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ON THE DIHEDRAL (C0)HOMOLOGY FOR SCHEMES

H. N. Alaa and Y. Gh. Gouda

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Abstract. In this paper we define the dihedral (co)homology of schemes over a commutative ring k by sheafifying the (co)dihedral complex. We study the Mayer-Vietoris sequence of dihedral (co)homology and introduce the relation between the cyclic and dihedral (co)homology of schemes

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1. Introduction

We recall some basic facts of schemes from [8] and dihedral homology groups of [3,4,6,7].

Definition 1.1. Let k be an algebraic closed field and R be a ring with unit. We shall denote the set of all prime ideals over R by $spec(R)$ and, for any ideal P in R , we denote by $V(P)$ the set of all prime ideal in $spec(R)$ containing P . These sets define a topology. It's called a *Zariski topology.*

Definition 1.2. A *ringed space* is a pair (X, θ_X) where X is a topological space and θ_X is the structure of sheaf on X. The space (X, θ_X) is called *locally ringed space* if the stalks $\theta_{x|X}$ are locally rings for any $x \in X$.

Definition 1.3. A locally ringed space (X, θ_X) is called *affine space* if $(X, \theta_X) = (spec(R), \theta_{spec(R)})$ and a *scheme* if it has an open covering $X = \cup_{i \in I}$ such that is an affine scheme for $i \in I$.

Definition 1.4. Given a pair $(spec(R), \theta_{spec(R)})$ called locally ringed space. For any scheme X, the structure sheaf θ_X is defined to be the ring of all regular functions denoted by $\theta_X(U) = \{f \mid f : U \to U_{p \in U} A_p, U \in X\}$, where A_p is the local ring on X at p.

We define an involution $*$ on the sheaf $\theta_X$as an involution is an automorphism of order two by considering the inverse regular function $* : \theta_X \to \theta_X$, that is satisfy:

$(*)^2 = id$, $f^* = f^{-1}$,and $(fg)^* = g^{-1}f^{-1}$. The scheme X with this property on sheaf θ_X is called a scheme with an involution.

Now we review briefly the notion of dihedral modules before we consider it in the context of schemes.

Definition 1.5. [2] The dihedral category ΔD has objects $[n], n \in N$ and the following family of morphisms:

$$\delta_n^i : [n-1] \to [n], \sigma_n^j : [n+1] \to [n] \tag{1}$$

$$\tau_n : [n] \to [n], \rho_n : [n] \to [n] \tag{2}$$

such that the following framework are hold.:

$$\delta_{n+1}^j \delta_n^i = \delta_{n+1}^i \delta_n^{j-1},\ if\ i \prec j \tag{3}$$

$$\sigma_n^j \sigma_{n+1}^i = \sigma_n^i \sigma_{n+1}^{j+1},\ if\ i \leq j \tag{4}$$

$$\sigma_n^j \delta_{n+1}^i = \begin{cases} \delta_{n-1}^i \sigma_{n-2}^{j-1}, & if \quad i \prec j \\ Id_{[n]}, & if \quad i = j\ or\ i = j+1 \\ \delta_{n+1}^{i-1} \sigma_n^j, & if \quad i \succ j+1 \end{cases} \tag{5}$$

$$\tau_n \delta_n^{i-1} = \delta_n^{i-1} \tau_{n-1}\ ,\ 1 \leq i \leq n \tag{6}$$

$$\tau_n \sigma_n^j = \delta_n^{j-1} \tau_{n+1}\ ,\ 1 \leq j \leq n \tag{7}$$

$$\tau_n^{n+1} = Id_{[n]} \tag{8}$$

$$\rho_n \delta_n^i = \delta_n^{i-1} \rho_{n-1}\ ,\ 0 \leq i \leq n \tag{9}$$

$$\rho_n \sigma_n^j = \sigma_n^{j-1} \rho_{n+1}\ ,\ 0 \leq j \leq n \tag{10}$$

$$\rho_n^2 = Id_{[n]} \tag{11}$$

$$\tau_n \rho_n = \rho_n \tau_n^{-1}. \tag{12}$$

Definition 1.6. Let ζ be an arbitrary category. A *dihedral object of the category* ζ is a functor $V : \Delta D^{op} \to \zeta$, such that $V(n) = X_n, V(\delta_n^i) = d_n^i, V(\sigma_n^j) = S_n^j, V(\tau_n) = t_n$ and $V(\rho_n) = r_n($ ΔD^{op} is the inverse of ΔD). If ζ is a category of modules, then the dihedral object is called a *dihedral module*.

Note that the morphisms $\{d_n^i, S_n^j, t_n, r_n\}$ satisfy the relations 3-12.

Definition 1.7. Let $M = \{M_n\}$ be a dihedral $k-module$. The *dihedral homology groups* of M is given by:

$$HD_n = Tor_n^{k[\Delta D^{op}]}(K^D, M)\ ,\ n \geq 0 \tag{13}$$

where K^D is trivial dihedral $k-module$.

2. Dihedral cohomology of schemes

In this section we define the dihedral homology of schemes and study some of its properties.

Definition 2.1. [8] A sheaf of θ_X-modules is a sheaf $\Im$ on X, such that for each open set $u \in X$, the group $\Im(u)$ is a $\theta_X(u)$-module and for each inclusion of open sets $v \subseteq u$ the restriction homomorphism $\Im(u) \to \Im(v)$ is compatible with the module structures via the ring. The set of all sheaves of θ_X-module defines a category, called a category of sheaves of modules and denoted by $Mod(\theta_X)$.

Definition 2.2. The dihedral module of sheaves is a functor $F : \Delta D^{op} \to Mod(\theta_X)$ such that

$$F([n]) = \theta_X^{\otimes(n+1)} \tag{14}$$

$$F(\delta_n^i) = d_n^i : \theta_X^{\otimes(n+1)} \to \theta_X^{\otimes(n)} \tag{15}$$

$$d_n^i(f_0 \otimes f_1 \otimes ... \otimes f_n) = (f_0 \otimes f_1 \otimes ... \otimes f_i f_j \otimes ... \otimes f_n),\ 0 \le i \le 1 \tag{16}$$

$$F(\sigma_n^j) = S_n^j : \theta_X^{\otimes(n-1)} \to \theta_X^{\otimes(n)} \tag{17}$$

$$S_n^j(f_0 \otimes f_1 \otimes ... \otimes f_n) = (f_0 \otimes f_1 \otimes ... \otimes f_i \otimes id \otimes f_{i+1} \otimes ... \otimes f_n),\ 0 \le i \le 1 \tag{18}$$

$$F(\tau_n) = t_n : \theta_X^{\otimes(n)} \to \theta_X^{\otimes(n)} \tag{19}$$

$$t_n(f_0 \otimes f_1 \otimes ... \otimes f_n) = (f_n \otimes f_0 \otimes ... \otimes f_{n-1}), \tag{20}$$

$$F(\rho_n) = r_n : \theta_X^{\otimes(n)} \to \theta_X^{\otimes(n)} \tag{21}$$

$$t_n(f_0 \otimes f_1 \otimes ... \otimes f_n) = \alpha(f_0^{-1} \otimes f_n^{-1} \otimes ... \otimes f_1^{-1}),\ \alpha = \pm 1 \tag{22}$$

with the following:

$$b_n = \sum_{i=0}^{n} (-1)^i d_i,\ b_n^{'} = \sum_{i=0}^{n-1} (-1)^i d_i \tag{23}$$

$$T_n = (-1)^n t^n\ ,\ N = 1 + t + ... + t^{n-1} \tag{24}$$

$$R_n = (-1)^{\frac{n(n+1)}{2}} r_n. \tag{25}$$

We can construct the tricomplex of sheaves $({}^{\alpha}CD^n(\theta_X, \delta))$, $\alpha = \pm 1$, (see [5]) where $\delta = \delta_1 + \delta_2 + \delta_3$, and:

$$\delta_1 = \begin{cases} b_n \\ -b_n^{'} \end{cases} : \theta_X^{\otimes(n)} \to \theta_X^{\otimes(n-1)} \tag{26}$$

$$\delta_2 = \begin{cases} 1 - T_n \\ N \end{cases} : \theta_X^{\otimes(n)} \to \theta_X^{\otimes(n)} \tag{27}$$

$$\delta_3 = \begin{cases} 1 - R_n \\ -1 - R_n T_n \\ 1 + R_n T_n \\ 1 - R_n \end{cases} : \theta_X^{\otimes(n)} \to \theta_X^{\otimes(n)} \tag{28}$$

Clearly $(\delta_i)^2 = 0$, $i = 1, 2, 3$.

In order to define the dihedral cohomology of schemes, we make a use of the hyperhomology of define in [9].

Definition 2.3. The dihedral homology of scheme X over a commutative ring k is the hyperhomology of the total complex of the tricomplex of sheaves $({}^{\alpha}CD^n(\theta_X, \delta))$:

$${}^{\alpha}HD^*(X) = H^*(Tot(CD^n(\theta_x), X)) \tag{29}$$

where

$$Tot(CD^n(\theta_x)) = \frac{\theta_X^n}{\mathrm{Im}(1-R) + \mathrm{Im}(1-T)}, \; n = 0, 1, 2, ..., \; \alpha = \pm 1. \tag{30}$$

Remark 2.4. If we droped the operator $\rho_n : [n] \to [n]$ in Definition 1.5 we get the cyclic category, cyclic module and for a scheme X we obtain the cyclic homology of a scheme (see [9]).

3. The Mayer-Vietoris sequence for dihedral homology of schemes

In this part we, first establish a lemma which use to prove a theorem of Mayer-Vietoris sequence for dihedral homology of schemes.

Lemma 3.1. [1] *The following sequence is exact :*

$$0 \to_{\alpha} CD^n(\theta_X) \underset{\longrightarrow}{J}{}_{\alpha}CD^n(\theta_{X_1}) \oplus_{\alpha} CD^n(\theta_{X_2}) \underset{\longrightarrow}{I}{}_{\alpha}CD^n(\theta_{X_1} \cap \theta_{X_2}) \to 0, \; \alpha = \pm 1 \tag{31}$$

where $\theta_X = \theta_{X_1} \cap \theta_{X_2}$ *,* ${}_{\alpha}CD^n$ *is the dihedral complex, and* $J = J_1 - J_2, I = I_1 + I_2$*, are defined by:*

$$I_1 : X_1 \cup X_2 \to X_1 \;,\; I_2 : X_1 \cup X_2 \to X_2 \;,\; J_1 : X_1 \to X_1 \cap X_2 \;,\; J_2 : X_2 \to X_1 \cap X_2. \tag{32}$$

Proof. Clearly, J is an epimorphism and I is a monomorphism and $JoI = o$. Let $(\theta_{X_1} \cap \theta_{X_2})(u) \in CD^n(\theta_X)(u)$,then,

$$(JoI)(\theta_{X_1 \cup X_2})(u) = J(\theta_{X_1}(u), \theta_{X_2}(u)) = 0$$

where $(\theta_{X_1}(u), \theta_{X_2}(u)) \in [CD^n(\theta_{X_1}) \oplus CD^n(\theta_{X_2})]$ and $(\theta_{X_1} \cap \theta_{X_2})(u) \in {}_{\alpha}CD^n(\theta_{X_1} \cap \theta_{X_2})$. □

Theorem 3.2. *If $X = X_1 \cup X_2$ where X_1 and X_2 are open subsets of scheme X and the diagram*

$$\begin{array}{ccc} X & \rightarrow & X_1 \\ \uparrow & & \uparrow \\ X_2 & \rightarrow & X_1 \cap X_2 \end{array} \tag{33}$$

is commutative, then there exists the following long exact sequence:

$$... \rightarrow {}^{\alpha}HD_n(X_1 \cup X_2) \overset{J^*}{\rightarrow} HD_n(X_1) \oplus HD_n(X_2) \overset{I^*}{\rightarrow} HD_n(X_1 \cap X_2) \overset{E}{\rightarrow} HD_{n-1}(X_1 \cup X_2) \rightarrow ... \tag{34}$$

where $I^* = (I_1^*, I_2^*), J^* = (J_1^*, J_2^*), E$ is a connecting homomorphism.

Proof. The exact sequences

$$0 \rightarrow {}_{\alpha}CD^n(\theta_X) \overset{J}{\rightarrow} {}_{\alpha}CD^n(\theta_{X_1}) \oplus {}_{\alpha}CD^n(\theta_{X_2}) \overset{I}{\rightarrow} {}_{\alpha}CD^n(\theta_{X_1} \cap \theta_{X_2}) \rightarrow 0, \ \alpha = \pm 1 \tag{35}$$

induce the following long exact sequence of dihedral groups:

$$... \rightarrow {}^{\alpha}HD_n(X_1 \cup X_2) \overset{J^*}{\rightarrow} HD_n(X_1) \oplus HD_n(X_2) \overset{I^*}{\rightarrow} HD_n(X_1 \cap X_2) \overset{E}{\rightarrow} HD_{n-1}(X_1 \cup X_2) \rightarrow ... \tag{36}$$

Since $EoJ^* = 0$, this ends the proof. □

4. The relation between cyclic and dihedral homology of schemes

We extend the relation between the cyclic and dihedral homology from algebra [5] or [7] to all schemes over ring with identity and involution. This fact can be given by the following theorem.

Theorem 4.1. *Let $\Im$ be a sheaf of θ_X-modules, and let X be a scheme over unital ring k with involution. Then the relation between the cyclic and dihedral cohomology groups is given by:*

$$... \rightarrow {}_{-\alpha}HD_n(X, \Im) \overset{i^*}{\rightarrow} HC_n(X, \Im) \overset{j^*}{\rightarrow} {}_{\alpha}HD_n(X, \Im) \rightarrow {}_{\alpha}HD_{n-1}(X, \Im) \rightarrow ... \tag{37}$$

where j^* is a connecting homomorphism.

Proof. For a scheme X, let $\zeta(X, \Im)$ be the total complex of Connes double complex [9]. We embed the complex $\zeta(X, \Im)$ in the tricomplex $D(X, \Im)$ (see[4]), passing to the total complexes associated with and, we get the following short exact sequence

$$0 \rightarrow Tot\zeta(X, \Im) \rightarrow Tot\,{}^{\alpha}D(X, \Im) \rightarrow Tot\,{}^{-\alpha}D(X, \Im)[-4] \rightarrow 0 \tag{38}$$

This sequence induces the following long exact sequence which relates the cyclic and dihedral cohomology groups:

$$... \to {}_{-\alpha}HD_n(X,\Im) \xrightarrow{i^*} HC_n(X,\Im) \xrightarrow{j^*} {}_{\alpha}HD_n(X,\Im) \to {}_{\alpha}HD_{n-1}(X,\Im) \to ... \tag{39}$$

when 2 is invertible in the ground ring k we get the following exact sequence

$$0 \to {}_{-\alpha}HD_n(X,\Im) \to HC_n(X,\Im) \to {}_{\alpha}HD_n(X,\Im) \to 0. \tag{40}$$

□

5. Cohomology groups

In this part we are concerned with the dihedral cohomology groups. It's necessary to translate the definitions and results of a pervious discussion in the cohomological framework because there is an interesting pairing between homology and cohomology groups. It's well known, in cyclic cohomology case, that if A is a unital associated $k-algebra$ and $A^* = Hom(A,k)$, then its cochain complex is $C^*(A) = Hom(A^{\otimes(n+1)},k)$. The dualizing of the Connes bicomplex $CC_*(A)$ gives a bicomplex of cochains $CC^{**}(A)$ and its homology gives the cyclic cohomology group. The dihedral cohomology group can be defined in the same manner. Achive this, we replace the category ΔD^{op} by ΔD in the Definitions 1.6, 1.7 and 2.2, then we get the dihedral cohomology group $HD^n(M) = Ext^n_{k[\Delta D]}(M,k^D), n \geq 0$, where K^D is trivial dihedral $k-module$. Also the dihedral cohomology of schemes X over a commutative ring k is the hypercohomology of the total complex of the tricomplex of sheaves $({}^{\alpha}CD^n(\theta_X,\delta_i)) : {}^{\alpha}HD^*(X) = H^*(Tot(CD^n(\theta_x),X))$, where

$$Tot(CD^n(\theta_x)) = \frac{\theta_X^n}{\mathrm{Im}(1-R)+\mathrm{Im}(1-T)},\ n = 0,1,2,...,\ \alpha = \pm 1. \tag{41}$$

The Theorem 3.2 of the Mayer-Vietoris sequence for dihedral homology and Theorem 4.1 of the relation between cyclic and dihedral homology of schemes can be translated to cohomology case.

Similar arguments as those used in the proof of Theorem 3.2 give the following.

Theorem 5.1. [6] *If $X = X_1 \cup X_2$ where $X_1$and X_2 are open subsets of scheme X and the diagram*

$$\begin{array}{ccc} X & \to & X_1 \\ \uparrow & & \uparrow \\ X_2 & \to & X_1 \cap X_2 \end{array} \tag{42}$$

is commutative, then there exist the following long exact sequence:

$$... \to {}^{\alpha}HD^n(X_1 \cup X_2) \xrightarrow{I^*} HD^n(X_1) \oplus HD^n(X_2) \xrightarrow{J^*} HD^n(X_1 \cap X_2) \xrightarrow{E} HD^{n+1}(X_1 \cup X_2) \to \dots \quad (43)$$

where $I^* = (I_1^*, I_2^*)$, $J^* = (J_1^*, J_2^*)$, E *is a connecting homomorphism.*

Similar arguments as those used in the proof of Theorem 4.1 give the following.

Theorem 5.2. *Let* $\Im$ *be a sheaf of* θ_X*-modules, and let* X *be a scheme over unital ring* k *with involution. Then the relation between the cyclic and dihedral cohomology groups is given by:*

$$... \to {}^{-\alpha}HD^n(X, \Im) \xrightarrow{i^*} HC^n(X, \Im) \xrightarrow{j^*} {}^{\alpha}HD^n(X, \Im) \to {}^{\alpha}HD^{n+1}(X, \Im) \to ... \quad (44)$$

where j^* *is a connecting homomorphism.*

Remark 5.3. When 2 is invertible in the ground ring k we get the following exact sequence

$$0 \to {}^{-\alpha}HD^n(X, \Im) \to HC^n(X, \Im) \to {}^{\alpha}HD^n(X, \Im) \to 0.$$

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Alaa Hassan Nor-Eldean and **Y. Gh. Gouda**
Department of Mathematics
Faculty of Science
Aswan-South Valley University
Aswan, Egypt
e-mails: ala2222000@yahoo.com (Alaa Hassan Nor-Eldean)
yasiengouda@yahoo.com (Y.Gh. Gouda)