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ON NEAR PSEUDO-VALUATION RINGS AND THEIR EXTENSIONS

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Abstract. Recall that a commutative ring R is said to be a pseudo-valuation ring (PVR) if every prime ideal of R is strongly prime. We say that a commutative ring R is near pseudo-valuation ring if every minimal prime ideal is a strongly prime ideal.

We also recall that a prime ideal P of a ring R is said to be divided if it is comparable (under inclusion) to every ideal of R. A ring R is called a divided ring if every prime ideal of R is divided.

Let R be a commutative ring, σ an automorphism of R and δ a σ-derivation of R. We say that a prime ideal P of R is δ-divided if it is comparable (under inclusion) to every σ-stable and δ- invariant ideal I of R. A ring R is called a δ-divided ring if every prime ideal of R is δ-divided. We say that a ring R is almost δ-divided ring if every minimal prime ideal of R is δ-divided. With this we prove the following:

Let R be a commutative Noetherian $\mathbb{Q}$-algebra ($\mathbb{Q}$ is the field of rational numbers), σ and δ as usual. Then:

(1) If R is a near pseudo valuation $\sigma(*)$- ring, then $R[x;\sigma,\delta]$ is a near pseudo valuation ring.
(2) If R is an almost δ-divided $\sigma(*)$-ring, then $R[x;\sigma,\delta]$ is an almost divided ring.

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1. Introduction

We follow the notation as in Bhat [9], but to make the paper self contained, we have the following:

All rings are associative with identity. Throughout this paper R denotes a commutative ring with identity $1 \neq 0$. The set of all nilpotent elements of R and the prime radical of R are denoted by $N(R)$ and $P(R)$ respectively. The set of prime ideals of R is denoted by $Spec(R)$ and the set of minimal prime ideals of R is denoted by $Min.Spec(R)$. The centre of R is denoted by $Z(R)$. The field of rational numbers and the ring of integers are denoted by $\mathbb{Q}$ and $\mathbb{Z}$ respectively unless otherwise stated. Let I and J be any two ideals of a ring R. Then $I \subset J$ means that I is strictly contained in J.

We recall that as in Hedstrom and Houston [11], an integral domain R with quotient field F, is called a pseudo-valuation domain (PVD) if each prime ideal P of R is strongly prime ($ab \in P$, $a \in F$, $b \in F$ implies that either $a \in P$ or $b \in P$).

For example let F = $\mathbb{Q}(\sqrt{2})$. Set V = $F + xF[[x]] = F[[x]]$. Then V is a pseudo-valuation domain. We also note that $S = \mathbb{Q} + \mathbb{Q}x + x^2V$ is not a pseudo-valuation domain Badawi [7]. For more details on pseudo-valuation rings and almost-pseudo valuation rings, the reader is refered to Badawi [7].

In Badawi, Anderson and Dobbs [3], the study of pseudo-valuation domains was generalized to arbitrary rings in the following way:

A prime ideal P of R is said to be strongly prime if aP and bR are comparable (under inclusion; i.e. $aP \subseteq bR$ or $bR \subseteq aP$) for all $a, b \in R$. A ring R is said to be a pseudo-valuation ring (PVR) if each prime ideal P of R is strongly prime. We note that a PVR is quasilocal by Lemma 1(b) of Badawi, Anderson and Dobbs [3]. An integral domain is a PVR if and only if it is a PVD by Proposition 3.1 of Anderson [1], Proposition 4.2 of Anderson [2] and Proposition 3 of Badawi [4]. We denote the set of strongly prime ideals of R by $S.Spec(R)$.

In Badawi [5], another generalization of PVDs is given in the following way:

For a ring R with total quotient ring Q such that $N(R)$ is a divided prime ideal of R, let $\phi : Q \to R_{N(R)}$ such that $\phi(a/b) = a/b$ for every $a \in R$ and every $b \in R\backslash Z(R)$. Then ϕ is a ring homomorphism from Q into $R_{N(R)}$, and ϕ restricted to R is also a ring homomorphism from R into $R_{N(R)}$ given by $\phi(r) = r/1$ for every $r \in R$. Denote $R_{N(R)}$ by T. A prime ideal P of $\phi(R)$ is called a T-strongly prime ideal if $xy \in P$, $x \in T$, $y \in T$ implies that either $x \in P$ or $y \in P$. $\phi(R)$ is said to be a T-pseudo-valuation ring (T-PVR) if each prime ideal of $\phi(R)$ is T-strongly prime. A prime ideal S of R is called ϕ-strongly prime ideal if $\phi(S)$ is a T-strongly prime ideal of $\phi(R)$. If each prime ideal of R is ϕ-strongly prime, then R is called a ϕ-pseudo-valuation ring ($\phi - PVR$).

This article concerns the study of skew polynomial rings over PVDs. Let R be a ring, σ an automorphism of R and δ a σ-derivation of R ($\delta : R \to R$ is an additive map with $\delta(ab) = \delta(a)\sigma(b) + a\delta(b)$, for all $a, b \in R$).

For example let σ be an automorphism of a ring R and $\delta : R \to R$ any map. Let $\phi : R \to M_2(R)$ be defined by

$$\phi(r) = \begin{pmatrix} \sigma(r) & 0 \\ \delta(r) & r \end{pmatrix}, \text{ for all } r \in R.$$

Then δ is a σ-derivation of R.

We denote the Ore extension $R[x;\sigma,\delta]$ by $O(R)$. If I is an ideal of R such that I is σ-stable; i.e. $\sigma(I) = I$ and I is δ-invariant; i.e. $\delta(I) \subseteq I$, then we denote $I[x;\sigma,\delta]$ by $O(I)$. We would like to mention that $R[x;\sigma,\delta]$ is the usual set of polynomials with coefficients in R, i.e. $\{\sum_{i=0}^{n} x^i a_i,\ a_i \in R\}$ in which multiplication is subject to the relation $ax = x\sigma(a) + \delta(a)$ for all $a \in R$.

In case δ is the zero map, we denote the skew polynomial ring $R[x;\sigma]$ by $S(R)$ and for any ideal I of R with $\sigma(I) = I$, we denote $I[x;\sigma]$ by $S(I)$.
In case σ is the identity map, we denote the differential operator ring $R[x;\delta]$ by $D(R)$ and for any ideal J of R with $\delta(J) \subseteq J$, we denote $J[x;\delta]$ by $D(J)$.

Ore-extensions (skew-polynomial rings and differential operator rings) have been of interest to many authors. For example see [9,10,12].

In this note we define a near pseudo-valuation ring (NPVR) in the following way:

Definition 1.1. Let R be a ring. We say that R is a *near pseudo-valuation ring (NPVR)* if each minimal prime ideal P of R is strongly prime. For example a reduced ring is NPVR.

Here the term near may not be interpreted as near ring Bell and Mason [8]. We note that a near pseudo-valuation ring (NPVR) is a pseudo-valuation ring (PVR), but the converse is not true. For example a reduced ring is a NPVR, but need not be a PVR.

We recall that a prime ideal P of R is said to be divided if it is comparable (under inclusion) to every ideal of R. A ring R is called a divided ring if every prime ideal of R is divided Badawi [6]. It is known Lemma 1 of Badawi, Anderson and Dobbs [3] that a pseudo-valuation ring is a divided ring.

We define an almost divided ring in the following way:

Definition 1.2. Let R be a ring. We say that R is an *almost divided ring* if every minimal prime ideal of R is divided.

We also recall that a prime ideal P of R is σ-divided if it is comparable (under inclusion) to every σ-stable ideal I of R. A ring R is called a σ-divided ring if every prime ideal of R is σ-divided (see Bhat [9]).

We define an almost σ-divided ring and an almost δ-divided ring in the following way:

Definition 1.3. Let R be a ring. We say that R is *almost σ-divided ring* if every minimal prime ideal of R is σ-divided.

We say that a prime ideal P of R is δ-divided if it is comparable (under inclusion) to every σ-stable and δ- invariant ideal I of R. A ring R is called a δ-divided ring if every prime ideal of R is δ-divided.

Definition 1.4. Let R be a ring. We say that R is *almost δ-divided ring* if every minimal prime ideal of R is δ-divided.

The author of this paper has proved in Theorems 2.6 and Theorem 2.8 of [9] the following:

Let R be a ring and σ an automorphism of R. Then:

(1) If R is a commutative pseudo-valuation ring such that $x \notin P$ for any $P \in Spec(S(R))$, then $S(R)$ is also a pseudo-valuation ring.
(2) If R is a σ-divided ring such that $x \notin P$ for any $P \in Spec(S(R))$, then $S(R)$ is also a σ-divided ring.

In Theorems 2.10 and Theorem 2.11 of [9] the following results have been proved:

Let R be a commutative Noetherian $\mathbb{Q}$-algebra and δ a derivation of R. Then:

(1) If R is a pseudo-valuation ring, then $D(R)$ is also a pseudo-valuation ring.
(2) If R is a divided ring, then $D(R)$ is also a divided ring.

Main Results

In this paper an analogue of the above results for near Pseudo-valuation rings, almost divided rings and almost δ-divided rings has been given. Before we state the results, we recall that in [12], Kwak defines a $\sigma(*)$-ring R to be a ring in which $a\sigma(a) \in P(R)$ implies $a \in P(R)$ for $a \in R$.

Example 1.5. Let R $= \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$, where F is a field. Then P(R) $= \begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix}$

Let $\sigma : R \to R$ be defined by $\sigma\Big(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\Big) = \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix}$. Then it can be seen that R is a $\sigma(*)$-ring.

With this we prove the following:

Let R be a Noetherian ring, which is also an algebra over $\mathbb{Q}$. Let σ be an automorphism of R such that R is a $\sigma(*)$-ring and δ be a σ-derivation of R. Then:

(1) If R be a near pseudo-valuation ring, then $O(R)$ is also a near pseudo-valuation ring.
(2) If R be an almost δ-divided ring, then $O(R)$ is also an almost δ-divided ring.

These results have been proved in Theorems 2.5 and 2.7 respectively.

2. Polynomial rings

We recall that an ideal I of a ring R is called completely semiprime if $a^2 \in I$ implies $a \in I$, where $a \in R$.

Proposition 2.1. *Let R be a Noetherian ring which is also an algebra over $\mathbb{Q}$. Let σ be an automorphism of R such that R is a $\sigma(*)$-ring and δ a σ-derivation of R. Then $\delta(U) \subseteq U$ for all $U \in Min.Spec(R)$.*

Proof. We will first show that P(R) is completely semiprime. Let $a \in R$ be such that $a^2 \in P(R)$. Then

$$a\sigma(a)\sigma(a\sigma(a)) = a\sigma(a)\sigma(a)\sigma^2(a) \in \sigma(P(R)) = P(R).$$

Therefore $a\sigma(a) \in P(R)$ and hence $a \in P(R)$.

We now show that $\sigma(U) = U$ for all $U \in Min.Spec(R)$. Let $U = U_1$ be a minimal prime ideal of R. Let $U_2, U_3, ..., U_n$ be the other minimal primes of R. Suppose that $\sigma(U) \neq U$. Then $\sigma(U)$ is also a minimal prime ideal of R. Renumber so that $\sigma(U) = U_n$. Let $a \in \cap_{i=1}^{n-1} U_i$. Then $\sigma(a) \in U_n$, and so $a\sigma(a) \in \cap_{i=1}^{n} U_i = P(R)$. Now P(R) is completely semiprime implies that $a \in P(R)$, and thus $\cap_{i=1}^{n-1} U_i \subseteq U_n$, which implies that $U_i \subseteq U_n$ for some $i \neq n$, which is impossible. Hence $\sigma(U) = U$.

Let now $T = \{a \in U \mid \text{such that } \delta^k(a) \in U \text{ for all integers } k \geq 1\}$. Then T is a δ-invariant ideal of R. Now it can be seen that $T \in Spec(R)$. Now $T \subseteq U$, so $T = U$ as $U \in Min.Spec(R)$. Hence $\delta(U) \subseteq U$. □

Lemma 2.2. *Let R be a Noetherian ring which is also an algebra over $\mathbb{Q}$. Let σ be an automorphism of R such that R is a $\sigma(*)$-ring and δ a σ-derivation of R. Then*

(1) *If U is a minimal prime ideal of R, then O(U) is a minimal prime ideal of of O(R) and $O(U) \cap R = U$.*

(2) *If P is a minimal prime ideal of O(R), then $P \cap R$ is a minimal prime ideal of R.*

Proof. (1) Let U be a minimal prime ideal of R. Then by Proposition 2.1 $\sigma(U) = U$ and $\delta(U) \subseteq U$. Now on the same lines as in Theorem 2.22 of Goodearl and Warfield [10] we have $O(U) \in Spec(O(R))$. Suppose $L \subset O(U)$ be a minimal prime ideal of $O(R)$. Then $L \cap R \subset U$ is a prime ideal of R, a contradiction. Therefore $O(U) \in Min.Spec(O(R))$. Now it is easy to see that $O(U) \cap R = U$.

(2) We note that $x \notin P$ for any prime ideal P of $O(R)$ as it is not a zero divisor. Now the proof follows on the same lines as in Theorem 2.22 of Goodearl and Warfield [10] using Lemma 2.1 and Lemma 2.2 of Bhat [9] and Proposition 2.1. □

We note that the above Lemma is true even if R is noncommutative.

Theorem 2.3. *(Hilbert Basis Theorem): Let R be a right/left Noetherian ring. Let σ and δ be as usual. Then the ore extension $O(R) = R[x, \sigma, \delta]$ is right/left Noetherian.*

Proof. See Theorem 1.12 of Goodearl and Warfield [10]. □

Proposition 2.4. *Let R be a ring, σ an automorphism of R and δ a σ-derivation of R. Then:*

(1) *For any strongly prime ideal P of R with $\delta(P) \subseteq P$ and $\sigma(P) = P$, O(P) is a strongly prime ideal of O(R).*
(2) *For any strongly prime ideal U of O(R), $U \cap R$ is a strongly prime ideal of R.*

Proof. See Proposition 2.5 of Bhat [9]. □

We note that the above Proposition is true even if R is noncommutative.

It is known (Theorem 2.6 of Bhat [9]) that if R is a commutative PVR such that $x \notin P$ for any $P \in Spec(S(R))$. Then $S(R)$ is also a PVR. It is also known (Theorem 2.10 of Bhat [9]) that if R is a commutative Noetherian $\mathbb{Q}$-algebra which is also a PVR. Then $D(R)$ is also a PVR. We generalize these results over a NPVR for $O(R)$ without using the hypothesis that $x \notin P$ for any $P \in Spec(S(R))$. Towards this we prove the following:

Theorem 2.5. *Let R be a Noetherian near pseudo valuation ring which is also an algebra over $\mathbb{Q}$. Let σ be an automorphism of R such that R is a $\sigma(*)$-ring and δ a σ-derivation of R. Then $O(R)$ is a Noetherian near pseudo-valuation ring.*

Proof. $O(R)$ is Noetherian by Theorem 2.3. We note that $x \notin P$ for any prime ideal P of $O(R)$ as it is not a zero divisor. Let $J \in Min.Spec(S(R))$. Then by Lemma 2.2 $J \cap R \in Min.Spec(R)$. Now R is a near commutative Noetherian pseudo-valuation $\mathbb{Q}$-algebra, therefore $J \cap R \in S.Spec(R)$. Also $\sigma(J \cap R) = J \cap R$ and $\delta(J \cap R) \subseteq J \cap R$ by Proposition 2.1. Now Proposition 2.4 implies that $O(J \cap R) \in S.Spec(O(R))$. Now it is easy to see that $O(J \cap R) = J$. Therefore $J \in S.Spec(O(R))$. Hence $O(R)$ is a Noetherian pseudo-valuation ring. □

Corollary 2.6. *Let R be a Noetherian near pseudo valuation ring which is also an algebra over $\mathbb{Q}$, σ and δ as usual such that $\sigma(U) = U$ for all $U \in Min.Spec(R)$. Then $O(R)$ is a Noetherian near pseudo-valuation ring.*

Proof. $O(R)$ is Noetherian by Theorem 2.3. We note that $x \notin P$ for any prime ideal P of $O(R)$ as it is not a zero divisor. Let $J \in Min.Spec(O(R))$. Now $\sigma(J) = J$, therefore, Proposition 2.1 and Lemma 2.2 imply that $J \cap R \in Min.Spec(R)$. Now R is a near commutative Noetherian pseudo-valuation $\mathbb{Q}$-algebra, therefore $J \cap R \in S.Spec(R)$. Also $\sigma(J \cap R) = J \cap R$ and $\delta(J \cap R) \subseteq J \cap R$ by Proposition 2.1. Now Proposition 2.4 implies that $O(J \cap R) \in S.Spec(O(R))$. Now it is easy to see that $O(J \cap R) = J$. Therefore $J \in S.Spec(O(R))$. Hence $O(R)$ is a Noetherian pseudo-valuation ring. □

It is known (Theorem 2.8 of Bhat [9]) that if R is a σ-divided Noetherian ring such that $x \notin P$ for any $P \in Spec(S(R))$. Then $S(R)$ is also σ-divided Noetherian. It is also known (Theorem 2.11 of Bhat [9]) that if R be a divided commutative Noetherian $\mathbb{Q}$-algebra. Then $D(R)$ is also divided Noetherian. We generalize these results and prove the following:

Theorem 2.7. *Let R be a Noetherian ring which is also an algebra over $\mathbb{Q}$. Let σ be an automorphism of R such that R is a $\sigma(*)$-ring and δ a σ-derivation of R such that R is an almost δ-divided ring. Then $O(R)$ is a Noetherian almost δ-divided ring.*

Proof. $O(R)$ is Noetherian by Theorem 2.3. Now we note that σ can be extended to an automorphism of $O(R)$ such that $\sigma(x) = x$ and δ can be extended to a σ-derivation of $O(R)$ such that $\delta(x) = 0$. Let $J \in Min.Spec(O(R))$ and $0 \neq K$ be a proper ideal of $O(R)$ such that $\sigma(K) = K$ and $\delta(K) \subseteq K$. Now by Lemma 2.2 $J \cap R \in Min.Spec(R)$. Also Proposition 2.1 implies that $\sigma(J \cap R) = (J \cap R)$ and $\delta(J \cap R) \subseteq (J \cap R)$. Also $K \cap R$ is an ideal of R with $\sigma(K \cap R) = (K \cap R)$ and $\delta(K \cap R) \subseteq (K \cap R)$. Now R is almost δ-divided, therefore $J \cap R$ and $K \cap R$ are comparable under inclusion. Say $(J \cap R) \subseteq (K \cap R)$. Therefore $O(J \cap R) \subseteq O(K \cap R)$. Thus $J \subseteq K$. Hence $O(R)$ is a Noetherian almost δ-divided ring. □

Corollary 2.8. *If R is a Noetherian almost σ-divided $\sigma(*)$-ring, then $S(R)$ is a Noetherian almost σ-divided ring.*

Question 2.9. Let R be a NPVR. Let σ be an automorphism of R and δ a σ-derivation of R. Is O(R) $= R[x; \sigma, \delta]$ a NPVR (even if R is commutative Noetherian)?

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