International Electronic Journal of Algebra
Volume 7 (2010) 110-119

CLIFFORD SEMIGROUPS AND SEMINEAR-RINGS OF ENDOMORPHISMS

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Received: 18 April 2009; Revised: 7 September 2009
Communicated by Surender K. Jain

Abstract. We consider the structure of the semigroup of self-mappings of a semigroup S under pointwise composition, generated by the endomorphisms of S. We show that if S is a Clifford semigroup, with underlying semilattice Λ, then the endomorphisms of S generate a Clifford semigroup $E^+(S)$ whose underlying semilattice is the set of endomorphisms of Λ. These results contribute to the wider theory of seminear-rings of endomorphisms, since $E^+(S)$ has a natural structure as a distributively generated seminear-ring.

Mathematics Subject Classification (2000): 16Y30
Keywords: endomorphism, Clifford semigroup, seminear-ring

Introduction

Let G be a group, and let $M(G)$ be the set of all functions $G \to G$. Then $M(G)$ admits two natural binary operations: it is a semigroup under composition of functions (written multiplicatively) and a group under pointwise composition (written additively) using the group operation in G. If we write maps on the right, we find that function composition distributes on the left over pointwise composition, so that $f(g+h) = fg + fh$ for all $f, g, h \in M(G)$. This endows the set $M(G)$ with the structure of a *near-ring* (see [9]). Within $M(G)$ we have the subnear-ring $M_0(G)$ consisting of all functions $G \to G$ that map the identity element of G to itself. Then $M_0(G)$ contains the set $\mathrm{End}(G)$ of endomorphisms of G (a semigroup under composition of functions), and these are precisely the elements that always distribute on the right: $(f + g)h = fh + gh$ for all $f, g \in M_0(G)$ if and only if $h \in \mathrm{End}(G)$ (see [9, Lemma 9.6]). We let $E(G)$ be the subnear-ring of $M_0(G)$ generated by the subset $\mathrm{End}(G)$. The fact that $\mathrm{End}(G)$ is a right distributive semigroup implies that $E(G)$ is generated by $\mathrm{End}(G)$ as a group (that is, using only the pointwise composition). An important result about this construction, and a motivation for the more general theory of *distributively generated* near-rings (originating in [12]) is Fröhlich's theorem [3] that, for a finite non-abelian simple group G we have $E(G) = M_0(G)$. Further specific computations have been carried out for dihedral groups [7] and general linear groups [8].

If we replace the group G by a semigroup S we may attempt to generalise these ideas. The set $M(S)$ of all functions $S \to S$ is now a *seminear-ring*: it is a semigroup under both composition of functions and pointwise composition, and left distributivity holds. We remark that the algebra of seminear-rings underlies one approach to *process algebra* in the Bergstra-Klop axiomatization of the algebra of communicating processes, see [1], and has also been considered in the context of reversible computation [2]. We consider the subsemigroup $E^+(S)$ of $M(S)$ generated by $\mathrm{End}(S)$ using pointwise composition. Since the elements of $\mathrm{End}(S)$ are also right-distributive in $M(S)$, it follows that $E^+(S)$ is in fact a subseminear-ring of $M(S)$. The structure of $E^+(S)$ for a Brandt semigroup S was considered in [4].

In this paper, building on results in [13] and [11], we study the structure of $E^+(S)$ when S is a *Clifford semigroup*, that is an inverse semigroup with central idempotents. The structure of Clifford semigroups is well known: they are precisely the strong semilattices of groups. Our main result shows that if S is a strong semilattice of groups in which all the linking maps are isomorphisms, then $(E^+(S), +)$ has the same kind of structure, and moreover, if Λ is the semilattice underlying S, then the semilattice underlying $E^+(S)$ is $\mathrm{End}(\Lambda)$.

1. Preliminaries

A (left) *seminear-ring* is a set L admitting two associative binary operations, which we shall write as addition and multiplication, such that the left distributive law is satisfied: for all $a, b, c \in L$, we have $a(b + c) = ab + ac$. An element $d \in L$ is called *distributive* if it also distributes on the right, so that for all $a, b \in L$ we have $(a + b)d = ad + bd$. The set of distributive elements is clearly a subsemigroup of $(L, \cdot)$.

Let S be a semigroup (written multiplicatively). Then the set $M(S)$ of all functions $S \to S$ is a seminear-ring under the multiplication operation given by function composition, and the addition operation given by pointwise composition: so for all $f, g \in M(S)$ and $s \in S$ we have

$$s(f + g) = (sf)(sg) \quad \text{and} \quad s(fg) = (sf)g.$$

Following [9, Lemma 9.6], we have:

Lemma 1.1. *The semigroup of distributive elements in* $M(S)$ *is the semigroup* $\mathrm{End}(S)$ *of endomorphisms of* S.

Proof. It is clear that an endomorphism is distributive, so suppose that $d : S \to S$ is a distributive element of $M(S)$. For $s \in S$ let $c_s : S \to S$ be the constant map to s. Then for any $a, s, t \in S$ we have

$$a((c_s + c_t)d) = (st)d \ \text{ and } \ a(c_s d + c_t d) = (sd)(td)$$

and hence d is an endomorphism. □

A seminear-ring L is called *distributively generated* if $(L, \cdot)$ contains a subsemigroup of distributive elements that generates $(L, +)$. Distributively generated seminear-rings were first studied in [10]. Now let $E^+(S)$ be the subsemigroup of $(M(S), +)$ generated by $\text{End}(S)$. It is clear that $E^+(S)$ is then a distributively generated seminear-ring, called the *endomorphism seminear-ring* of S.

Now if S is commutative, then $E^+(S) = \text{End}(S)$ and $(E^+(S), +, \cdot)$ is a semiring (see [5]). In particular, we have the following special case, which will be important for our subsequent considerations.

Lemma 1.2. *Let Λ be a semilattice. Then $(\text{End}(\Lambda), +)$ is also a semilattice.*

A study of the structure of the endomorphism semiring of a semilattice can be found in [6].

We recall that the partial order on a semilattice Λ is determined by the multiplication as follows: if $\alpha, \beta \in \Lambda$, then $\alpha \geqslant \beta$ if and only if $\alpha\beta = \beta$. A *Clifford semigroup*, or a *strong semilattice of groups*, is a disjoint union of groups $S = \bigsqcup_{\alpha \in \Lambda} G_\alpha$ indexed by a semilattice Λ, together with a group homomorphism $\phi_{\alpha,\beta} : G_\alpha \to G_\beta$ whenever $\alpha \geqslant \beta$ in Λ, such that

- for each $\alpha \in \Lambda$, the homomorphism $\phi_{\alpha,\alpha}$ is the identity,
- if $\alpha \geqslant \beta \geqslant \gamma$ then $\phi_{\alpha,\gamma} = \phi_{\alpha,\beta}\phi_{\beta,\gamma}$.

The semigroup operation on S is defined by $ab = (a\phi_{\alpha,\alpha\beta})(b\phi_{\beta,\alpha\beta})$ if $a \in G_\alpha$ and $b \in G_\beta$.

2. Endomorphisms of Clifford semigroups

We begin this section with a routine lemma on endomorphisms of Clifford semigroups.

Lemma 2.1. *Let $S = (\Lambda, G_\alpha, \phi_{\alpha,\beta})$ be a strong semilattice of groups and let $f \in \text{End}(S)$. Then the following hold:*

(1) *f induces an endomorphism of the semilattice Λ,*
(2) *for each $\alpha \in G_\alpha$ we have $G_\alpha f \subseteq G_{\alpha f}$.*

Proof. Let e_α be the identity element of G_α. Now $G_\alpha f \subseteq G_\gamma$ for some γ, and since $e_\alpha f$ is an idempotent, we have $e_\alpha f = e_\gamma$, and we set $\alpha f = \gamma$. Since $e_\alpha e_\beta = e_{\alpha\beta}$ it follows that f is an endomorphism of Λ. □

The endomorphisms of Clifford semigroups were studied in detail in [11], under various restrictions on the properties of the linking maps $\phi_{\alpha,\beta}$. To pursue our study of the structure of $E^+(S)$, we shall assume the strongest of the conditions considered in [11], namely that the linking maps are all isomorphisms. In this case, we can simplify the description of S.

Lemma 2.2. *Let $S = (\Lambda, G_\alpha, \phi_{\alpha,\beta})$ be a strong semilattice of groups in which all the linking maps $\phi_{\alpha,\beta}$ are isomorphisms. For any $\lambda \in \Lambda$, let S_λ be the strong semilattice of groups over Λ in which each group $G_\alpha, \alpha \in \Lambda$ is equal to G_λ and all the linking maps are the identity. Then S is isomorphic to S_λ.*

Proof. We define an isomorphism $\psi : S \to S_\lambda$ as follows. Its restriction ψ_α to G_α is defined to be $\psi_\alpha = \phi_{\alpha,\alpha\lambda}\phi^{-1}_{\lambda,\alpha\lambda}$. Then ψ is clearly bijective and we need only check that it is a homomorphism. To this end, let $a \in G_\alpha$ and $b \in G_\beta$, so that in S we have $ab = (a\phi_{\alpha\alpha\beta})(b\phi_{\beta,\alpha\beta}) \in G_{\alpha\beta}$. Then

$$\begin{aligned}(a\psi)(b\psi) &= (a\psi_\alpha)(b\psi_\beta)\\ &= (a\phi_{\alpha,\alpha\lambda}\phi^{-1}_{\lambda,\alpha\lambda})(b\phi_{\beta,\beta\lambda}\phi^{-1}_{\lambda,\beta\lambda})\end{aligned}$$

whereas

$$\begin{aligned}(ab)\psi &= ((a\phi_{\alpha,\alpha\beta})(b\phi_{\beta,\alpha\beta}))\psi_{\alpha\beta}\\ &= (a\phi_{\alpha,\alpha\beta})\psi_{\alpha\beta}(b\phi_{\beta,\alpha\beta})\psi_{\alpha\beta}.\end{aligned}$$

Now

$$\begin{aligned}(a\phi_{\alpha,\alpha\beta})\psi_{\alpha\beta} &= (a\phi_{\alpha,\alpha\beta})\phi_{\alpha\beta,\alpha\beta\lambda}\phi^{-1}_{\lambda,\alpha\beta\lambda}\\ &= a\phi_{\alpha,\alpha\beta\lambda}\phi^{-1}_{\lambda,\alpha\beta\lambda}\\ &= a(\phi_{\alpha,\alpha\lambda}\phi_{\alpha\lambda,\alpha\beta\lambda})(\phi^{-1}_{\alpha\lambda,\alpha\beta\lambda}\phi^{-1}_{\lambda,\alpha\lambda})\\ &= a\phi_{\alpha,\alpha\lambda}\phi^{-1}_{\lambda,\alpha\lambda}.\end{aligned}$$

Similarly, $(b\phi_{\beta,\alpha\beta})\psi_{\alpha\beta} = b\phi_{\beta,\beta\lambda}\phi^{-1}_{\lambda,\beta\lambda}$ and ψ is indeed a homomorphism. □

By virtue of Lemma 2.2 we may now assume that S is a strong semilattice of groups over Λ in which every group is equal to a fixed group G and with each linking map equal to the identity. Hence S is the disjoint union of copies G_α of G, indexed by $\alpha \in \Lambda$. If $g \in G$, then we denote by $g^{(\alpha)}$ the copy of element g in G_α. In this notation the multiplication in S is given by

$$g^{(\alpha)}h^{(\beta)} = (gh)^{(\alpha\beta)}. \tag{1}$$

Proposition 2.3. *Any $\sigma \in \mathrm{End}(G)$ and $f \in \mathrm{End}(\Lambda)$ determine an endomorphism $\sigma_f \in \mathrm{End}(S)$ defined by $g^{(\alpha)}\sigma_f = (g\sigma)^{(\alpha f)}$ and every endomorphism of S arises in this way. Hence we have*

$$\mathrm{End}(S) \cong \mathrm{End}(G) \times \mathrm{End}(\Lambda)$$

as semigroups of mappings.

Proof. To show that $\sigma_f \in \mathrm{End}(S)$ we have to check the preservation of the multiplication given in (1), but this is almost trivial:

$$\begin{aligned}(g^{(\alpha)})\sigma_f(h^{(\beta)})\sigma_f &= (g\sigma)^{(\alpha f)}(h\sigma)^{(\beta f)} \\ &= (g\sigma h\sigma)^{((\alpha f)(\beta f))} \\ &= ((gh)\sigma)^{((\alpha\beta)f)} \\ &= ((gh)^{(\alpha\beta)})\sigma_f.\end{aligned}$$

Now let $\sigma \in \mathrm{End}(S)$ and let f be the induced endomorphism of Λ. For each $\alpha \in \Lambda$ we have $\sigma : G_\alpha \to G_{\alpha f}$, and since $G_\alpha = G = G_{\alpha f}$, the restriction of σ to G_α induces an endomorphism σ_α of G. Now for any $g \in G$ and $\alpha, \beta \in \Lambda$ we have $1_G^{(\alpha\beta)} = g^{(\alpha)}(g^{-1})^{(\beta)}$. Applying σ, we obtain

$$\begin{aligned}1_G^{((\alpha\beta)f)} &= g^{(\alpha)}\sigma(g^{-1})^{(\beta)}\sigma \\ &= ((g\sigma_\alpha)(g^{-1}\sigma_\beta))^{((\alpha f)(\beta f))}.\end{aligned}$$

Therefore $g\sigma_\alpha = g\sigma_\beta$ and $\sigma \in \mathrm{End}(S)$ induces the same endomorphism ρ on each group G_α, with $g^{(\alpha)}\sigma = (g\rho)^{(\alpha f)}$. Therefore $\sigma = \rho_f$. It is now clear that $(\rho, f) \mapsto \rho_f$ is a bijection $\mathrm{End}(G) \times \mathrm{End}(\Lambda) \to \mathrm{End}(S)$, and since $g^\alpha \rho_f \sigma_k = (g\rho)^{(\alpha f)}\sigma_k = (g\rho\sigma)^{\alpha f k}$ this bijection is a semigroup isomorphism. □

These considerations allow us to recover one of the main results of [11], by reintroducing the isomorphic linking maps into $S = (\Lambda, G_\alpha, \phi_{\alpha,\beta})$. For any $\lambda \in \Lambda$, we may write an endomorphism τ of S in the form $\tau = \psi\sigma_f\psi^{-1}$ where $\sigma_f \in \mathrm{End}(S_\lambda)$, and hence for $g \in G_\alpha$ we have

$$\begin{aligned}g\tau &= g\psi_\alpha\sigma_f\psi_{\alpha f}^{-1} \\ &= g\phi_{\alpha,\alpha\lambda}\phi_{\lambda,\alpha\lambda}^{-1}\sigma_f\phi_{\alpha f,(\alpha f)\lambda}\phi_{\lambda,(\alpha f)\lambda}^{-1}\end{aligned}$$

which is the formula for τ given in [11].

3. Seminear-rings of endomorphisms

For the rest of the paper, we shall assume that the group G is finite. This implies that any mapping in the group $E(G)$ is a positive combination of endomorphisms, and hence that the semigroup $E^+(G)$ generated by $\mathrm{End}(G)$ coincides with $E(G)$.

For fixed $f \in \mathrm{End}(\Lambda)$ we have an embedding $\mathrm{End}(G) \to \mathrm{End}(S)$ by $\alpha \mapsto \alpha_f$. We claim that this embedding induces a homomorphism $\gamma_f : E(G) \to E^+(S)$. Suppose that $\xi = \sigma_1 + \cdots + \sigma_m \in E(G)$. We define $\xi_f = (\sigma_1)_f + \cdots + (\sigma_m)_f$. Then for each $\alpha \in \Lambda$ and each $g^{(\alpha)} \in G_\alpha$ we have

$$g^{(\alpha)}\xi_f = ((g\sigma_1)^{(\alpha f)}) \dots ((g\sigma_m)^{(\alpha f)}) = (g\xi)^{(\alpha f)}.$$

Hence ξ_f depends only on ξ and f, and $\gamma_f : \xi \mapsto \xi_f$ is a well-defined embedding $E(G) \to E^+(S)$. Moreover, if $\xi_f = \eta_k$ then for all $g \in G$ and $\alpha \in \Lambda$ we have $(g\xi)^{(\alpha f)} = (g\eta)^{(\alpha k)}$. Hence $f = k$, and the images of the distinct embeddings γ_f $(f \in \mathrm{End}(\Lambda))$ are disjoint. We write $E(G)_f$ for the image of $E(G)$ under the embedding γ_f. For each $f \in \mathrm{End}(\Lambda)$, $E(G)_f$ is a subgroup of $E^+(S)$ isomorphic to $E(G)$.

Now if $\theta \in E^+(S)$ we have $\theta = \theta_1 + \theta_2 + \cdots + \theta_m$ for some $\theta_j \in \mathrm{End}(S)$ and hence there exist $\sigma_j \in \mathrm{End}(G)$ and $f_j \in \mathrm{End}(\Lambda)$ such that $\xi = (\sigma_1)_{f_1} + \cdots + (\sigma_m)_{f_m}$. Therefore $(E^+(S), +)$ is generated by the collection of disjoint subgroups $E(G)_f$ where $f \in \mathrm{End}(\Lambda)$.

Now take $\xi_1, \xi_2 \in E(G)$ and $f_1, f_2 \in \mathrm{End}(\Lambda)$. Then for all $g \in G$ and $i = 0, 1, \dots n$ we have

$$\begin{aligned} g^{(\alpha)}((\xi_1)_{f_1} + (\xi_2)_{f_2}) &= (g\xi_i)^{\alpha f_1}(g\xi_2)^{\alpha f_2} \\ &= ((g\xi_1)(g\xi_2))^{((\alpha f_1)(\alpha f_2))} = (g(\xi_1 + \xi_2))^{(\alpha(f_1+f_2))}. \end{aligned}$$

A straightforward induction argument then shows that

$$(\xi_1)_{f_1} + \cdots + (\xi_m)_{f_m} = (\xi_1 + \cdots + \xi_m)_{f_1+\cdots+f_m}.$$

Therefore

$$E^+(S) = \bigsqcup_{f \in \mathrm{End}(\Lambda)} E(G)_f$$

and so $E^+(S)$ is a semilattice of groups.

We first look at the composition of maps in $E^+(S)$. For $g^{(\alpha)} \in G_\alpha$ we have

$$g^\alpha \xi_f \eta_k = (g\xi)^{(\alpha f)} \eta_k = (g\xi\eta)^{(\alpha f k)}$$

and hence

$$\xi_f \eta_k = (\xi\eta)_{fk} \tag{2}$$

Now we have linking homomorphisms $\phi_{f_1,f_2} : E(G)_{f_1} \to E(G)_{f_2}$ whenever $f_1 \geqslant f_2$, defined by $\xi_{f_1} \mapsto \xi_{f_2}$. So the linking homomorphisms are identity maps between the indexed copies of $E(G)$ in $E^+(S)$, and for the addition of $\xi_{f_1}, \eta_{f_2} \in E^+(S)$ we have

$$\xi_{f_1} + \eta_{f_2} = (\xi + \eta)_{f_1+f_2} \tag{3}$$

$$= (\xi_{f_1})\phi_{f_1,f_1+f_2} + (\eta_{f_2})\phi_{f_2,f_1+f_2} \tag{4}$$

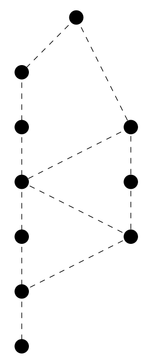
and so $E^+(S)$ is a strong semilattice of its subgroups $E(G)_f$. We summarize our conclusions in the following theorem, returning to the case of a strong semilattice of groups whose linking maps are isomorphisms:

Theorem 3.1. *Let $S = (\Lambda, G_\alpha, \phi_{\alpha,\beta})$ be a strong semilattice of finite groups in which all the linking maps $\phi_{\alpha,\beta}$ are isomorphisms.*

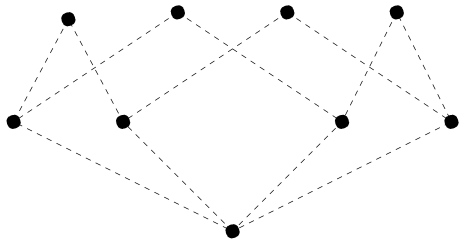
(1) *As a semigroup under composition of maps, $E^+(S)$ is isomorphic to $E(G) \times E(\Lambda) = E(G) \times \mathrm{End}(\Lambda)$.*
(2) *As a semigroup under addition of maps, $E^+(S)$ is isomorphic to a strong semilattice of groups over the semilattice* $\mathrm{End}(\Lambda)$*, with each group isomorphic to $E(G)$.*

4. Examples

4.1. Finite chains of finite groups. Let Λ be the finite chain $0 < 1 < \cdots < n$. It is well-known that in this case $|\mathrm{End}(\Lambda)| = \binom{2n+1}{n}$. If $n = 1$ there are 3 endomorphisms, and in this case $(\mathrm{End}(\Lambda), +)$ is again a finite chain. Hence if $S = G_0 \sqcup G_1$ with an isomorphism $\phi : G_1 \to G_0$ then $E^+(S) = E(G) \sqcup E(G) \sqcup E(G)$. For $n > 1$ the semilattice $(\mathrm{End}(\Lambda), +)$ will not be a finite chain: for $n = 2$ it is the 10-element semilattice:



4.2. Finite Clifford semigroups over the free 2–generator semilattice. Let $\Lambda = \{\alpha, \beta, \alpha\beta\}$ be the free 2–generator semilattice, with isomorphisms $G_\alpha \to G_{\alpha\beta} \leftarrow G_\beta$. Then $\mathrm{End}(\Lambda)$ is the 9–element semilattice



The maximal elements at the left and right-hand end of this picture are the two automorphisms of Λ. There are four endomorphisms in the principal order ideal of $\mathrm{End}(\Lambda)$ generated by the identity id. In addition to id itself, we have $a : \alpha \mapsto \alpha, \beta \mapsto \alpha\beta$, $b : \alpha \mapsto \alpha\beta, \beta \mapsto \beta$ and the constant map $c = c_{\alpha\beta}$ at $\alpha\beta$. Consider the subseminear-ring

$$E_{\downarrow(\mathrm{id})}(S) = \bigsqcup_{\substack{f\in\mathrm{End}(\Lambda)\\ f\leqslant \mathrm{id}}} E(G)_f = E(G)_{\mathrm{id}} \sqcup E(G)_a \sqcup E(G)_b \sqcup E(G)_c.$$

A quick check reveals that the multiplication table for the subsemilattice $\{\mathrm{id}, a, b, c\}$ (under $+$) coincides with its multiplication table under composition of maps. It follows that $E_{\downarrow(\mathrm{id})}(S)$ is a strong semilattice of near-rings (see [13]).

4.3. Non-isomorphic linking maps. If the linking maps $\phi_{\alpha,\beta}$ in S are not isomorphisms, then further complications arise in the analysis of $E^+(S)$. As an illustration, consider the case $n = 1$ in Example 4.1, so that $S = G_0 \sqcup G_1$ with G_0 and G_1 finite, but with an arbitrary homomorphism $\phi : G_1 \to G_0$. The three endomorphisms of the chain $0 < 1$ give rise to three types of endomorphism of S.

Let $f \in \mathrm{End}(S)$. If f induces the endomorphism c_0 which is constant at 0 on the chain $0 < 1$, then f is determined by $f_0 \in \mathrm{End}(G_0)$, so that $f|_{G_0} = f_0$ and $f|_{G_1} = \phi f_0$. Similarly, if f induces the endomorphism c_1 which is constant at 1 on the chain $0 < 1$, then f is determined by $f_0 : G_0 \to G_1$, and again we have $f|_{G_0} = f_0$ and $f|_{G_1} = \phi f_0$. However, if f induces the identity on the chain $0 < 1$ then it is determined by two endomorphisms $f_1 \in \mathrm{End}(G_1)$ and $f_0 \in \mathrm{End}(G_0)$ such that $f_1\phi = \phi f_0$. Hence as a set $\mathrm{End}(S)$ can be identified with the disjoint union

$$\mathrm{End}(G_0) \sqcup \Phi \sqcup \mathrm{Hom}(G_0, G_1) \tag{5}$$

where

$$\Phi = \{(f_0, f_1) \in \mathrm{End}(G_0) \times \mathrm{End}(G_1) : f_1\phi = \phi f_0\}.$$

Clearly if $(f_0, f_1) \in \Phi$ then $(\operatorname{im}\phi)f_0 \subseteq \operatorname{im}\phi$ and $(\ker\phi)f_1 \subseteq \ker\phi$. If ϕ is surjective, then the condition $(\ker\phi)f_1 \subseteq \ker\phi$ also implies that f_1 determines f_0, and so we may simplify the description of Φ to $\Phi = \{f \in \operatorname{End}(G_1) : (\ker\phi)f_1 \subseteq \ker\phi\}$. If ϕ is injective, then f_0 determines f_1 and so we may simplify Φ to $\Phi = \{f \in \operatorname{End}(G_0) : (\operatorname{im}\phi)f_0 \subseteq \operatorname{im}\phi\}$.

Now each subset shown in the partition (5) is a subsemigroup of $(\operatorname{End}(S), \cdot)$: the composition in $\operatorname{End}(G_0)$ and in Φ is the obvious one in each case, and if $a, b \in \operatorname{Hom}(G_0, G_1)$ then $a \cdot b = a\phi b$. We let $E_{c_0}(S)$ be the subsemigroup of $(E^+(S), +)$ generated by $\operatorname{End}(G_0)$, $E_{\mathrm{id}}(S)$ be the subsemigroup of $(E^+(S), +)$ generated by Φ, and $E_{c_1}(S)$ be the subsemigroup of $(E^+(S), +)$ generated by $\operatorname{Hom}(G_0, G_1)$. An element of $E_{c_0}(S)$ is represented by some function $\xi : G_0 \to G_1$: then ξ acts on G_0, and its action on S is given by defining $g\xi = g\phi\xi$ if $g \in G_1$. An element of $E_{\mathrm{id}}(S)$ is represented by a pair of maps (ξ, η) with $\xi : G_0 \to G_0$ and $\eta : G_1 \to G_1$, that satisfy $\phi\xi = \eta\phi$. Finally an element of $E_{c_0}(S)$ is represented by $\xi \in E(G_0)$ acting on G_0, and its action on S is again given by defining $g\xi = g\phi\xi$ if $g \in G_1$. Then we have a decomposition

$$E^+(S) = E_{c_0}(S) \sqcup E_{\mathrm{id}}(S) \sqcup E_{c_1}(S)$$

(with $c_0 < \mathrm{id} < c_1$ as endomorphisms of the chain $0 < 1$) of $(E^+(S), +)$ as a strong semilattice of groups with linking maps

$$\phi_{c_1,\mathrm{id}} : E_{c_1}(S) \to E_{\mathrm{id}}(S),\ \xi \mapsto (\xi\phi, \xi)\,,$$
$$\phi_{\mathrm{id},c_0} : E_{\mathrm{id}}(S) \to E_{c_0},\ (\xi, \eta) \mapsto \xi\,,$$

and

$$\phi_{c_1,\mathrm{id}} : E_{c_1}(S) \to E_{c_0}(S),\ \xi \mapsto \xi\phi\,.$$

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