International Electronic Journal of Algebra
Volume 5 (2009) 17-26

SOME CHARACTERIZATIONS OF EF-EXTENDING RINGS

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Received: 3 March 2008; Revised: 8 October 2008
Communicated by W. Keith Nicholson

Abstract. In [19], Thuyet and Wisbauer considered the extending property for the class of (essentially) finitely generated submodules. A module M is called *ef-extending* if every closed submodule which contains essentially a finitely generated submodule is a direct summand of M. A ring R is called right ef-extending if R_R is an ef-extending module. We show that a ring R is QF if and only if R is a left Noetherian, right GP-injective and right ef-extending ring. Moreover, we prove that R is right PF if and only if R is a right cogenerator, right ef-extending and I-finite.

Mathematics Subject Classification (2000): 16D50, 16D70, 16D80.
Keywords: ef-extending rings, extending (or CS) rings, PF rings, QF rings.

1. Introduction

Throughout the paper, R represents an associative ring with identity $1 \neq 0$ and all modules are unitary R-modules. We write M_R (resp., ${}_RM$) to indicate that M is a right (resp., left) R-module. We also write J (resp., Z_r) for the Jacobson radical (resp., the right singular ideal) and $E(M_R)$ (resp., $\mathrm{Rad}(M_R)$) for the injective hull of M_R (resp., radical of M_R). If X is a subset of R, the right (resp., left) annihilator of X in R is denoted by $r_R(X)$ (resp., $l_R(X)$) or simply $r(X)$ (resp., $l(X)$) if no confusion appears. If N is a submodule of M (resp., proper submodule), we denote by $N \leq M$ (resp., $N < M$). Moreover, we write $N \leq^e M$ and $N \leq^{\oplus} M$ to indicate that N is an essential submodule and a direct summand of M, respectively. A module M is called uniform if $M \neq 0$ and every non-zero submodule of M is essential in M. A module M is finitely dimensional (or has finite rank) if $E(M)$ is a finite direct sum of indecomposable submodules; or equivalently, if M contains no infinite independent family of non-zero submodules.

A ring R is called *right P-injective* if $lr(a) = Ra$ for each $a \in R$. A ring R is called *right GP-injective* (resp., *right AGP-injective*) if for each $0 \neq a \in R$, there

The work was supported by the Natural Science Council of Vietnam.

exists $n \in \mathbb{N}$ such that $a^n \neq 0$ and $lr(a^n) = Ra^n$ (resp., $lr(a^n) = Ra^n \oplus X_a$ with $X_a \leq {}_RR$).

In [9] J. L. Gómez Pardo and P. A. Guil Asensio proved that every right Kasch right CS ring has finitely generated essential right socle, and hence R is a right PF ring if and only if R is a right cogenerator right CS ring. Their work extends a well-known theorem of B. Osofsky which states that a right Kasch right self-injective ring is semiperfect with finitely generated essential right socle (i.e. R_R is an injective cogenerator). In this paper, we show that R is QF iff R is a left Noetherian, right GP-injective and right ef-extending ring. Moreover, we prove that R is right PF iff R is right cogenerator, right ef-extending and I-finite.

General background material can be found in [1], [6], [14], [20].

2. Definitions and results.

Definition 2.1. [19] A module M is called *ef-extending* if every closed submodule which contains essentially a finitely generated submodule is a direct summand of M. A ring R is called right ef-extending if R_R is an ef-extending module.

We refer to the following conditions on a module M_R:

C1: Every submodule of M is essential in a direct summand of M.

C2: Every submodule of M that is isomorphic to a direct summand of M is itself a direct summand of M.

C3: $M_1 \oplus M_2$ is a direct summand of M for any two direct summand M_1, M_2 of M with $M_1 \cap M_2 = 0$.

A module M_R is called extending or CS (quasi-continuous, continuous), if it satisfies C1 (both C1 and C3; both C1 and C2). A ring R is called right CS (right quasi-continuous; right continuous), if R_R is CS-module (quasi-continuous, continuous).

From the definition of ef-extending module and ring, we have:

i) A right CS ring is a right ef-extending ring. But the converse is not true in general.

Example. Let K be a division ring and ${}_KV$ be a left K-vector space of infinite dimension. Take $S = End({}_KV)$, then it is well-known that S is regular but not right self-injective. Let

$$R = \begin{pmatrix} S & S \\ S & S \end{pmatrix},$$

then R is also regular, which implies R is right P-injective and every finitely generated right ideal of R is a direct summand of R. Thus, R is a right C2, right ef-extending ring. But R can not be right CS. For if R is right CS, then R is right continuous. Hence R is right self-injective by [14, Theorem 1.35], a contradiction.

ii) Every finitely generated submodule of an ef-extending module M is essential in direct summand of M.

Some properties of ef-extending module is studied in [5], [16], [17], [19]. In this paper, we consider some other properties of ef-extending modules with condition C3.

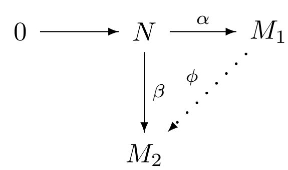
Let M, N be R-modules. M is said to be N- F *-injective* if for each R-homomorphism $f : H \to M$ from a finitely generated submodule H of N into M extends to N.

Modification in proving [10, Lemma 5], we have:

Lemma 2.2. *Let a module $M = M_1 \oplus M_2$ be a direct sum of submodules M_1, M_2. Then the following conditions are equivalent:*

(1) *M_2 is M_1-F-injective.*
(2) *For each finitely generated submodule N of M with $N \cap M_2 = 0$, there exists a submodule M' of M such that $M = M' \oplus M_2$ and $N \leq M'$.*

Proof. (1) $\Rightarrow$ (2). For $i = 1, 2$, let $\pi_i : M \to M_i$ denote the projection mapping. Consider the following diagram:



where $\alpha = \pi_1|_N$, $\beta = \pi_2|_N$. It is easy to see that α is a monomorphism. By (1), there exists a homomorphism $\phi : M_1 \to M_2$ such that $\phi\alpha = \beta$. Let $M' = \{x + \phi(x) | x \in M_1\}$. It is easy to check that $M = M' \oplus M_2$ and $N \leq M'$.

(2) $\Rightarrow$ (1). Let K be a finitely generated submodule of M_1, and $f : K \to M_2$ a homomorphism. Let $L = \{y - f(y) | y \in K\}$. Since K is finitely generated, then L is also a finitely generated submodule of M with $L \cap M_2 = 0$. By (ii), $M = L' \oplus M_2$ for some submodule L' of M such that $L \leq L'$. Let $\pi : M \to M_2$ denote the canonical projection (for the direct sum $M = L' \oplus M_2$). Let $\bar{f} = \pi|_{M_1} : M_1 \to M_2$ and, for any $y \in K$, we have $\bar{f}(y) = \bar{f}(y - f(y) + f(y)) = f(y)$. It means that $\bar{f}$ is an extension of f and so M_2 is M_1-F-injective. □

Lemma 2.3. [10, Lemma 6] *The following statements are equivalent for a module M.*

1. *M satisfies C3.*
2. *For all direct summands P, Q of M with $P \cap Q = 0$, there exists a submodule P' of M such that $M = P \oplus P'$ and $Q \leq P'$.*

Proposition 2.4. *An ef-extending module M has C3 if and only if whenever $M = M_1 \oplus M_2$ is a direct sum of submodules M_1, M_2, then M_2 is M_1-F-injective.*

Proof. ($\Rightarrow$) Assume that M is ef-extending satisfying C3. Let N be a finitely generated submodule N of M with $N \cap M_2 = 0$. Since M is ef-extending, there exists a direct summand N' of M such that N is essential in N'. Clearly $N' \cap M_2 = 0$. By Lemma 2.3, $M = M' \oplus M_2$ for some submodule M' such that $N' \leq M'$. Note that $N \leq N'$. Thus M_2 is M_1-F-injective by Lemma 2.2.

($\Leftarrow$) Assume that M_2 is M_1-F-injective whenever $M = M_1 \oplus M_2$. By Lemma 2.2 and Lemma 2.3, M satisfies C3. □

Corollary 2.5. *If $M = M_1 \oplus M_2$ is ef-extending, satisfies C3, then M_i is M_j-F-injective for all $i, j \in \{1, 2\}$, $i \neq j$.*

From this we have the following result.

Theorem 2.6. *The following conditions are equivalent for ring R:*

1. *R is QF.*
2. *$(R \oplus R)_R$ is ef-extending, satisfies C3 and R has ACC on right annihilators.*

Remark. Let p be a prime number. Then $\mathbb{Z}$-modules $\mathbb{Z}/p\mathbb{Z}$, $\mathbb{Z}/p^3\mathbb{Z}$ are ef-extending. But $\mathbb{Z}$-module $M = \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p^3\mathbb{Z}$ is not ef-extending. Because $(1 + p\mathbb{Z}, p + p^3\mathbb{Z})\mathbb{Z}$ is a closed submodule of M (which contains a finitely generated, essential submodule) and not a direct summand of M.

We next consider some properties of ef-extending rings.

Lemma 2.7. [19] *Every direct summand of an ef-extending module is ef-extending.*

Lemma 2.8. *Assume that $R_R = e_1R \oplus e_2R \oplus \cdots \oplus e_nR$, where each e_iR is uniform for all $i = 1, 2, \ldots, n$. If every monomorphism $R_R \longrightarrow R_R$ is an epimorphism, then R is semiperfect.*

Proof. By [14, Lemma 4.26]. □

A ring R is called *I-finite* if R contains no infinite orthogonal sets of idempotents (see [14]).

Lemma 2.9. *Assume that R is right AGP-injective, right ef-extending and I-finite. Then R is semiperfect.*

Proof. Since R is I-finite, there exists an orthogonal set of primitive idempotents $\{e_i\}_{i=1}^n$ such that $R_R = e_1R \oplus e_2R \oplus \cdots \oplus e_nR$. Since R is right ef-extending, e_iR is ef-extending and so e_iR is uniform for all $i = 1, 2, \ldots, n$. We will claim that every monomorphism $f : R \longrightarrow R$ is an epimorphism. Let $a = f(1)$. Then $r(a^n) = 0, \forall n \geq 1$. Assume that $aR \neq R$. Since R is right AGP-injective, there exist a positive integer $m \geq 1$ and $X_1 \leq {}_RR$ such that $a^m \neq 0$ and $lr(a^m) = Ra^m \oplus X_1$. It implies that $R = Ra^m \oplus X_1$ (since $r(a^m) = 0$) and so $Ra^m = Re$ for some $e^2 = e \in R$. Then

$$0 = r(a^m) = r(Ra^m) = r(Re) = r(e) = (1 - e)R,$$

and hence $e = 1$ or $Ra^m = R$. It implies that $R = Ra$, i.e., $ba = 1$ for some $b \in R$. If $ab \neq 1$, then by [12, Example 21.26], there some $e_{ij} = a^ib^j - a^{i+1}b^{j+1} \in R$, $i, j \in \mathbb{N}$ such that $e_{ij}e_{kl} = \delta_{jk}e_{il}$ for all $i, j, k \in \mathbb{N}$ where δ_{jk} are the Kronecker deltas. Notice $e_{ij} \neq 0$ for all $i, j \in \mathbb{N}$, by construction. Set $e_i = e_{ii}$. Then $e_ie_j = \delta_{ij}e_i$, $\forall i, j \in \mathbb{N}$. Therefore we have

$$e_1R \oplus e_2R \oplus \cdots \oplus e_nR \oplus \cdots,$$

this is a contradiction(because R has finite dimensional). Hence $ab = 1$ and so $aR = R$. This is a contradiction by our assumption. In short, f is an epimorphism. Then R is semiperfect by Lemma 2.8. □

From this lemma we have:

Theorem 2.10. *The following conditions are equivalent:*

(1) *R is QF.*
(2) *R is a left Noetherian, right GP-injective and right ef-extending ring.*
(3) *R is a right GP-injective, right ef-extending ring and satisfies ACC on right annihilators.*

Proof. (1) $\Rightarrow$ (2), (3) is clear.

(2) $\Rightarrow$ (1) By Lemma 2.9, R is semiperfect. But R is right GP-injective, $J = Z_r$ and so R is right C2 by [14, Example 7.18].

We have $R = e_1R \oplus \cdots \oplus e_nR$, $\{e_i\}_{i=1}^n$ is an orthogonal set of local idempotents. For every $i \neq j$ $(i, j \in \{1, 2, \ldots, n\})$ and $f : e_iR \to e_jR$ is a monomorphism.

Then $e_iR \cong f(e_iR) \leq e_jR$. Moreover, R satisfies the right C2, $f(e_iR)$ is a direct summand of e_jR or $f(e_iR) = e_jR$ (because e_jR is indecomposable). Hence f is an isomorphism. Since R is right ef-extending, then every uniform right ideal of R is essential in a direct summand of R_R. Therefore for every $i_0 \in \{1, 2, \ldots, n\}$, $\bigoplus_{\{1,2,\ldots,n\}\setminus\{i_0\}} e_iR$ is $e_{i_0}R$-injective by [6, Corollary 8.9]. Since e_iR is ef-extending, indecomposable and so e_iR is quasi-continuous. By [13, Theorem 2.13], R is right quasi-continuous. Thus R is QF by [4, Corollary 5].

(3) ⇒ (1) By [2, Theorem 3.7], R is left Artinian. Argument of proving (2) ⇒ (1) and [4, Theorem 5], it follows that R is QF. □

A ring R is called *left Johns* if R is left Noetherian such that every left ideal is a left annihilator. Since every left Johns ring is left Noetherian right P-injective, the next corollary follows from Theorem 2.10.

Corollary 2.11. *If R is left Johns, right ef-extending, then R is QF.*

Corollary 2.12. [3, Theorem 2.21] *If R is left Noetherian, right P-injective and right CS, then R is QF.*

A ring R is called *right mininjective* if $lr(a) = Ra$, where aR is a simple right ideal of R.

Proposition 2.13. *Let R be a right GP-injective, right ef-extending ring and satisfies ACC on left annihilators. If $Soc(R_R) \leq^e R_R$, then R is QF.*

Proof. By a similar proof of Theorem 2.10, R is semiperfect. Since R is right GP-injective, R is right mininjective. Hence R is right Kasch by [14, Theorem 3.12]. It follows that $\mathrm{Soc}(R_R) = \mathrm{Soc}({}_RR)$ by [2, Theorem 2.3]. Now will claim that R is left mininjective. In fact that, for every idempotent local $e \in R$. Since R is right ef-extending, eR is an ef-extending module and so uniform. It is easy to see that $\mathrm{Soc}(eR)$ is simple (because $\mathrm{Soc}(R_R) \leq^e R_R$). We have $e\mathrm{Soc}({}_RR) = eR \cap \mathrm{Soc}({}_RR) = eR \cap \mathrm{Soc}(R_R) = \mathrm{Soc}(eR)$ is simple. Therefore R is left mininjective by [14, Theorem 3.2]. Thus R is QF by [16, Theorem 2.7]. □

Note that in [17], the authors proved that if R is a right AGP-injective ring, satisfying ACC on left (or right) annihilators and $(R \oplus R)_R$ is ef-extending, then R is QF. But we do not know whether the condition "$\mathrm{Soc}(R_R) \leq^e R_R$" in above proposition can omit or not.

A ring R is called *left CF*, if every cyclic left R-module can be embedded in a free module. Now we consider the property of left CF, right ef-extending ring:

Proposition 2.14. *Let R be left CF, right ef-extending ring. Then following conditions are equivalent:*

(1) *R is QF.*
(2) *$J \leq Z_l$.*
(3) *$S_l \leq S_r$.*
(4) *R is a left mininjective ring.*

Proof. (1) $\Rightarrow$ (2), (1) $\Rightarrow$ (4) $\Rightarrow$ (3) are obvious.

(3) $\Rightarrow$ (1) Since R is left CF, R is right P-injective and left Kasch. Let T be a maximal left ideal of R. Since R is left Kasch, $r(T) \neq 0$. There exists $0 \neq a \in r(T)$ or $T \leq l(a)$ which yields $T = l(a)$ by maximality of T and so $r(T) = rl(a)$. Since R is right ef-extending, then $aR \leq^e eR$ for some $e^2 = e \in R$. On the other hand, $aR \leq rl(a) \leq eR$ and then $rl(a) \leq^e eR$. Hence $r(T) \leq^e eR$. It implies that R is semiperfect by [14, Lemma 4.1]. By Theorem 2.10, R is right continuous. Therefore $S_l \leq^e R_R$ by [21, Theorem 10]. By (3) $S_r \leq^e R_R$. It is easy to see that S_r is finitely generated as right R-module. Hence R is left finitely cogenerated by [14, Theorem 5.31]. Since R is left CF, it follows that R is left Artinian. Thus R is QF.

(2) $\Rightarrow$ (1) As above, R is semiperfect. So, by (2), $S_r = l(J) \geq l(Z_l) \geq S_l$. Arguing as above proves (1). □

J. L. Gómez Pardo and P. A. Guil Asensio proved that R is right PF iff R is injective cogenerator in Mod-R. For a right ef-extending ring R, we have:

Firstly we have the following lemma:

Lemma 2.15. [14, Lemma 1.54] *Let $P_R \neq 0$ be projective. Then the following are equivalent:*

(1) *$Rad(P)$ is a maximal submodule of P that is small in P.*
(2) *$End(P)$ is local.*

Now we prove the main result:

Theorem 2.16. *The following conditions are equivalent for a ring R:*

(1) *R is right PF.*
(2) *R is a right cogenerator, right ef-extending and I-finite.*

Proof. (1) $\Rightarrow$ (2) is clear.

(2) $\Rightarrow$ (1) By hypothesis, $R = u_1R\oplus\cdots\oplus u_nR$ where each u_iR is indecomposable. Since R is right ef-extending, u_iR is uniform for every $i = 1, 2, \ldots, n$. Hence R has right finite dimensional and right Kasch, let $\{K_1, K_2, \cdots, K_n\}$ be a set of representatives of the simple right R-modules. If we write $E_i = E(K_i)$, then $E_1, \cdots, E_n$ are pairwise nonisomorphic indecomposable injective modules. For each i, since R_R is cogenerator, there exists an embedding $\sigma : E(K_i) \longrightarrow R^{(I)}$ for some set I. Then $\pi\sigma \neq 0$ for some projection $\pi : R^{(I)} \longrightarrow R$, so $(\pi\sigma)|_{K_i} \neq 0$ and hence is monic. Thus $\pi\sigma : E(K_i) \longrightarrow R$ is monic, and so $E(K_i)$ is projective. Hence $End(E_i)$ is local for each i, and so by Lemma 2.15 shows that $Rad(E_i)$ is maximal and small in E_i. Hence $T_i = E_i/Rad(E_i)$ is simple and E_i is a projective cover of T_i. Moreover, if $T_i \cong T_j$ then $E_i \cong E_j$ by [14, Corollary B.17], and hence $i = j$. Thus $\{T_1, \cdots, T_n\}$ is a set of distinct representatives of the simple right R-modules and it follows that every simple right R-module has a projective cover. Thus R is semiperfect by [1, Lemma 25.4]. Let $\{e_1, \ldots, e_n\}$ be a basic set of local idempotents in R. Since each $E_i = E(e_iR/Rad(e_iR))$ is indecomposable and projective we have $E_i \cong e_{\sigma(i)}R$ for some $\sigma(i) \in \{1, \ldots, n\}$. Since the E_i are pairwise nonisomorphic, it follows that σ is a bijection and hence that each e_iR is injective with simple essential socle. Thus R is right self-injective with $\mathrm{Soc}(R_R) \leq^e R_R$ and so it is a right PF ring. □

Question. Whether the condition "I-finite" in Theorem 2.16 can omit or not?

Theorem 2.17. *The following conditions are equivalent:*

(1) *R is right and left PF.*
(2) *R is a left cogenerator and $(R \oplus R)_R$ is ef-extending.*
(3) *R is a right cogenerator and ${}_R(R \oplus R)$ is ef-extending.*

Proof. (1) $\Rightarrow$ (2), (3) is clear.

(2) $\Rightarrow$ (1) Since R is left cogenerator, R is left Kasch. Then by proving of [17, Theorem 2.8] or by proving of Theorem 2.10 and [14, Example 7.18], R is right self-injective. By [11, Theorem 12.1.1], R is two-sided PF.

(3) $\Rightarrow$ (1) By a similar proof of (2) $\Rightarrow$ (1). □

Acknowledgment. The authors would like to thank the referee for the valuable suggestions and comments.

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