 6.10 will show that it is not necessary to treat i-dim_U separately when the module U with local endomorphism ring is uniserial.

Lemma 6.8. *Let U be a uniform module such that every monomorphism $f\colon U \to U$ is an isomorphism. Then, for every $k \geq 0$, every monomorphism $\varphi\colon U^k \to U^k$ is an isomorphism.*

Proof. Induction on k, the case $k = 1$ being trivial. Let $\varphi\colon U^k \to U^k$ be a monomorphism. Let $\iota\colon U \to U^k$ be the inclusion into the first component. Since U is uniform and $\varphi\iota\colon U \to U^k$ is injective, there exists an index $i = 1, \dots, k$ such that $\pi_i\varphi\iota\colon U \to U$ is a monomorphism (here $\pi_i\colon U^k \to U$ denotes the i-th canonical projection). Thus $\pi_i\varphi\iota$ is an automorphism of U, so that $U^k = \varphi\iota(U) \oplus \ker \pi_i$, with $\ker \pi_i \cong U^{k-1}$. We have a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \to & U & \xrightarrow{\iota} & U^k & \to & U^k/\iota(U) & \to & 0 \\ & & {\scriptstyle\varphi\iota}\downarrow & & \downarrow{\scriptstyle\varphi} & & \downarrow & & \\ 0 & \to & \varphi\iota(U) & \hookrightarrow & U^k & \to & U^k/\varphi\iota(U) & \to & 0, \end{array}$$

in which all vertical arrows are monomorphisms. Since $U^k/\iota(U) \cong U^{k-1}$ and $U^k/\varphi\iota(U) \cong \ker \pi_i \cong U^{k-1}$, the vertical arrow on the right is an isomorphism by the inductive hypothesis. The vertical arrow on the left $\varphi\iota\colon U \to \varphi\iota(U)$ is an isomorphism by construction, so that the vertical arrow in the middle $\varphi\colon U^k \to U^k$ is an isomorphism as well, as desired. $\square$

Proposition 6.9. *Let U be a uniform module such that every monomorphism $f\colon U \to U$ is an isomorphism. Then, for every $\alpha\colon A \to B$, we have* $\text{m-dim}_U(\alpha) = \text{i-dim}_U(\alpha)$.

Proof. Obviously, $\text{i-dim}_U(\alpha) \leq \text{m-dim}_U(\alpha)$ for any α. For the opposite inequality, suppose that there are $f\colon U^k \to A$ and $g\colon B \to U^k$ with $g\alpha f$ a monomorphism. By Lemma 6.8, $g\alpha f$ is an isomorphism. Hence $\text{i-dim}_U(\alpha) = \text{m-dim}_U(\alpha)$. $\square$

Of course, there is a dual version also.

Proposition 6.10. *Let U be a couniform module such that every epimorphism $f\colon U \to U$ is an isomorphism. Then, for every $\alpha\colon A \to B$, we have* $\text{e-dim}_U(\alpha) = \text{i-dim}_U(\alpha)$.

7. The Weak Krull-Schmidt Theorem

A right R-module N_R is *small* if for every family $\{ M_i \mid i \in I \}$ of right R-modules and every homomorphism $\varphi\colon N_R \to \oplus_{i\in I} M_i$, there is a finite subset $F \subseteq I$ such that $\pi_j\varphi = 0$ for every $j \in I \setminus F$. Here the $\pi_j\colon \oplus_{i\in I} M_i \to M_j$ are the canonical projections. Clearly, every small module is quasi-small.

Proposition 7.1. *Let U_i, $i \in \mathbb{N}$, be a countable family of non-zero uniserial modules. Let $M = \oplus_{i\in\mathbb{N}} U_i$, and let $M = A \oplus B = A' \oplus B'$ be two decompositions of M with $B \cong B'$. Suppose that there exist two morphisms $f\colon A \to A'$ and $g\colon A' \to A$ such that:*

(1) $\text{i-dim}_{U_i}(1_A - gf) = \text{i-dim}_{U_i}(1_{A'} - fg) = 0$ *for every index $i \in \mathbb{N}$ with U_i of type 1, and*
(2) $\text{m-dim}_{U_i}(1_A - gf) = \text{e-dim}_{U_i}(1_A - gf) = \text{m-dim}_{U_i}(1_{A'} - fg) = \text{m-dim}_{U_i}(1_{A'} - fg) = 0$ *for every index $i \in \mathbb{N}$ with U_i of type 2.*

Then $A \cong A'$.

Proof. We claim that if the hypotheses of the statement are satisfied and we fix a submodule X of A with X a small module, then there exists a morphism $g'\colon A' \to A$ satisfying the same hypotheses as g and with the further property that $X \subseteq \ker(1_A - g'f)$.

The proof of the claim is similar to [11, Lemma 2.1]. Let $\alpha\colon B \to B'$ and $\alpha'\colon B' \to B$ be mutually inverse isomorphisms. Let $\varphi, \psi \in \text{End}_R(M)$ be defined by

$$\varphi = \begin{pmatrix} f & 0 \\ 0 & \alpha \end{pmatrix} : A \oplus B \to A' \oplus B', \quad \psi = \begin{pmatrix} g & 0 \\ 0 & \alpha' \end{pmatrix} : A' \oplus B' \to A \oplus B.$$

Observe that $\text{i-dim}_{U_i}(1_M - \psi\varphi) = \text{i-dim}_{U_i}(1_M - \varphi\psi) = 0$ if U_i is of type 1, and $\text{m-dim}_{U_i}(1_M - \psi\varphi) = \text{e-dim}_{U_i}(1_M - \psi\varphi) = \text{m-dim}_{U_i}(1_M - \varphi\psi) = \text{e-dim}_{U_i}(1_M - \varphi\psi) = 0$ if U_i is of type 2. (In order to check this, observe that $1_M - \psi\varphi = (\iota_A\pi_A + \iota_B\pi_B)(1_M - \psi\varphi)(\iota_A\pi_A + \iota_B\pi_B) = \iota_A(1_A - gf)\pi_A$.) Since X is small and $\psi\varphi(X)$ is also small, there exists $k \in \mathbb{N}$ such that $X + \psi\varphi(X) \subseteq \oplus_{j=1}^k U_j$. Now consider $\iota\colon \oplus_{j=1}^k U_j \to M$ and $\pi\colon M \to \oplus_{j=1}^k U_j$, the embedding and the canonical projection. Set $h := \pi(1_M - \psi\varphi)\iota$. Then $h \in J(\text{End}_R(\oplus_{j=1}^k U_j))$, because (proof of [3, Theorem 4.4]), for every $1 \leq j, l \leq k$, if U_j is of type 1 and $U_j \cong U_l$, then $\pi_l h \iota_j$ is not an isomorphism; if U_j is of type 2 and $[U_j]_m = [U_l]_m$, then the homomorphism $\pi_l h \iota_j$ is not a monomorphism; and if U_j is of type 2 and $[U_j]_e = [U_l]_e$, then the homomorphism $\pi_l h \iota_j$ is not an epimorphism. Let h' be the inverse of $1 - h \in \text{End}_R(\oplus_{j=1}^k U_j)$ and let τ be the automorphism of M given by $\tau := \begin{pmatrix} h' & 0 \\ 0 & 1_{\oplus_{j>k} U_j} \end{pmatrix}$. Put $g' := \pi_A \tau \iota_A g$. Notice that, for any $x \in X$, $h(x) = x - gf(x)$, so $(1 - h)(x) = \iota_A gf(x)$ and, consequently, $x = \tau\iota_A gf(x)$. Since $x \in A$, we get that $g'f(x) = x$ for every $x \in X$. Moreover, $\text{i-dim}_{U_j}(1_A - \pi_A\tau\iota_A) = 0$ if U_j is of type 1, and $\text{m-dim}_{U_j}(1_A - \pi_A\tau\iota_A) = \text{e-dim}_{U_j}(1_A - \pi_A\tau\iota_A) = 0$ whenever $j \in \mathbb{N}$ and U_j is of type 2. This is because $1_A - \pi_A\tau\iota_A = \pi_A(1_M - \tau)\iota_A$ and

$1_M - \tau$ has all those m-dim's, e-dim's and i-dim's zero. Now we can finish the proof of the claim easily. For example, consider $\text{m-dim}_{U_i}(1_A - g'f) = \text{m-dim}_{U_i}(1_A - gf + (1_A - \pi_A\tau\iota_A)gf)$. If U_j is of type 2, then $\text{m-dim}_{U_j}(1_A - gf) = 0$ and also $\text{m-dim}_{U_j}((1_A - \pi_A\tau\iota_A)gf)$, therefore, by Lemma 4.1, $\text{m-dim}_{U_j}(1_A - g'f) = 0$. The proof for $\text{e-dim}_{U_j}(1_A - g'f) = 0$ and $\text{i-dim}_{U_j}(1_A - g'f) = 0$ are similar. Finally, one can use $1_{A'} - fg' = (1_{A'} - fg) + f(1_A - \pi_A\tau\iota_A)g$ to prove the remaining equalities. This proves the claim.

Every uniserial module is either countaby generated or small [2, Proposition 2.45]. Hence, for every $i \in \mathbb{N}$, there exists a countable filtration $V_{i,1} \subseteq V_{i,2} \subseteq \dots$ of U_i such that $U_i = \cup_{j\in\mathbb{N}} V_{i,j}$ and $V_{i,j}$ is small for every $i, j \in \mathbb{N}$.

We now apply a standard back and forth (this is the analog of [11, Lemma 2.2]). First of all, fix a bijection $\gamma\colon \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ and set $X_i := \pi_A(V_{\gamma(i)})$ and $Y_i := \pi_{A'}(V_{\gamma(i)})$. Observe that $A = \sum_{i\in\mathbb{N}} X_i$, and $A' = \sum_{i\in\mathbb{N}} Y_i$. By induction we construct ascending chains $A_1 \subseteq A_2 \subseteq \cdots$ of submodules of A, $B_1 \subseteq B_2 \subseteq \cdots$ of submodules of A' and homomorphisms $f_1, f_2, f_3, \dots \colon A \to A'$, $g_1, g_2, g_3, \dots \colon A' \to A$ such that:

- (i) A_i and B_i are small modules for every $i \in \mathbb{N}$, and $\sum_{i\in\mathbb{N}} A_i = A$, $\sum_{i\in\mathbb{N}} B_i = A'$;
- (ii) $g_i f_i(x) = x$ for every $x \in A_i$;
- (iii) $f_{i+1} g_i(x) = x$ for every $x \in B_i$;
- (iv) for every $i \in \mathbb{N}$ and every $j \in \mathbb{N}$ such that U_j is of type 2, we have that $\text{m-dim}_{U_j}(1_A - g_i f_i) = \text{e-dim}_{U_j}(1_A - g_i f_i) = \text{m-dim}_{U_j}(1_{A'} - f_i g_i) = \text{e-dim}_{U_j}(1_{A'} - f_i g_i) = 0$ and $\text{m-dim}_{U_j}(1_A - g_i f_{i+1}) = \text{e-dim}_{U_j}(1_A - g_i f_{i+1}) = \text{m-dim}_{U_j}(1_{A'} - f_{i+1} g_i) = \text{e-dim}_{U_j}(1_{A'} - f_{i+1} g_i) = 0$;
- (v) for every $i \in \mathbb{N}$ and every $j \in \mathbb{N}$ such that U_j is of type 1, we have that $\text{i-dim}_{U_j}(1_A - g_i f_i) = \text{i-dim}_{U_j}(1_{A'} - f_i g_i) = 0$ and $\text{i-dim}_{U_j}(1_A - g_i f_{i+1}) = \text{i-dim}_{U_j}(1_{A'} - f_{i+1} g_i) = 0$;
- (vi) for every $n \in \mathbb{N}$, we have $g_n(B_n) \subseteq A_{n+1}$ and $f_n(A_n) \subseteq B_n$.

The induction process is as follows. Set $f_1 := f$, $g_1 := g$, $A_1 := 0$ and $B_1 := Y_1$. Suppose we have constructed $f_1, \dots, f_n$, $g_1, \dots, g_n$, $A_1, \dots, A_n$ and $B_1, \dots, B_n$. Define $A_{n+1} = g_n(B_n) + X_n$. Now, for every $j \in \mathbb{N}$, $\text{m-dim}_{U_j}(1_A - g_n f_n) = \text{m-dim}_{U_j}(1_{A'} - f_n g_n) = \text{e-dim}_{U_j}(1_A - g_n f_n) = \text{e-dim}_{U_j}(1_{A'} - f_n g_n) = 0$ when U_j is of type 2, and $\text{i-dim}_{U_j}(1_A - g_n f_n) = \text{i-dim}_{U_j}(1_{A'} - f_n g_n) = 0$ when U_j is of type 1. Thus the claim guarantees the existence of a morphism $f_{n+1}\colon A \to A'$ such that, for every $j \in \mathbb{N}$, $\text{m-dim}_{U_j}(1_{A'} - f_{n+1} g_n) = \text{m-dim}_{U_j}(1_A - g_n f_{n+1}) = \text{e-dim}_{U_j}(1_{A'} - f_{n+1} g_n) = \text{e-dim}_{U_j}(1_A - g_n f_{n+1}) = 0$ (or $\text{i-dim}_{U_j}(1_A - g_n f_{n+1}) = \text{i-dim}_{U_j}(1_{A'} - f_{n+1} g_n) = 0$)

when U_j is of type 2 (or of type 1), and $f_{n+1}g_n(x) = x$ for every $x \in B_n$. Set $B_{n+1} := Y_{n+1} + f_{n+1}(A_{n+1})$. Again, by the claim, there exists $g_{n+1}\colon A' \to A$ with $\text{m-dim}_{U_j}(1_A - g_{n+1}f_{n+1}) = \text{e-dim}_{U_j}(1_A - g_{n+1}f_{n+1}) = \text{m-dim}_{U_j}(1_{A'} - f_{n+1}g_{n+1}) = \text{e-dim}_{U_j}(1_{A'} - f_{n+1}g_{n+1}) = 0$ (or $\text{i-dim}_{U_j}(1_A - g_{n+1}f_{n+1}) = \text{i-dim}_{U_j}(1_{A'} - f_{n+1}g_{n+1}) = 0$) for every $j \in \mathbb{N}$ with U_j of type 2 (or of type 1), and $g_{n+1}f_{n+1}(x) = x$ for every $x \in A_{n+1}$.

Notice that f_n and f_{n+1} agree on A_n, because $f_{n+1}(x) = (f_{n+1}g_n)f_n(x) = f_n(x)$ for every $x \in A_n$. Therefore we can define $f\colon A \to A'$ such that $f|_{A_n} = f_n|_{A_n}$ for every $n \in \mathbb{N}$. Similarly, g_n and g_{n+1} agree on B_n and we can define $g\colon A' \to A$ with $g|_{B_n} = g_n|_{B_n}$. Now it is obvious, from (ii),(iii) and (vi), that f and g are mutually inverse. □

Recall that if U, V are non-zero uniserial modules and $[U]_m = [V]_m$, then U is of type 1 if and only if V is of type 1. Similarly, when $[U]_e = [V]_e$, we get that U is of type 1 if and only if V is of type 1 (see [5, Lemma 5.2]).

Lemma 7.2. *Let U_i, $i \in \mathbb{N}$, be a countable family of non-zero quasi-small uniserial modules. Let A_1, A_2 be direct summands of $\oplus_{i\in\mathbb{N}} U_i$. Then $A_1 \cong A_2$ if and only if the following conditions hold.*

(i) $\text{i-dim}_{U_i}(A_1) = \text{i-dim}_{U_i}(A_2)$ *for every $i \in \mathbb{N}$ with U_i of type 1.*
(ii) $\text{m-dim}_{U_i}(A_1) = \text{m-dim}_{U_i}(A_2)$ *for every $i \in \mathbb{N}$ with U_i of type 2.*
(iii) $\text{e-dim}_{U_i}(A_1) = \text{e-dim}_{U_i}(A_2)$ *for every $i \in \mathbb{N}$ with U_i of type 2.*

Proof. Assume $\oplus_{i\in\mathbb{N}} U_i = A_1 \oplus B_1 = A_2 \oplus B_2$. Without loss of generality we can suppose $B_1 \cong B_2$ (if X is a direct summand of $\oplus_{i\in\mathbb{N}} U_i$, then $X \oplus (\oplus_{i\in\mathbb{N}} U_i)^{(\aleph_0)} \cong (\oplus_{i\in\mathbb{N}} U_i)^{(\aleph_0)}$).

By Proposition 7.1, it is enough to find $f\colon A_1 \to A_2$ and $g\colon A_2 \to A_1$ such that, for every $i \in \mathbb{N}$,

(1) $\text{i-dim}_{U_i}(1_{A_1} - gf) = \text{i-dim}_{U_i}(1_{A_2} - fg) = 0$ if U_i is of type 1, and
(2) $\text{m-dim}_{U_i}(1_{A_1} - gf) = \text{e-dim}_{U_i}(1_{A_1} - gf) = \text{m-dim}_{U_i}(1_{A_2} - fg) = \text{e-dim}_{U_i}(1_{A_2} - fg) = 0$ if U_i is of type 2.

Define three subsets of $\mathbb{N}$ as follows. Set $N_i := \{\, i \in \mathbb{N} \mid U_i$ is of type 1 and there is no $j < i,\ j \in \mathbb{N}$, with $U_j \cong U_i \,\}$, $N_m := \{\, i \in \mathbb{N} \mid U_i$ is of type 2 and there is no $j < i,\ j \in \mathbb{N}$, with $[U_j]_m = [U_i]_m \,\}$, and $N_e := \{\, i \in \mathbb{N} \mid U_i$ is of type 2 and there is no $j < i,\ j \in \mathbb{N}$, with $[U_j]_e = [U_i]_e \,\}$. Notice that there are one-to-one correspondences between N_i and the set of all isomorphism classes of the U_i's of type 1 and between N_m (resp. N_e) and the set of all monogeny (resp. epigeny) classes of the U_i's of type 2.

Fix any $n \in N_m$. By Lemma 4.9, there are endomorphisms f_n and g_n in $\mathrm{End}_R(\oplus_{i\in\mathbb{N}}U_i)$ with the following three properties: (i) for every $i,j \in \mathbb{N}$ with either $\pi_j f_n \iota_i$ or $\pi_j g_n \iota_i$ non-zero, one has $[U_i]_m = [U_j]_m = [U_n]_m$; (ii) $\text{m-dim}_{U_n}(1_{A_1} - \pi_{A_1} g_n \iota_{A_2} \pi_{A_2} f_n \iota_{A_1}) = 0$ and $\text{m-dim}_{U_n}(1_{A_2} - \pi_{A_2} f_n \iota_{A_1} \pi_{A_1} g_n \iota_{A_2}) = 0$; (iii) for every $i,j \in \mathbb{N}$, the morphisms $\pi_j f_n \iota_i$ and $\pi_j g_n \iota_i$ are not epimorphisms. Then $\{ f_n \mid n \in N_m \}$ is a summable family of homomorphisms, because for every $i \in \mathbb{N}$ and every $x \in U_i$ there is a finite subset F_x of N_m with $f_n \iota_i(x) = 0$ for every $n \in N_m \setminus F_x$. (Given $i \in \mathbb{N}$ and $x \in U_i$, let F_x be the subset of cardinality 1 of N_m containing the unique element t of N_m with $[U_t]_m = [U_i]_m$. Then $[U_n]_m \neq [U_i]_m$ for every $n \in N_m \setminus F_x$, so that $\pi_j f_n \iota_i = 0$ for every $n \in N_m \setminus F_x$ and every $j \in \mathbb{N}$ by Property (i). Thus $f_n \iota_i = 0$ for every $n \in N_m \setminus F_x$.) Similarly, $\{ g_n \mid n \in N_m \}$ also is a summable family of homomorphisms. Notice that each f_n sends a U_i to 0 if $[U_i]_m \neq [U_n]_m$, and sends U_i to the direct sum of the U_j's with $[U_j]_m = [U_i]_m$ if $[U_i]_m = [U_n]_m$. Similarly for the g_n's.

Set $\alpha_m := \sum_{n\in N_m} f_n$, $\beta_m := \sum_{n\in N_m} g_n$, so that α_m, β_m are endomorphisms of $\oplus_{i\in\mathbb{N}}U_i$. Notice that α_m and β_m send, for every $n \in N_m$, the direct sum of the U_i's with $[U_i]_m = [U_n]_m$ into itself. Thus, for any $k \in N_m$, $\text{m-dim}_{U_k}(\alpha_m - f_k) = 0$ and $\text{m-dim}_{U_k}(\beta_m - g_k) = 0$ (because $\alpha_m - f_k$ sends the direct sum of the U_i's with $[U_i]_m = [U_k]_m$ to zero, and it is possible to apply Lemma 4.1).

Now fix any $n \in N_e$. By Lemma 5.10, we can find endomorphisms f'_n, g'_n of $\oplus_{i\in\mathbb{N}}U_i$ with the following three properties: (i) if $i,j \in \mathbb{N}$ and either $\pi_j f'_n \iota_i$ or $\pi_j g'_n \iota_i$ is non-zero, then $[U_i]_e = [U_j]_e = [U_n]_e$; (ii) $\text{e-dim}_{U_n}(1_{A_1} - \pi_{A_1} g'_n \iota_{A_2} \pi_{A_2} f'_n \iota_{A_1}) = 0$ and $\text{e-dim}_{U_n}(1_{A_2} - \pi_{A_2} f'_n \iota_{A_1} \pi_{A_1} g'_n \iota_{A_2}) = 0$; (iii) for every $i,j \in \mathbb{N}$, the morphisms $\pi_j f'_n \iota_i$ and $\pi_j g'_n \iota_i$ are not monomorphisms. Observe that $\{ f'_n \mid n \in N_e \}$ and $\{ g'_n \mid n \in N_e \}$ are summable families of homomorphisms. Set $\alpha_e := \sum_{n\in N_e} f'_n$, $\beta_e := \sum_{n\in N_e} g'_n$. For any $k \in N_e$, we have that $\text{e-dim}_{U_k}(\alpha_e - f'_k) = 0$ and $\text{e-dim}_{U_k}(\beta_e - g'_k) = 0$.

Now fix $n \in N_i$. If $K_{U_n} \subseteq I_{U_n}$, then $\text{i-dim}_U = \text{m-dim}_U$ by Proposition 6.9. We can use Lemma 4.9 and get that there are endomorphisms $f''_n, g''_n \in \mathrm{End}_R(\oplus_{i\in\mathbb{N}}U_i)$ with the following two properties: (i) for every $i,j \in \mathbb{N}$ with either $\pi_j f''_n \iota_i$ or $\pi_j g''_n \iota_i$ non-zero, one has $U_i \cong U_j \cong U_n$; and (ii) $\text{i-dim}_U(1_{A_1} - \pi_{A_1} g''_n \iota_{A_2} \pi_{A_2} f''_n \iota_{A_1}) = 0$ and $\text{i-dim}_U(1_{A_2} - \pi_{A_2} f''_n \iota_{A_1} \pi_{A_1} g''_n \iota_{A_2}) = 0$. Similarly we proceed in the case $I_{U_n} \subseteq K_{U_n}$. By Proposition 6.10 and Lemma 5.10, we find endomorphisms $f''_n, g''_n \in \mathrm{End}_R(\oplus_{i\in\mathbb{N}}U_i)$ such that: (i) for every $i,j \in \mathbb{N}$ with either $\pi_j f''_n \iota_i$ or $\pi_j g''_n \iota_i$ non-zero, one has $U_i \cong U_j \cong U_n$; and (ii) $\text{i-dim}_U(1_{A_1} - \pi_{A_1} g''_n \iota_{A_2} \pi_{A_2} f''_n \iota_{A_1}) = 0$ and $\text{i-dim}_U(1_{A_2} - \pi_{A_2} f''_n \iota_{A_1} \pi_{A_1} g''_n \iota_{A_2}) = 0$. Again, the f''_n's and the g''_n send all the

U_i's with $U_i \not\cong U_n$ to 0, and send the direct sum of all the U_i's with $U_i \cong U_n$ into iteself. Therefore the families $\{ f''_n \mid n \in N_e \}$ and $\{ g''_n \mid n \in N_e \}$ are summable. Set $\alpha_i := \sum_{n\in N_i} f''_n$ and $\beta_i := \sum_{n\in N_i} g''_n$. Also, for any $k \in N_i$, $\text{i-dim}_{U_k}(\alpha_i - f''_k) = 0$ and $\text{i-dim}_{U_n}(\beta_i - g''_k) = 0$.

Put $f := \pi_{A_2}(\alpha_m + \alpha_e + \alpha_i)\iota_{A_1}$ and $g := \pi_{A_1}(\beta_m + \beta_e + \beta_i)\iota_{A_2}$. It remains to prove that f and g have the required properties (1) and (2) for every $i \in \mathbb{N}$. Clearly, it suffices to check property (1) when i is an element $k \in N_i$, to check that $\text{m-dim}_{U_i}(1_{A_1} - gf) = \text{m-dim}_{U_i}(1_{A_2} - fg) = 0$ when i is an element $k \in N_m$, and to check that $\text{e-dim}_{U_i}(1_{A_1} - gf) = \text{e-dim}_{U_i}(1_{A_2} - fg) = 0$ when i is an element $k \in N_e$. For every non-zero uniserial module U, let $\mathcal{M}_U$ ($\mathcal{E}_U$, $\mathcal{I}_U$) be the ideal in Mod-R consisting of all morphisms in Mod-R with m-dim_U (e-dim_U, i-dim_U) zero.

Fix an index $k \in N_i$. We have that $\pi_j f_n \iota_i \in \mathcal{I}_{U_k}$ for every $n \in N_m$ and every $i, j \in \mathbb{N}$, because they are not epimorphisms. It follows that $f_n \in \mathcal{I}_{U_k}$ for every $n \in N_m$, so that $\alpha_m \in \mathcal{I}_{U_k}$, that is, $\text{i-dim}_{U_k}(\alpha_m) = 0$. Similarly, $\text{i-dim}_{U_k}(\alpha_e) = \text{i-dim}_{U_k}(\beta_m) = \text{i-dim}_{U_k}(\beta_e) = 0$, that is, $\alpha_e, \beta_m, \beta_e \in \mathcal{I}_{U_k}$. Therefore $1_{A_1} - gf \in \mathcal{I}_{U_k}$ if and only if $1_{A_1} - \pi_{A_1}\beta_i\iota_{A_2}\pi_{A_2}\alpha_i\iota_{A_1} \in \mathcal{I}_{U_k}$. Now we have $\text{i-dim}_{U_k}(\alpha_i - f''_k) = 0$ and $\text{i-dim}_{U_k}(\beta_i - g''_k) = 0$, that is, $\alpha_i - f''_k, \beta_i - g''_k \in \mathcal{I}_k$. Therefore $1_{A_1} - \pi_{A_1}\beta_i\iota_{A_2}\pi_{A_2}\alpha_i\iota_{A_1} \in \mathcal{I}_{U_k}$ if and only if $1_{A_1} - \pi_{A_1} g''_k \iota_{A_2}\pi_{A_2} f''_k \iota_{A_1} \in \mathcal{I}_{U_k}$. This last assertion is true, because $\text{i-dim}_{U_k}(1_{A_1} - \pi_{A_1} g''_k \iota_{A_2}\pi_{A_2} f''_k \iota_{A_1}) = 0$. The proof for $\text{i-dim}_{U_k}(1_{A_2} - fg) = 0$ is similar.

Now let $k \in N_m$. For every $i, j \in \mathbb{N}$ and $n \in N_e$, the morphism $\pi_j f'_n \iota_i$ is not a monomorphism, hence belongs to $\mathcal{M}_{U_k}$. Thus $f'_n \in \mathcal{M}_{U_k}$ for every $n \in N_e$ (Remark 2.9), so that $\alpha_e = \sum_{n\in N_e} f'_n \in \mathcal{M}_{U_k}$, that is, $\text{m-dim}_{U_k}(\alpha_e) = 0$. Similarly, $\text{m-dim}_{U_k}(\alpha_i) = \text{m-dim}_{U_k}(\beta_e) = \text{m-dim}_{U_k}(\beta_i) = 0$. Therefore $1_{A_1} - gf \in \mathcal{M}_{U_k}$ if and only if $1_{A_1} - \pi_{A_1}\beta_m\iota_{A_2}\pi_{A_2}\alpha_m\iota_{A_1} \in \mathcal{M}_{U_k}$. Now $\text{m-dim}_{U_k}(\alpha_m - f_k) = 0$ and $\text{m-dim}_{U_k}(\beta_m - g_k) = 0$, so that $1_{A_1} - \pi_{A_1}\beta_m\iota_{A_2}\pi_{A_2}\alpha_m\iota_{A_1} \in \mathcal{M}_{U_k}$ if and only if $1_{A_1} - \pi_{A_1} g_k \iota_{A_2}\pi_{A_2} f_k \iota_{A_1} \in \mathcal{M}_{U_k}$. The last assertion is true, because $\text{m-dim}_{U_k}(1_{A_1} - \pi_{A_1} g_k \iota_{A_2}\pi_{A_2} f_k \iota_{A_1}) = 0$. The proof for $\text{m-dim}_{U_k}(1_{A_2} - fg) = 0$ is similar.

Finally, let $k \in N_e$. As before, we find that $\text{e-dim}_{U_k}(\alpha_m) = \text{e-dim}_{U_k}(\alpha_i) = \text{e-dim}_{U_k}(\beta_m) = \text{e-dim}_{U_k}(\beta_i) = 0$, hence $1_{A_1} - gf \in \mathcal{E}_{U_k}$ if and only if $1_{A_1} - \pi_{A_1}\beta_e\iota_{A_2}\pi_{A_2}\alpha_e\iota_{A_1} \in \mathcal{E}_{U_k}$. Now $\text{e-dim}_{U_k}(\alpha_e - f'_k) = 0$ and $\text{e-dim}_{U_k}(\beta_e - g'_k) = 0$, so that $1_{A_1} - \pi_{A_1}\beta_e\iota_{A_2}\pi_{A_2}\alpha_e\iota_{A_1} \in \mathcal{E}_{U_k}$ if and only if $1_{A_1} - \pi_{A_1} g'_k \iota_{A_2}\pi_{A_2} f'_k \iota_{A_1} \in \mathcal{E}_{U_k}$. The last statement is true because $\text{e-dim}_{U_k}(1_{A_1} - \pi_{A_1} g'_k \iota_{A_2}\pi_{A_2} f'_k \iota_{A_1}) = 0$. The proof for $\text{e-dim}_{U_k}(1_{A_2} - fg) = 0$ is similar. □

Recall that, for every uniserial module U that is not quasi-small, there exists a cyclic submodule V of U with $[V]_m = [U]_m$, and for any such submodule V, U

turns out to be isomorphic to a direct summand of $V^{(\aleph_0)}$ (To see this, notice that by [1, Lemma 4.5(b)] there exists a cyclic submodule V of U with $[V]_m = [U]_m$. Now apply [10, Theorem 2.6] to show that $U \oplus V^{(\aleph_0)} \cong V^{(\aleph_0)}$.)

Corollary 7.3. *Let U_i, $i \in \mathbb{N}$, be a family of non-zero uniserial modules. Set $K := \{ i \in \mathbb{N} \mid U_i$ is quasi-small$\}$. Let A and A' be direct summands of $\oplus_{i\in\mathbb{N}} U_i$. Then $A \cong A'$ if and only if:*

(i) $\text{i-dim}_{U_i}(A) = \text{i-dim}_{U_i}(A')$ *for every* $i \in \mathbb{N}$ *with* U_i *of type 1,*
(ii) $\text{m-dim}_{U_i}(A) = \text{m-dim}_{U_i}(A')$ *for every* $i \in \mathbb{N}$ *with* U_i *of type 2,*
(iii) $\text{e-dim}_{U_i}(A) = \text{e-dim}_{U_i}(A')$ *for every* $i \in K$.

Proof. In the direct sum $\oplus_{i\in\mathbb{N}} U_i$ we can substitute each summand U_i that is not quasi-small with a countable family of cyclic pairwise isomorphic submodules of U_i in the same monogeny class of U_i. Thus we get a countable family V_j, $j \in \mathbb{N}$, of uniserial modules, in which every V_j is quasi-small and $\oplus_{i\in\mathbb{N}} U_i$ is isomorphic to a direct summand of $\oplus_{j\in\mathbb{N}} V_j$.

Then A, A' are direct summands of $\oplus_{j\in\mathbb{N}} V_j$ and, by Lemma 7.2, it is enough to prove that $\text{i-dim}_{V_j}(A) = \text{i-dim}_{V_j}(A')$ for every $j \in \mathbb{N}$ with V_j is of type 1, and $\text{m-dim}_{V_j}(A) = \text{m-dim}_{V_j}(A')$, $\text{e-dim}_{V_j}(A) = \text{e-dim}_{V_j}(A')$ for every $j \in \mathbb{N}$ with V_j of type 2. The first equalities are obvious. The equalities $\text{m-dim}_{V_j}(A) = \text{m-dim}_{V_j}(A')$ hold because m-dim_V depends only on the monogeny class of V. Finally, assume that $j \in \mathbb{N}$ is such that V_j is of type 2. If $[V_j]_e = [U_i]_e$ for some $i \in K$, we are done by (iii). Now suppose that $[V_j]_e \neq [U_i]_e$ for any $i \in \mathbb{N}$ such that U_i is a quasi-small module of type 2. Since any module of the same epigeny class as V_j is quasi-small (this follows from [1, Lemma 4.5]) and of type 2 [5, Lemma 5.2], we get that $\text{e-dim}_{V_j}(\oplus_{i\in\mathbb{N}} U_i) = 0$ by Remark 5.2 and Lemma 5.3. Then necessarily $\text{e-dim}_{V_j}(A) = \text{e-dim}_{V_j}(A') = 0$, and we are done. □

We are ready to prove our final categorical version of the Weak Krull-Schmidt Theorem. Let SUsr be the category of all serial right modules over a fixed ring R. For every uniserial module U of type 1, let $F\colon \text{add(SUsr)} \to \text{add(SUsr)}/\mathcal{I}_U$ be the canonical functor. We know that there is an equivalence $G\colon \text{add(SUsr)}/\mathcal{I}_U \to \text{Mod-}(\text{End}_R(U)/I_U)$. For an object $A \in \text{add(SUsr)}$, define $\text{I-dim}_U(A)$ as the dimension of $GF(A)$ over the division ring $\text{End}_R(U)/J_U$. Similarly, for every uniserial module U of type 2, define $\text{M-dim}_U(A)$ (resp., $\text{E-dim}_U(A)$) as the dimension of $GF(A)$, where $F\colon \text{add(SUsr)} \to \text{add(SUsr)}/\mathcal{M}_U$ (resp., $F\colon \text{add(SUsr)} \to \text{add(SUsr)}/E_U$) is the canonical functor and G is the categorical equivalence

$G\colon \mathrm{add(SUsr)}/\mathcal{M}_U \to \text{Mod-}(\mathrm{End}_R(U)/I_U)$ (resp., $G\colon \mathrm{add(SUsr)}/\mathcal{E}_U \to \text{Mod-}(\mathrm{End}_R(U)/K_U)$).

Notice the difference between the invariants m-$\dim_U$ and M-$\dim_U$. The invariant m-$\dim_U(A)$ was defined in [4] for any right R-module A to consider the finite case: it is either a non-negative integer or ∞. The invariant M-$\dim_U(A)$ introduced now is defined only for the objects of add(SUsr), it has been introduced to treat infinite direct sums, and its value is a cardinal number. The two invariants m-$\dim_U(A)$ and M-$\dim_U(A)$ coincide for the modules A that are direct summands of countable direct sums of uniserial modules (Lemma 4.8).

Theorem 7.4. *Let A and A' be direct summands of serial right modules over a ring R. Then $A' \cong A$ if and only if*

(i) I-$\dim_U(A)$ = I-$\dim_U(A')$ *for every uniserial right R-module U of type 1, and*

(ii) M-$\dim_U(A)$ = M-$\dim_U(A')$ *and* E-$\dim_U(A)$ = E-$\dim_U(A')$ *for every quasi-small uniserial right R-module U of type 2.*

Proof. As in the proof of Corollary 7.3, let U_i, $i \in I$, be a family of non-zero quasi-small uniserial modules such that both A and A' are direct summands of $\oplus_{i\in I} U_i$. The modules A and A' have decompositions $A = \oplus_{x\in X} A_x$ and $A' = \oplus_{y\in Y} A'_y$, where every A_x, $x \in X$, and every A'_y, $y \in Y$, is isomorphic to a direct summand of $\oplus_{i\in I'} U_i$ for some countable subset I' of I [2, Corollary 2.49]. Clearly, we can suppose that all the A_x's and all the A'_y's are non-zero.

Let U be a non-zero quasi-small uniserial module. Observe that

$$\sum_{x\in X} \text{I-}\dim_U(A_x) = \text{I-}\dim_U(A) \qquad \text{and} \qquad \sum_{y\in Y} \text{I-}\dim_U(A'_y) = \text{I-}\dim_U(A'),$$

where the sums indicate the cardinality of the disjoint union of the cardinals I-$\dim_U(A_x)$, $x \in X$ (resp., I-$\dim_U(A'_y)$, $y \in Y$) . Similarly $\sum_{x\in X}$ M-$\dim_U(A_x)$ = M-$\dim_U(A)$, $\sum_{y\in Y}$ M-$\dim_U(A'_y)$ = M-$\dim_U(A')$, $\sum_{x\in X}$ E-$\dim_U(A_x)$ = E-$\dim_U(A)$ and $\sum_{y\in Y}$ E-$\dim_U(A'_y)$ = E-$\dim_U(A')$.

Therefore we can construct the following bipartite, non-directed graphs with multiple edges. Fix an index $i \in I$ with U_i of type 1. Fix a set E_i of cardinality $|E_i|$ = I-$\dim_{U_i}(A)$ and two mappings $p\colon E_i \to X$, $q\colon E_i \to Y$ with $|p^{-1}(x)|$ = I-$\dim_{U_i}(A_x)$, $|q^{-1}(y)|$ = I-$\dim_{U_i}(A'_y)$ for every $x \in X$, $y \in Y$. Define a graph G_i with set of vertices the disjoint union $X \dot\cup Y$ of X and Y, set of edges E_i, and any edge $e \in E_i$ connecting the vertices $p(e) \in X$ and $q(e) \in Y$. Notice that the graph $G_i = (X \dot\cup Y, E_i)$ is bipartite because there are no edges between two vertices in X

or between two vertices in Y, that for every $x \in X$ there are I-$\dim_{U_i}(A_x)$ edges adjacent to x and that for every $y \in Y$ there are I-$\dim_{U_i}(A'_y)$ edges adjacent to y.

If $i \in I$ and U_i is of type 2, define two bipartite, non-directed graphs G'_i and G''_i in a similar way. Both graphs have $X \dot{\cup} Y$ as set of vertices. The graph G'_i has a set E'_i of edges of cardinality $|E'_i| =$ M-$\dim_{U_i}(A)$, and any edge of G'_i connects a vertex $x \in X$ and a vertex $y \in Y$. Further, for every $x \in X$ there are M-$\dim_{U_i}(A_x)$ edges of G'_i adjacent to x and M-$\dim_{U_i}(A'_y)$ edges adjacent to y. Similarly, for G''_i. For every $x \in X$ there are E-$\dim_{U_i}(A_x)$ edges of G''_i adjacent to x and E-$\dim_{U_i}(A'_y)$ edges adjacent to y.

Let $I_0 \subseteq I$ be such that U_i is of type 1 for every $i \in I_0$, and for any $i \in I$ such that U_i is of type 1 there exists exactly one $j \in I_0$ such that $U_i \cong U_j$. Similarly, let $I'_0 \subseteq I$ (resp., $I''_0 \subseteq I$) be such that U_i is of type 2 for every $i \in I'_0$ (resp., $i \in I''_0$), and for any $i \in I$ such that U_i is of type 2 there exists exactly one $j \in I'_0$ (resp., $j \in J''_0$) with $[U_i]_m = [U_j]_m$ (resp., $[U_i]_e = [U_j]_e$). Consider the collection of graphs $\mathcal{C} = \{G_i \mid i \in I_0\} \cup \{G'_i \mid i \in I'_0\} \cup \{G''_i \mid i \in I''_0\}$.

Notice that in any of these graphs every vertex has degree $\leq \aleph_0$, that is, at most countably many edges adjacent to it. Also, any vertex has non-zero degree in at most countably many of the graphs of the collection $\mathcal{C}$.

Let κ be an infinite ordinal of cardinality greater than the cardinality of X. We will now construct two families of subsets X_λ, $\lambda \leq \kappa$, of X and Y_λ, $\lambda \leq \kappa$, of Y with the following properties:

(i) $X = \cup_{\lambda \leq \kappa} X_\lambda$, $Y = \cup_{\lambda \leq \kappa} Y_\lambda$.

(ii) If $\lambda \leq \kappa$ is a limit ordinal, then $X_\lambda = \cup_{\lambda' < \lambda} X_{\lambda'}$ and $Y_\lambda = \cup_{\lambda' < \lambda} Y_{\lambda'}$.

(iii) For any $\lambda < \kappa$, $X_\lambda \subseteq X_{\lambda+1}$, $Y_\lambda \subseteq Y_{\lambda+1}$ and the sets $X_{\lambda+1} \setminus X_\lambda$ and $Y_{\lambda+1} \setminus Y_\lambda$ are at most countable.

(iv) For any $\lambda < \kappa$, each graph of the collection $\mathcal{C}$ is the disjoint union of its two full subgraphs with set of vertices $X_\lambda \dot{\cup} Y_\lambda$ and $(X \setminus X_\lambda) \dot{\cup} (Y \setminus Y_\lambda)$, that is, there is no edge between any vertex in $X_\lambda \dot{\cup} Y_\lambda$ and any vertex in $(X \setminus X_\lambda) \dot{\cup} (Y \setminus Y_\lambda)$.

The construction of the X_λ's and the Y_λ's is by induction on λ. For $\lambda = 0$, define $X_\lambda := \emptyset$ and $Y_\lambda := \emptyset$. If λ is a limit ordinal and $X_{\lambda'}, Y_{\lambda'}$ have been already constructed for every $\lambda' < \lambda$, set $X_\lambda := \cup_{\lambda' < \lambda} X_{\lambda'}$ and $Y_\lambda := \cup_{\lambda' < \lambda} Y_{\lambda'}$ (notice that (iv) is true for λ if it is true for every $\lambda' < \lambda$). Now suppose that we have defined X_λ and Y_λ and we want to define $X_{\lambda+1}$ and $Y_{\lambda+1}$. If $X_\lambda = X$, then by (iv) there is no edge between any vertex in $X \dot{\cup} Y_\lambda$ and any vertex in $Y \setminus Y_\lambda$. But every vertex in Y has non-zero degree in at least one graph of $\mathcal{C}$, so that $Y_\lambda = Y$ also. Hence, in this case $X_\lambda = X$, define $X_{\lambda+1} = X$ and $Y_{\lambda+1} = Y$. Otherwise, fix $x \in X \setminus X_\lambda$. Now construct by induction subsets $C_1 \subseteq C_2 \subseteq \dots$ of X and $D_1 \subseteq D_2 \subseteq \dots$ of Y as follows. Set $C_1 := \{x\}$, and let D_1 be the set of all $y \in Y$ that are connected to x in at least one of the graphs in $\mathcal{C}$. Suppose that $C_1, \dots, C_k$ and $D_1, \dots, D_k$ have been defined. Define C_{k+1} as the subset of all elements of Y that are connected to some element of D_k in at least one of the graphs in $\mathcal{C}$. Similarly, let D_{k+1} be the set of elements of X that are connected to some element of C_{k+1} in at least one of the graphs in $\mathcal{C}$. Notice that $C_k \subseteq C_{k+1}$ and $D_k \subseteq D_{k+1}$. Define $C := \cup_{k\in\mathbb{N}} C_k$ and $D := \cup_{k\in\mathbb{N}} D_k$. Since Condition (iv) is true for λ, we have that $C \subseteq X \setminus X_\lambda$ and $D \subseteq Y \setminus Y_\lambda$.

Clearly, any of the defined graphs is the disjoint union of its three full subgraphs with set of vertices $X_\lambda \dot\cup Y_\lambda$, $C \dot\cup D$ and $(X \setminus (C \cup X_\lambda)) \dot\cup (Y \setminus (D \cup Y_\lambda))$ respectively. Define $X_{\lambda+1} := C \cup X_\lambda$ and $Y_{\lambda+1} := D \cup Y_\lambda$, so that (iii) and (iv) hold for $\lambda + 1$.

Since $X_\lambda \subset X_{\lambda+1}$ when $X_\lambda \neq X$, obviously $X_\kappa = X$, so that $Y_\kappa = Y$ also.

Lemma 7.2 guarantees that $\oplus_{x\in X_{\lambda+1}\setminus X_\lambda} A_x \cong \oplus_{y\in Y_{\lambda+1}\setminus Y_\lambda} A'_y$ for every $\lambda < \kappa$. Since $X = \dot\cup_{\lambda<\kappa} X_{\lambda+1} \setminus X_\lambda$ and $Y = \dot\cup_{\lambda<\kappa} Y_{\lambda+1} \setminus Y_\lambda$, we conclude that $A \cong A'$. □

The following result was proved in [9]. Now it is almost obvious.

Corollary 7.5. *Let U_i, $i \in I$, be uniserial modules that are not quasi-small for every $i \in I$. Then any direct summand of $\oplus_{i\in I} U_i$ is serial.*

Proof. Suppose that A is a direct summand of $X = \oplus_{i\in I} U_i$. Let V be a quasi-small uniserial module of type 2. As any non-zero factor of a uniserial module that is not quasi-small cannot be quasi-small [1, Lemma 4.5], we have that $\text{e-dim}_V(X) = 0$ and hence also $\text{e-dim}_V(A) = 0$. Similarly, $\text{i-dim}_W(X) = 0$, hence $\text{i-dim}_W(A) = 0$, for every uniserial module W of type 1. Let I_0 be a subset of I such that for every $j \in I$ there is exactly one $i \in I_0$ with $U_i \cong U_j$, so that $\{\, U_i \mid i \in I_0 \,\}$ is a set of representatives up to isomorphism of $\{\, U_i \mid i \in I_0 \,\}$. Then $A \cong \oplus_{i\in I_0} U_i^{(\text{M-dim}_{U_i}(A))}$. □

Remarks 7.6. (1) Let us explain why we can consider Theorem 7.4 a generalization of [10, Theorem 2.6]. Suppose we have a family of non-zero uniserial modules U_i, $i \in I$, and let $I' \subseteq I$ be the set of the $i \in I$ for which U_i is quasi-small. Let V be a uniserial module of type 2. Then $\text{M-dim}_V(\oplus_{i\in I} U_i)$ is the cardinality of set $\{\, i \in I \mid [U_i]_m = [V]_m \,\}$. If V is quasi-small, then also any uniserial module of the same epigeny class is quasi-small [1, Lemma 4.5]. Therefore $\text{E-dim}_V(\oplus_{i\in I} U_i) = |\{i \in I \mid [U_i]_e = [V]_e\}| = |\{i \in I' \mid [U_i]_e = [V]_e\}|$. Now suppose that V is a nonzero uniserial module of type 1. If every monomorphism in $\text{End}_R(V)$ is an isomorphism, then

$[U]_m = [V]_m$ if and only if $U \cong V$. Therefore I-dim$_V(\oplus_{i\in I} U_i) = |\{i \in I \mid [U_i]_m = [V]_m\}|$. If every epimorphism in $\mathrm{End}_R(V)$ is an isomorphism, then $[U]_e = [V]_e$ if and only if $U \cong V$. Therefore I-dim$_V(\oplus_{i\in I} U_i) = |\{i \in I \mid [U_i]_e \cong [V]_e\}|$. Having realized this, it is easy to prove the following: Let U_i, $i \in I$, and V_j, $j \in J$, be two families of nonzero uniserial modules. Let $I' = \{i \in I \mid U_i$ is quasi-small$\}$ and let $J' = \{j \in J \mid U_j$ is quasi-small$\}$. Then the following are equivalent

(i) There are bijections $\sigma\colon I \to J$ and $\tau\colon I' \to J'$ such that $[U_i]_m = [V_{\sigma(i)}]_m$ for every $i \in I$ and $[U_i]_e = [V_{\tau(i)}]_e$ for every $i \in I'$.

(ii) I-dim$_U(\oplus_{i\in I} U_i) =$ I-dim$_U(\oplus_{j\in J} V_j)$ for every uniserial module U of type 1, M-dim$_U(\oplus_{i\in I} U_i) =$ M-dim$_U(\oplus_{j\in J} V_j)$ for every uniserial module U of type 2, and E-dim$_U(\oplus_{i\in I} U_i) =$ E-dim$_U(\oplus_{j\in J} V_j)$ for every quasi-small uniserial module U of type 2.

(2) We conclude the paper with an analysis in this setting of the example given by Puninski in [13]. Our analysis is a continuation of [11, Section 5]. Essentially, Puninski found a uniserial ring R such that, for any $0 \neq r, s \in J(R)$, the modules R/rR and R/sR are isomorphic. Set $U := R_R$ and let V be a uniserial module isomorphic to R/rR for some non-zero $r \in J(R)$. The module V is of type 2 and there exists a uniserial direct summand V' of $V^{(\aleph_0)}$ not isomorphic to V. So V' is not quasi-small, but $[V']_m = [V]_m$. Moreover, $U^{(\aleph_0)} \oplus V \cong V' \oplus W$. Let us calculate the m-dim, e-dim, i-dim of W. Firstly, i-dim$_U(V') = 0$, therefore i-dim$_U(W) = \infty$. If U' is a uniserial module of type 1 not isomorphic to U, then i-dim$_{U'}(U^{(\aleph_0)} \oplus V) = 0$, so that also i-dim$_{U'}(W) = 0$. Now m-dim$_V(U^{(\aleph_0)} \oplus V) = 1 =$ m-dim$_V(V')$, therefore m-dim$_V(W) = 0$. If U' is a uniserial module of type 2 and $[U']_m \neq [V]_m$, then m-dim$_{U'}(U^{(\aleph_0)} \oplus V) = 0 =$ m-dim$_{U'}(W)$. Finally, e-dim$_V(U^{(\aleph_0)} \oplus V) = 1$ and e-dim$_V(V') = 0$ implies e-dim$_V(W) = 1$. Also, if U' is a quasi-small uniserial module of type 2 such that $[U']_e \neq [V]_e$, then e-dim$_{U'}(W) = 0$. Recall that if X, Y, Z are uniserial modules such that $[X]_m = [Y]_m$ and $[Y]_e = [Z]_e$, then X and Z have the same type (type 1 or type 2) [5, Lemma 5.2]. To see that W is not serial, assume the contrary, in which case W would contain a uniserial direct summand Y with e-dim$_V(Y) = 1$, but m-dim$_{U'}(Y) = 0$ for every uniserial module U' of type 2. Now e-dim$_V(Y) = 1$ and V of type 2 imply that Y also is of type 2 [5, Lemma 5.2]. Thus m-dim$_Y(Y) = 0$, which is not possible.

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