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DIAGONALIZATION OF REGULAR MATRICES OVER EXCHANGE RINGS

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Abstract. In this paper, we establish several necessary and sufficient conditions under which every regular matrix admits a diagonal reduction. We prove that every regular matrix over an exchange ring R admits diagonal reduction if and only if for any $m, n \in \mathbb{N}$ $(m \geq n+1)$ and any regular $X \in M_{m\times n}(R)$, $\begin{pmatrix} X & \mathbf{0}_{m\times(m-n)} \end{pmatrix} \in M_m(R)$ is unit-regular if and only if for any $m, n \in \mathbb{N}$ $(m \geq n+1)$ and any regular $X \in M_{m\times n}(R)$, there exist an idempotent $E \in M_m(R)$ and a completed $U \in M_{m\times n}(R)$ such that $X = EU$ if and only if for any idempotents $e \in R, f \in M_2(R)$, $\varphi : eR \cong f(2R)$ implies that there exists a completed $u \in {}^2R$ such that $\varphi(e) = ue = fu$. These shows that diagonal reduction over exchange rings behaves like stable ranges.

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1. Introduction

A ring R is an exchange ring if for every right R-module A and any two decompositions $A = M \oplus N = \bigoplus_{i\in I} A_i$, where $M_R \cong R$ and the index set I is finite, then there exist submodules $A_i' \subseteq A_i$ such that $A = M \oplus (\bigoplus_{i\in I} A_i')$. The class of exchange rings is very large. It includes all regular rings, all π-regular rings, all strongly π-regular rings, all semiperfect rings, all left or right continuous rings, all clean rings, all unit C^*-algebras of real rank zero and all right semi-artinian rings (cf. [12]).

The $m \times n$ matrix A is said to admit diagonal reduction provided that there exist $P \in GL_m(R), Q \in GL_n(R)$ such that PAQ is a diagonal matrix. Many authors have studied diagonal reduction such as [1-2], [4], [6], [8-11] and [13]. The main purpose of this paper is to show that every regular matrix over an exchange ring admits diagonal reduction if and only if for every regular rectangular non-square matrix A, the square matrix obtained by adding 0-rows or columns to A is unit-regular. A characterization in terms of idempotents of the ring is also provided. This complements work of Henriksen, Menal, Moncasi, Ara, Goodearl, O'Meara and Pardo on diagonalization of matrices over regular rings and exchange rings. These also shows that diagonal reduction over exchange rings behaves like stable ranges.

Throughout, all rings are associative with identity. An element $x \in R$ is regular provided that $x = xyx$ for a $y \in R$. If $y \in R$ is invertible, we say that $x \in R$ is unit-regular. We use $GL_n(R)$ to denote the n-dimensional general linear group of a ring R.

2. The Main Results

A rectangular matrix $A \in M_{m\times n}(R)$ is completed provided that A can be completed to an invertible matrix by adding some columns or rows. We use $col_n(R)$ $\big(row_n(R)\big)$ to denote the set of all completed $n \times 1$ $(1 \times n)$ matrices over a ring R.

Lemma 2.1. *Let R be an exchange ring. Then the following are equivalent:*

(1) *Every regular matrix over R admits diagonal reduction.*

(2) *For any $m, n \in \mathbb{N}$ $(m \geq n+1)$, $M = A_1 \oplus B_1 = A_2 \oplus B_2$ with $A_1 \cong mR, nR \cong A_2$ implies that $(m-n)R \oplus B_1 \cong B_2$.*

(3) *For any $m, n \in \mathbb{N}$ $(m \geq n+1)$, $mR = A_1 \oplus B_1, nR = A_2 \oplus B_2$ with $A_1 \cong A_2$ implies that $B_1 \cong (m-n)R \oplus B_2$.*

Proof. (1) $\Rightarrow$ (2) Suppose that $M = A_1 \oplus B_1 = A_2 \oplus B_2$ with $A_1 \cong mR, nR \cong A_2$. Then $A_1 \cong A_{11} \oplus A_{12} \oplus \cdots \oplus A_{1m}$ and $A_2 \cong A_{21} \oplus A_{22} \oplus \cdots \oplus A_{2n}$, where each $A_{1i} \cong R \cong A_{2j}$. As $A_{11} \oplus A_{12} \cong 2R$, it follows by [1, Theorem 3.1] that $A_{12} \oplus \cdots \oplus A_{1m} \oplus B_1 \cong A_{22} \oplus \cdots \oplus A_{2n} \oplus B_2$. By iteration of this process, we get $(m-n)R \oplus B_1 \cong B_2$.

(2) $\Rightarrow$ (3) Suppose that $mR = A_1 \oplus B_1, nR = A_2 \oplus B_2$ with $A_1 \cong A_2$. Then $mR \oplus B_2 \cong A_1 \oplus B_1 \oplus B_2 \cong A_2 \oplus B_2 \oplus B_1 \cong nR \oplus B_1$. By assumption, we see that $(m-n)R \oplus B_2 \cong B_1$.

(3) $\Rightarrow$ (1) Choose $m = 2, n = 1$. Then the result follows from [4, Theorem 1]. $\square$

As is well known, an exchange ring R has stable range one if and only if every regular square matrix over R is unit-regular. For diagonal reduction over exchange rings, we can derive the following by virtue of regular rectangular matrices, which is an extension of [6, Theorem 2.2].

Lemma 2.2. *Let R be an exchange ring. Then the following are equivalent:*

(1) *Every regular matrix over R admits diagonal reduction.*

(2) *For any $m, n \in \mathbb{N}$ $(m \geq n+1)$ and any regular $X \in M_{m\times n}(R)$, there exists a completed $U \in M_{n\times m}(R)$ such that $X = XUX$.*

(3) *For any $m, n \in \mathbb{N}$ $(m \geq n+1)$ and any regular $X \in M_{n\times m}(R)$, there exists a completed $U \in M_{m\times n}(R)$ such that $X = XUX$.*

Proof. (1) $\Rightarrow$ (2) Since $X \in M_{m\times n}(R)$ is regular, there exists $Y \in M_{n\times m}(R)$ such that $X = XYX$. Clearly, $mR = XY(mR) \oplus (I_m - XY)(mR)$ and $nR = YX(nR) \oplus (I_n - YX)(nR)$. Obviously, we have a R-isomorphism $\phi : XY(mR) \cong YX(nR)$ given by $XY(r) \mapsto YXY(r)$ for any $r \in mR$. By virtue of Lemma 2.1, we have $\psi : (I_m - XY)(mR) \cong (m-n)R \oplus (I_n - YX)(nR)$. Now we construct a R-isomorphism $\varphi : mR = XY(mR) \oplus (I_m - XY)(nR) \to YX(nR) \oplus (I_n - YX)(nR) \oplus (m-n)R = mR$ given by $\varphi(s,t) = \phi(s) + \psi(t)$ for any $s \in XY(mR), t \in (I_m - XY)(mR)$. Let $\{e_1, \cdots, e_m\}$ be a basis of mR, and let $\varphi(e_1, \cdots, e_m) = (e_1, \cdots, e_m)A$ for a matrix A. Then $A \in GL_m(R)$. Let $U \in M_{n\times m}(R)$ be the first block of A. That is, $A = \begin{pmatrix} U \\ ** \end{pmatrix}$. Then

$$\begin{pmatrix} X & \mathbf{0}_{m\times(m-n)} \end{pmatrix} = \begin{pmatrix} X & \mathbf{0}_{m\times(m-n)} \end{pmatrix} A \begin{pmatrix} X & \mathbf{0}_{m\times(m-n)} \end{pmatrix},$$

and so $X = XUX$, as required.

(2) $\Rightarrow$ (1) Choose $m = 2, n = 1$. Then the result follows by [4, Lemma 5].

(1) $\Leftrightarrow$ (3) is proved by symmetry. □

Theorem 2.3. *Let R be an exchange ring. Then the following are equivalent:*

(1) *Every regular matrix over R admits diagonal reduction.*

(2) *For any $m, n \in \mathbb{N}$ $(m \geq n+1)$ and any regular $X \in M_{m\times n}(R)$,*

$$\begin{pmatrix} X & \mathbf{0}_{m\times(m-n)} \end{pmatrix} \in M_m(R) \text{ is unit-regular.}$$

(3) *For any $m, n \in \mathbb{N}$ $(m \geq n+1)$ and any regular $X \in M_{n\times m}(R)$,*

$$\begin{pmatrix} X \\ \mathbf{0}_{(m-n)\times n} \end{pmatrix} \in M_m(R) \text{ is unit-regular.}$$

Proof. (1) $\Rightarrow$ (2) For any $m, n \in \mathbb{N}$ $(m \geq n+1)$ and any regular $X \in M_{m\times n}(R)$, it follows from Lemma 2.2 that there exists a completed $U \in M_{n\times m}(R)$ such that $X = XUX$. Assume that $\begin{pmatrix} U \\ ** \end{pmatrix} \in GL_m(R)$. Then we see that

$$\begin{pmatrix} X & \mathbf{0}_{m\times(m-n)} \end{pmatrix} = \begin{pmatrix} X & \mathbf{0}_{m\times(m-n)} \end{pmatrix} \begin{pmatrix} U \\ ** \end{pmatrix} \begin{pmatrix} X & \mathbf{0}_{m\times(m-n)} \end{pmatrix},$$

and therefore $\begin{pmatrix} X & \mathbf{0}_{m\times(m-n)} \end{pmatrix}$ is unit-regular.

(2) $\Rightarrow$ (1) For any regular $x \in^2 R$, $(x,0) \in M_2(R)$ is unit-regular. Thus, we can find a $\begin{pmatrix} u \\ ** \end{pmatrix} \in GL_2(R)$ such that $(x,0) = (x,0)\begin{pmatrix} u \\ ** \end{pmatrix}(x,0)$. It is easy to see that $x = xux$, where $u \in row_2(R)$. According to Lemma [4, Lemma 5], every regular matrix over R admits diagonal reduction.

(1) $\Leftrightarrow$ (3) is proved in the same manner. □

Especially, we deduce that every matrix over a regular ring R if and only if for any (a,b), we have that $\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \in M_2(R)$ is unit-regular if and only if for any $\begin{pmatrix} a \\ b \end{pmatrix}$, we have that $\begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} \in M_2(R)$ is unit-regular. □

Lemma 2.4. *Let R be an exchange ring. Then the following are equivalent:*

(1) *Every regular matrix over R admits diagonal reduction.*
(2) *For any $m,n \in \mathbb{N}$ $(m \geq n+1)$ and any regular $X \in M_{m\times n}(R)$, there exists a completed $U \in M_{n\times m}(R)$ such that $XU \in M_m(R)$ is an idempotent.*
(3) *For any $m,n \in \mathbb{N}$ $(m \geq n+1)$ and any regular $X \in M_{n\times m}(R)$, there exists a completed $U \in M_{m\times n}(R)$ such that $XU \in M_n(R)$ is an idempotent.*

Proof. (1) $\Rightarrow$ (2) For any $m,n \in \mathbb{N}$ $(m \geq n+1)$ and any regular $X \in M_{m\times n}(R)$, it follows from Lemma 2.2 that there exists a completed $U \in M_{n\times m}(R)$ such that $X = XUX$. Let $XU = E$. Then $E = E^2 \in M_m(R)$, as desired.

(2) $\Rightarrow$ (1) For any regular $x \in R^2$, there exists a $y \in {}^2R$ such that $x = xyx$ and $y = yxy$. By assumption, we have a $u \in row_2(R)$ such that $e := yu \in M_2(R)$ is an idempotent. Since $xy + (1-xy) = 1$, we have $xe + (1-xy)u = u$, and so $\big(x+(1-xy)u\big)e+(1-xy)u(I_2-e) = u$. One easily checks that $\big(x+(1-xy)u\big)e = ue$ and $(1-xy)u(I_2-e) = u(I_2-e)$. As $u \in row_2(R)$, we see that $\begin{pmatrix} u \\ s \end{pmatrix} \in GL_2(R)$ for some $s \in M_{1\times 2}(R)$. Write $\begin{pmatrix} u \\ s \end{pmatrix}^{-1} = \begin{pmatrix} v & t \end{pmatrix}$ with $v,t \in col_2(R)$. Obviously, we have $u(I_2-e)vu(I_2-e) = u(I_2-e)$. Let $g = (I_2-e)vu(I_2-e)$. Then $g = g^2 \in M_2(R)$ and $u(e+g) = u$. Furthermore, we see that

$$\begin{aligned}
&u\big(I_2-(I_2-e)v(1-xy)ue\big)\big(y+(I_2+(I_2-e)v(1-xy)ue)(I_2-e)v(1-xy)\big)u\\
&= u\big((I_2-(I_2-e)v(1-xy)ue)e+(I_2-e)v(1-xy)u\big)\\
&= u\big(e+(I_2-e)v(1-xy)u(I_2-e)\big)\\
&= u(e+g)\\
&= u.
\end{aligned}$$

Let $w = y + \big(I_2 + (I_2 - e)v(1 - xy)ue\big)(I_2 - e)v(1 - xy)$. Then $u\big(I_2 - (I_2 - e)v(1 - xy)ue\big)wu = u$, and so $u\big(I_2 - (I_2 - e)v(1 - xy)ue\big)w = 1$. This implies that $\begin{pmatrix} u \\ s \end{pmatrix}\big(I_2-(I_2-e)v(1-xy)ue\big)\begin{pmatrix} w & (I_2+(I_2-e)v(1-xy)ue)t \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ ** & 1 \end{pmatrix} \in GL_2(R)$. Hence, $w \in col_2(R)$. In addition, $x = xwx$. In view of [4, Lemma 5], every regular matrix over R admits a diagonal reduction.

(1) $\Rightarrow$ (3) For any $m, n \in \mathbb{N}$ $(m \geq n+1)$ and any regular $X \in M_{n\times m}(R)$, there exists a completed $U \in M_{m\times n}(R)$ such that $X = XUX$. Let $XU = E$. Then $E = E^2 \in M_m(R)$, as required.

(3) $\Rightarrow$ (1) is obvious by [4, Lemma 5]. □

Theorem 2.5. *Let R be an exchange ring. Then the following are equivalent:*

(1) *Every regular matrix over R admits diagonal reduction.*
(2) *For any $m, n \in \mathbb{N}$ $(m \geq n+1)$ and any regular $X \in M_{m\times n}(R)$, there exist an idempotent $E \in M_m(R)$ and a completed $U \in M_{m\times n}(R)$ such that $X = EU$.*
(3) *For any $m, n \in \mathbb{N}$ $(m \geq n+1)$ and any regular $X \in M_{n\times m}(R)$, there exist an idempotent $E \in M_n(R)$ and a completed $U \in M_{n\times m}(R)$ such that $X = EU$.*

Proof. (1) $\Rightarrow$ (2) For any $m, n \in \mathbb{N}$ $(m \geq n+1)$ and any regular $X \in M_{m\times n}(R)$, it follows by Lemma 2.2 that there exists a completed $V \in M_{n\times m}(R)$ such that $X = XVX$. Assume that

$$\begin{pmatrix} V \\ ** \end{pmatrix}^{-1}_{m\times m} = \begin{pmatrix} U & * \end{pmatrix},$$

where $U \in M_{m\times n}(R)$. Then $VU = I_n$. Let $E = XV$. Then $X = X(VU) = EU$ with $E = E^2 \in M_m(R)$, as required.

(2) $\Rightarrow$ (1) For any regular $x \in R^2$, there exists $y \in^2 R$ such that $x = xyx$ and $y = yxy$. By assumption, there exist an idempotent $e \in M_2(R)$ and a $v \in col_2(R)$ such that $y = ev$. It follows from $yx+(I_2-yx) = I_2$ that $evx+(I_2-yx) = I_2$; hence, $evx(I_2 - e) + (I_2 - yx)(I_2 - e) = I_2 - e$. This implies that $e + (I_2 - yx)(I_2 - e) = I_2 - evx(I_2 - e) \in GL_2(R)$. Furthermore, we see that

$$\begin{aligned} u &:= y + (I_2 - yx)(I_2 - e)v \\ &= ev + (I_2 - yx)(I_2 - e)v \\ &= \big(I_2 - evx(I_2 - e)\big)v \\ &\in col_2(R). \end{aligned}$$

As a result, $xu = xy \in R$ is an idempotent. As in the proof of Lemma 2.4, every regular matrix over R admits diagonal reduction.

(1) $\Leftrightarrow$ (3) is proved in the same manner. $\square$

Lemma 2.6. *Suppose that $ax + b = 1$ with $a \in R^n, x \in {}^nR, b \in R$. Then the following are equivalent:*

(1) *There exists $y \in R^n$ such that $a + by \in row_n(R)$.*
(2) *There exists $z \in {}^nR$ such that $x + zb \in col_n(R)$.*

Proof. See [6, Lemma 4.1]. $\square$

Theorem 2.7. *Let R be an exchange ring. Then the following are equivalent:*

(1) *Every regular matrix over R admits diagonal reduction.*
(2) *For any idempotents $e \in R, f \in M_2(R)$, $\varphi : eR \cong f(2R)$ implies that there exists $u \in col_2(R)$ such that $\varphi(e) = ue = fu$.*

Proof. (1) $\Rightarrow$ (2) For any idempotents $e \in R, f \in M_2(R)$, $\varphi : eR \cong f(2R)$ implies that there exist some $r_1, r_2 \in R$ such that $fe_1 = \varphi(er_1)$ and $fe_2 = \varphi(er_2)$, where $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Thus, $f = f(e_1, e_2) = \varphi(e)(r_1, r_2) = f\varphi(e)e \cdot e(r_1, r_2)f$. Set $a = e(r_1, r_2)f$ and $b = f\varphi(e)e$. Then $f = ba$. Write $\varphi(e) = \begin{pmatrix} s_1 \\ s_2 \end{pmatrix}$. Then $\varphi(ab) = \varphi\big((er_1, er_2)\varphi(e)\big) = \varphi(er_1s_1 + er_2s_2) = fe_1s_1 + fe_2s_2 = (fe_1, fe_2)\varphi(e) = \varphi(e)$. This implies that $e = ab$. Since $ab + (1 - ab) = 1$, by virtue of Lemma 2.6, we can find a $z \in {}^2R$ such that $v := b + z(1 - ab) \in col_2(R)$. Hence, $b = \big(b + z(1 - ab)\big)ab = ve$. Let $u = (I_2 - f - va)v(1 - e - av)$. It is easy to verify that $(I_2 - f - va)^2 = I_2$ and $(1 - e - av)^2 = 1$. This implies that $u \in col_2(R)$. Further, $fu = -fvav(1 - e - av) = -fv(1 - e - av) = fve = b$. Also we have $ue = -(I_2 - f - va)vave = -(I_2 - f - va)b = -b + bab + ve = b$. Therefore $\varphi(e) = b = ue = fu$, as required.

(2) $\Rightarrow$ (1) Let $x \in R^2$ be regular. Then there exists a $y \in {}^2R$ such that $x = xyx$ and $y = yxy$. Thus, we get a R-isomorphism $\varphi : xyR \cong yx(2R)$ given by $\varphi(xyr) = yxyr$ for any $r \in R$. By hypothesis, there exists a $v \in {}^2R$ such that $y = yxv = vxy$. This implies that $y + (I_2 - yx)v(1 - xy) = v$. Thus, $x = xyx = x\big(y + (I_2 - yx)v(1 - xy)\big)x = xvx$. In view of [4, Lemma 5], we complete the proof. $\square$

Corollary 2.8. *Let R be an exchange ring. Then the following are equivalent:*

(1) *Every regular matrix over R admits diagonal reduction.*

(2) *For any idempotents* $e \in R, f \in M_2(R)$, $\varphi : eR \cong f(2R)$ *implies that there exists* $u \in col_2(R)$ *such that* $ue = fu$.

Proof. (1) $\Rightarrow$ (2) is trivial by Theorem 2.7.

(2) $\Rightarrow$ (1) See [4, Theorem 4]. □

An exchange ring R is strongly separative provided that for all finitely generated projective right R-modules A, B, $A \oplus A \cong A \oplus B$ implies $A \cong B$ (cf. [5] and [12]). Strongly separativity plays an important rule in non-stable K-theory. In [4, Theorem 5], the author showed that an exchange ring R is strongly separative if and only if every regular matrix over corners of R admits diagonal reduction. Therefore we also obtain many new characterizations of such exchange rings which possesses small cancellation of projectives.

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