n-$\mathscr{X}$-COHERENT RINGS

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ABSTRACT. This paper unifies several generalizations of coherent rings in one notion. Namely, we introduce n-$\mathscr{X}$-coherent rings, where $\mathscr{X}$ is a class of modules and n is a positive integer, as those rings for which the subclass $\mathscr{X}_n$ of n-presented modules of $\mathscr{X}$ is not empty, and every module in $\mathscr{X}_n$ is $n+1$-presented. Then, for each particular class $\mathscr{X}$ of modules, we find correspondent relative coherent rings. Our main aim is to show that the well-known Chase's, Cheatham and Stone's, Enochs', and Stenström's characterizations of coherent rings hold true for any n-$\mathscr{X}$-coherent rings.

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1. Introduction

Throughout this paper, R denotes a non-trivial associative ring with identity, and all R-modules are, if not specified otherwise, left R-modules. For an R-module M, we use M^* to denote the character module $\mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ of M. An R-module M is said to be n-presented, for a positive integer n, if there is an exact sequence of R-modules: $F_n \to F_{n-1} \to \cdots \to F_0 \to M \to 0$ where each F_i is finitely generated and free. In particular, 0-presented and 1-presented modules are finitely generated and finitely presented modules respectively. An R-module M is said to be infinitely presented, if it is m-presented for every positive integer m.

A ring R is called left coherent, if every finitely generated left ideal is finitely presented, equivalently every finitely presented R-module is 2-presented and so infinitely presented. The coherent rings were first appear in Chase's paper [4] without being mentioned by name. The term coherent was first used by Bourbaki in [1]. Since then, coherent rings have became a vigorously active area of research (see Glaz's book [14] for more details).

Several characterizations of coherent rings have been done by various notions. Here, we are interested in the following homological ones:

- In [4], Chase characterized left coherent rings as those rings over which every direct product of flat right modules is flat.
- In [22], Stenström proved that a ring R is left coherent if and only if every direct limit of FP-injective R-modules is also FP-injective.
- Cheatham and Stone [5, Theorem 1] showed that coherent rings can be characterized by the use of the notion of character module as follows:
  The following assertions are equivalent:
  (1) R is left coherent;
  (2) An R-module M is injective if and only if M^* is flat;
  (3) An R-module M is injective if and only if M^{**} is injective;
  (4) A right R-module M is flat if and only if M^{**} is flat.
- The notion of flat preenvelopes of modules is used by Enochs to characterize coherent rings. Recall that an R-module F in some class of R-modules $\mathscr{X}$ is said to be an $\mathscr{X}$-preenvelope of an R-module M, if there is a homomorphism $\varphi : M \to F$ such that, for any homomorphism $\varphi' : M \to F'$ with $F' \in \mathscr{X}$, there is a homomorphism $f : F \to F'$ such that $\varphi' = f\varphi$ (see [13] for more details about this notion). The homomorphism φ is also called an $\mathscr{X}$-preenvelope of M. If $\mathscr{X}$ is the class of flat R-modules, an $\mathscr{X}$-preenvelope of M is simply called a flat preenvelope of M. We have [13, Proposition 6.5.1]: a ring R is left coherent if and only if every right R-module has a flat preenvelope.

The above characterizations of coherent rings have led to introduce various relative coherent rings (see Examples 2.2 for some of these rings). Namely, for each relative coherent rings, relative flat and injective modules were introduced and so used to give characterizations of their corespondent relative coherent rings in the same way as Chase's, Cheatham and Stone's, Enochs', and Stenström's characterizations of coherent rings cited above (see references for more details). The idea of this paper is to unify these relative coherent rings in one notion which we call n-$\mathscr{X}$-coherent rings, where $\mathscr{X}$ is a class of modules and n is a positive integer (see Definition 2.1). As main results of this paper, we give a generalization of the above characterizations of coherent rings to the setting of n-$\mathscr{X}$-coherent rings (see Theorems 2.6, 2.13, 2.16). So, relative flat and injective modules are introduced (see Definition 2.5). Before giving the desired results, we begin with a characterization of n-$\mathscr{X}$-coherent rings in terms of the functors Tor and Ext (see Theorem 2.3).

2. Main results

In this paper we are concerned with the following generalization of the notion of coherent rings.

Definition 2.1. Let $\mathscr{X}$ be a class of R-modules.

- R is said to be left n-$\mathscr{X}$-coherent, for a positive integer n, if the subclass $\mathscr{X}_n$ of n-presented R-modules of $\mathscr{X}$ is not empty, and every R-module in $\mathscr{X}_n$ is $n+1$-presented.
- Similarly, the right n-$\mathscr{X}$-coherent rings are defined.
- A ring R is called n-$\mathscr{X}$-coherent if it is both left and right n-$\mathscr{X}$-coherent.

It is trivial to show that over n-$\mathscr{X}$-coherent rings the n-presented modules are in fact infinitely presented.

Examples 2.2.   (1) Clearly, for $n = 0$ and $\mathscr{C}$ is the class of all cyclic R-modules, the 1-$\mathscr{C}$-coherent rings are just the Noetherian rings.

(2) For $n = 1$, the 1-$\mathscr{C}$-coherent rings are just the coherent rings. Note that, from [14, Theorem 2.3.2], the 1-$\mathscr{C}$-coherence is the same as the 1-$\mathscr{M}$-coherence, where $\mathscr{M}$ denotes the class of all R-modules.

(3) An extension of the notion of coherent rings were introduced in [6] and [12] as follows: for any positive integer $n \geq 1$, a ring R is called n-coherent (resp., strong n-coherent), if it is n-$\mathscr{C}$-coherent (resp., n-$\mathscr{M}$-coherent). The strong n-coherent rings were introduced by Costa [6] who first called them n-coherent (see also [7,11,15]).

(4) Let s and t be two positive integer and let $\mathscr{M}_{(s,t)}$ be the class of finitely presented R-modules of the form R^s/K, where K is a t-generated submodule of the left R-module R^s. The 1-$\mathscr{M}_{(s,t)}$-coherent rings were introduced in [24] and they were called (s,t)-coherent rings. In particular, The 1-$\mathscr{M}_{(1,1)}$-coherent rings (equivalently, the rings that satisfy: every principal ideal is finitely presented) were introduced in [10] and they were called P-coherent rings.

(5) Also a left min-coherent ring were introduced in [20] as a particular case of 1-$\mathscr{M}_{(s,t)}$-coherent rings: a ring R is said to be left min-coherent if every simple left ideal of R is finitely presented. Then min-coherent rings are just the 1-$\mathscr{C}_S$-coherent, where $\mathscr{C}_S$ is the class of all cyclic R-modules of the form R/I, where I is a simple left ideal of R.

(6) In the case where $\mathscr{X}$ is the class of all submodules of the Jacobson radical, the 0-$\mathscr{X}$-coherent rings are called J-coherent (see [8]).

(7) In [3] (see also [19]), the class $\mathscr{T}$ of all torsionless R-module is of interest, such that the 0-$\mathscr{T}$-coherent rings are called Π-coherent ring.

(8) Finally, consider a class $\mathscr{P}_d$ of all modules of projective dimension at most a positive integer d. The m-$\mathscr{P}_d$-coherent rings were introduced in [9] and they were called (m, d)-coherent (this notion differs from the one in 4 above). The 1-$\mathscr{P}_d$-coherent rings were first introduced in [17] and they were called d-coherent (this notion also differs from the one in 3 above). Also, a particular case of 1-$\mathscr{P}_d$-coherent were introduced in [18].

All of the above relative coherent rings have analogous characterizations of Chase's, Cheatham and Stone's, Enochs', and Stenström's characterizations of coherent rings (see references). The aim of this paper is to show that all of these characterizations hold true for n-$\mathscr{X}$-coherent rings without any further condition on the class of modules $\mathscr{X}$.

We begin with the following characterization of n-$\mathscr{X}$-coherent rings in terms of the functors Tor and Ext.

Theorem 2.3. *Let $\mathscr{X}$ be a class of R-modules such that, for a positive integer $n \geq 1$, the subclass $\mathscr{X}_n$ of n-presented R-modules of $\mathscr{X}$ is not empty. Then, the following assertions are equivalent:*

(1) *R is left n-$\mathscr{X}$-coherent;*

(2) *For every set J, the canonical homomorphism $R^J \otimes_R M \to M^J$ is bijective for $M \in \mathscr{X}_n$, and we have $\mathrm{Tor}_i^R(R^J, M) = 0$ for every $0 < i \leq n$;*

(3) *For every family $(P_j)_{j \in J}$ of right R-modules, the canonical homomorphism:*
$$\prod_{j \in J} \mathrm{Tor}_i^R(P_j, M) \to \mathrm{Tor}_i^R(\prod_{j \in J} P_j, M)$$
is bijective for every $i \leq n$ and every $M \in \mathscr{X}_n$;

(4) *For every direct system $(N_j)_{j \in J}$ of R-modules over a directed index set J, the canonical homomorphism:*
$$\underrightarrow{\lim}\mathrm{Ext}_R^i(M, N_j) \to \mathrm{Ext}_R^i(M, \underrightarrow{\lim}N_j)$$
is bijective for every $i \leq n$ and every $M \in \mathscr{X}_n$.

Proof. All equivalences follow from the following result. $\square$

Lemma 2.4. [2, Exercise 3, page187] *Let M be an R-module. For a positive integer $n \geq 1$, the following assertions are equivalent:*

(1) *M is n-presented;*

(2) *For every set J, the canonical homomorphism $R^J \otimes_R M \to M^J$ is bijective,*
    and we have $\mathrm{Tor}_i^R(R^J, M) = 0$ for every $0 < i < n$;

(3) *For every family $(P_j)_{j \in J}$ of right R-modules, the canonical homomorphism:*

$$\prod_{j \in J} \mathrm{Tor}_i^R(P_j, M) \to \mathrm{Tor}_i^R(\prod_{j \in J} P_j, M)$$

is bijective for every $i < n$;

(4) *For every direct system $(N_j)_{j \in J}$ of R-modules over a directed index set J,*
    the canonical homomorphism:

$$\underrightarrow{lim}\,\mathrm{Ext}_R^i(M, N_j) \to \mathrm{Ext}_R^i(M, \underrightarrow{lim}\,N_j)$$

is bijective for every $i < n$.

We also use the above result to characterize n-$\mathscr{X}$-coherent rings by relative flatness and injectivity, which are defined as follows:

Definition 2.5. Let $\mathscr{X}$ be a class of R-modules such that, for a positive integer $n \geq 1$, the subclass $\mathscr{X}_n$ of n-presented R-modules of $\mathscr{X}$ is not empty.

- A right R-module M is called n-$\mathscr{X}$-flat if $\mathrm{Tor}_n^R(M, N) = 0$ for every $N \in \mathscr{X}_n$. The n-$\mathscr{X}$-flat left R-modules are defined similarly.
- An R-module M is called n-$\mathscr{X}$-injective if $\mathrm{Ext}_R^n(N, M) = 0$ for every $N \in \mathscr{X}_n$.

As in Example 2.2, we get, for each special class $\mathscr{X}$ of modules, a correspondent relative flatness and injectivity. For instance, if $\mathscr{C}$ is the class of all cyclic R-modules, the 1-$\mathscr{C}$-flat right R-modules are just the classical flat right R-module, and the 1-$\mathscr{C}$-injective R-modules are just the FP-injective R-modules [22]. If $\mathscr{M}$ is the class of all R-modules, then, for a positive integer $n \geq 1$, the n-$\mathscr{M}$-flat right R-modules were called in [7] n-flat right R-module, and the n-$\mathscr{M}$-injective R-modules were called n-injective modules.

Now we give our first main result, which is a generalization of Chase's and Stenström's characterizations of coherent rings.

Theorem 2.6. *Let $\mathscr{X}$ be a class of R-modules such that, for a positive integer $n \geq 1$, the subclass $\mathscr{X}_n$ of n-presented R-modules of $\mathscr{X}$ is not empty. Then, the following assertions are equivalent:*

(1) *R is left n-$\mathscr{X}$-coherent;*

(2) *For every set J, the right R-module R^J is n-$\mathscr{X}$-flat;*

(3) *For every family $(P_j)_{j \in J}$ of n-$\mathscr{X}$-flat right R-modules, the direct product $\prod\limits_{j \in J} P_j$ is n-$\mathscr{X}$-flat;*

(4) *For every direct system $(N_j)_{j \in J}$ of n-$\mathscr{X}$-injective R-modules over a directed index set J, the direct limit $\varinjlim N_j$ is n-$\mathscr{X}$-injective.*

Proof. The implication $1 \Rightarrow 3$ follows from Theorem 2.3 $(1 \Leftrightarrow 3)$. The implication $2 \Rightarrow 3$ is obvious.

We prove the implication $2 \Rightarrow 1$. Consider an R-module $M \in \mathscr{X}_n$. Then, there is an exact sequence of R-modules:

$$F_n \to F_{n-1} \to \cdots \to F_1 \to F_0 \to M \to 0$$

such that each F_i is finitely generated and free. Consider $K_n = \operatorname{Im}(F_n \to F_{n-1})$ and $K_{n-1} = \operatorname{Im}(F_{n-1} \to F_{n-2})$. Then, we have the following short exact sequence

$$0 \to K_n \to F_{n-1} \to K_{n-1} \to 0$$

Since $\operatorname{Tor}_1^R(N, K_{n-1}) \cong \operatorname{Tor}_n^R(N, M) = 0$ for every $N \in \mathscr{X}_n$, we get the following commutative diagram with exact rows:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & R^J \otimes_R K_n & \longrightarrow & R^J \otimes_R F_{n-1} & \longrightarrow & R^J \otimes_R K_{n-1} & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \\
0 & \longrightarrow & (K_n)^J & \longrightarrow & (F_{n-1})^J & \longrightarrow & (K_{n-1})^J & \longrightarrow & 0
\end{array}
$$

From Lemma 2.4 $(1 \Leftrightarrow 2)$, β and γ are isomorphisms. Then, using snake lemma [13, Proposition 1.2.13], we get that α is also an isomorphism. Then, by Lemma 2.4 $(2 \Leftrightarrow 1)$, K_n is finitely presented and therefore M is $n+1$-presented.

It remains to prove the equivalence $1 \Leftrightarrow 4$. The implication $1 \Rightarrow 4$ follows from Theorem 2.3 $(1 \Leftrightarrow 4)$. Using Lemma 2.4 $(4 \Leftrightarrow 1)$, the proof of the implication $4 \Rightarrow 1$ is similar to the one of the implication $2 \Rightarrow 1$ above. $\qquad \square$

Now we give a counterpart of Cheatham and Stone's characterization of n-$\mathscr{X}$-coherent rings using the notion of character module. For that we need some results.

Lemma 2.7. *Let $\mathscr{X}$ be a class of R-modules such that, for a positive integer $n \geq 1$, the subclass $\mathscr{X}_n$ of n-presented R-modules of $\mathscr{X}$ is not empty. Then, for a family $(M_j)_{j \in J}$ of R-modules, we have:*

(1) *$\bigoplus\limits_{j \in J} M_j$ is n-$\mathscr{X}$-flat if and only if each M_j is n-$\mathscr{X}$-flat.*

(2) *$\prod\limits_{j \in J} M_j$ is n-$\mathscr{X}$-injective if and only if each M_j is n-$\mathscr{X}$-injective.*

Proof. 1) Follows from the isomorphism $\operatorname{Tor}_n^R(N, \bigoplus_{j \in J} M_j) \cong \bigoplus_{j \in J} Tor_n^R(N, M_j)$ [21, Theorem 8.10].

2) Follows from the isomorphism $\operatorname{Ext}_R^n(N, \prod_{j \in J} M_j) \cong \prod_{j \in J} \operatorname{Ext}_R^n(N, M_j)$ [21, Theorem 7.14]. $\qquad\square$

We also need the following extension of the well-known Lambek's result [16].

Lemma 2.8. *Let $\mathscr{X}$ be a class of R-modules such that, for a positive integer $n \geq 1$, the subclass $\mathscr{X}_n$ of n-presented R-modules of $\mathscr{X}$ is not empty. Then, a left (resp. right) R-module M is n-$\mathscr{X}$-flat if and only if M^* is an n-$\mathscr{X}$-injective right (resp. left) R-module.*

Proof. Follows from the isomorphism $(\operatorname{Tor}_n^R(M, N))^* \cong \operatorname{Ext}_R^n(N, M^*)$ [21, page 360]. $\qquad\square$

The notion of pure submodules is also used. Recall that a short exact sequence of R-modules $0 \to A \to B \to C \to 0$ is said to be pure if, for every right R-module M, the sequence $0 \to M \otimes_R A \to M \otimes_R B \to M \otimes_R C \to 0$ is exact. In this case, A is called a pure submodule of B.

Lemma 2.9. [23, Exercise 40] *Let $0 \to A \to B \to C \to 0$ be a short exact sequence of R-modules. Then, the following assertions are equivalent:*

(1) *The exact sequence $0 \to A \to B \to C \to 0$ is pure;*

(2) *The exact sequence $0 \to \operatorname{Hom}_R(P, A) \to \operatorname{Hom}_R(P, B) \to \operatorname{Hom}_R(P, C) \to 0$ is exact for every finitely presented R-module P;*

(3) *The short sequence of right R-modules $0 \to C^* \to B^* \to A^* \to 0$ splits.*

Lemma 2.10. [23, Exercise 41] *Every R-module M is a pure submodule of M^{**} via the the canonical monomorphism $M \to M^{**}$.*

Lemma 2.11. [5, Lemma 1] *For every family $(P_j)_{j \in J}$ of left or right R-modules, we have:*

(1) *The sum $\bigoplus_{j \in J} P_j$ is a pure submodule of the product $\prod_{j \in J} P_j$.*

(2) *If each P_i is a pure submodule of an R-module Q_i, then $\prod_{j \in J} P_j$ is a pure submodule of $\prod_{j \in J} Q_j$.*

The following result is well-known for the classical flat case (see, for instance, [23, Theorem 11.1 $(a \Leftrightarrow c)$]).

Lemma 2.12. *Let $\mathscr{X}$ be a class of R-modules such that, for a positive integer $n \geq 1$, the subclass $\mathscr{X}_n$ of n-presented R-modules of $\mathscr{X}$ is not empty.*

   (1) *Every pure submodule of an n-$\mathscr{X}$-flat R-module is n-$\mathscr{X}$-flat.*

   (2) *Every pure submodule of an n-$\mathscr{X}$-injective R-module is n-$\mathscr{X}$-injective.*

Proof. 1) Let A be a pure submodule of an n-$\mathscr{X}$-flat R-module B. Then, by Lemma 2.9 (1 $\Leftrightarrow$ 3), the sequence $0 \to (B/A)^* \to B^* \to A^* \to 0$ splits. Then, by Lemma 2.7(2) and being a direct summand of the n-$\mathscr{X}$-injective right R-module B^* (Lemma 2.8), the right R-module A^* is n-$\mathscr{X}$-injective. Therefore, by Lemma 2.8, A is n-$\mathscr{X}$-flat.

2) Let A be a pure submodule of an n-$\mathscr{X}$-injective R-module B. Consider an R-module $M \in \mathscr{X}_n$. Then, there is an exact sequence of R-modules:

$$F_n \to F_{n-1} \to \cdots \to F_1 \to F_0 \to M \to 0$$

such that each F_i is finitely generated and free. Consider $K_n = \mathrm{Im}(F_n \to F_{n-1})$ and $K_{n-1} = \mathrm{Im}(F_{n-1} \to F_{n-2})$. Then, we have the short exact sequence

$$0 \to K_n \to F_{n-1} \to K_{n-1} \to 0$$

Since $\mathrm{Ext}^1_R(K_{n-1}, A) \cong Ext^n_R(M, A)$, we have only to prove that $\mathrm{Ext}^1_R(K_{n-1}, A) = 0$. Applying the functor $\mathrm{Hom}_R(K_{n-1}, -)$ to the short exact sequence $0 \to A \to B \to B/A \to 0$, we get the following exact sequence:

$$\mathrm{Hom}_R(K_{n-1}, B) \to \mathrm{Hom}_R(K_{n-1}, B/A) \to \mathrm{Ext}^1_R(K_{n-1}, A) \to \mathrm{Ext}^1_R(K_{n-1}, B)$$

Since A is n-$\mathscr{X}$-injective, $\mathrm{Ext}^1_R(K_{n-1}, B) \cong \mathrm{Ext}^n_R(M, A) = 0$. Thus, the exact sequence above becomes

$$(\alpha) \qquad \mathrm{Hom}_R(K_{n-1}, B) \to \mathrm{Hom}_R(K_{n-1}, B/A) \to \mathrm{Ext}^1_R(K_{n-1}, A) \to 0$$

On the other hand, since M is n-presented, K_{n-1} is finitely presented, and so, by Lemma 2.9 (1 $\Leftrightarrow$ 2), we have the following exact sequence:

$$(\beta) \qquad \mathrm{Hom}_R(K_{n-1}, B) \to \mathrm{Hom}_R(K_{n-1}, B/A) \to 0$$

Therefore, by the sequences (α) and (β) above, we get $\mathrm{Ext}^1_R(K_{n-1}, A) = 0$. $\qquad \square$

Now we are ready to prove our second main result.

Theorem 2.13. *Let $\mathscr{X}$ be a class of R-modules such that, for a positive integer $n \geq 1$, the subclass $\mathscr{X}_n$ of n-presented R-modules of $\mathscr{X}$ is not empty. Then, the following assertions are equivalent:*

   (1) *R is left n-$\mathscr{X}$-coherent;*

(2) *An R-module M is n-$\mathscr{X}$-injective if and only if M^* is n-$\mathscr{X}$-flat;*

(3) *An R-module M is n-$\mathscr{X}$-injective if and only if M^{**} is n-$\mathscr{X}$-injective;*

(4) *A right R-module M is n-$\mathscr{X}$-flat if and only if M^{**} is n-$\mathscr{X}$-flat.*

Proof. $(1 \Rightarrow 2)$ Since R is left n-$\mathscr{X}$-coherent, every n-presented module in $\mathscr{X}$ is infinitely presented, and so, from [21, Theorem 9.51 and the remark following it], we have:

$$\mathrm{Tor}_n^R(M^*, N) \cong (\mathrm{Ext}_R^n(N, M))^*$$

for every R-module $N \in \mathscr{X}_n$. This shows that an R-module M is n-$\mathscr{X}$-injective if and only if M^* is n-$\mathscr{X}$-flat.

$(2 \Rightarrow 3)$ Follows from the equivalence of (2) and Lemma 2.8.

$(3 \Rightarrow 4)$ Let M be n-$\mathscr{X}$-flat right R-module. Then, by Lemma 2.8, M^* is n-$\mathscr{X}$-injective, and by (3), M^{***} is n-$\mathscr{X}$-injective. Therefore, by Lemma 2.8, M^{**} n-$\mathscr{X}$-flat. Conversely, consider a right R-module M such that M^{**} is n-$\mathscr{X}$-flat. By Lemma 2.10, M is a pure submodule of M^{**}. Then, Lemma 2.12 (1) shows that M is n-$\mathscr{X}$-flat, too.

$(4 \Rightarrow 1)$ Using Theorem 2.6, we have to prove that every product of n-$\mathscr{X}$-flat right R-modules is n-$\mathscr{X}$-flat. Then, consider a family $(P_j)_{j \in J}$ of n-$\mathscr{X}$-flat right R-modules. From Lemma 2.7 (1), the sum $\bigoplus_{j \in J} P_j$ is n-$\mathscr{X}$-flat. Then, by (4), $(\bigoplus_{j \in J} P_j)^{**} \cong (\prod_{j \in J} P_j^*)^*$ is n-$\mathscr{X}$-flat. On the other hand, from Lemma 2.11 (1), the sum $\bigoplus_{j \in J} P_j^*$ is a pure submodule of the product $\prod_{j \in J} P_j^*$. Then, by Lemma 2.9 (1 $\Leftrightarrow$ 3), we deduce that $(\bigoplus_{j \in J} P_j^*)^*$ is a direct summand of $(\prod_{j \in J} P_j^*)^*$, and so $\prod_{j \in J} P_j^{**} \cong (\bigoplus_{j \in J} P_j^*)^*$ is n-$\mathscr{X}$-flat. Therefore, using Lemmas 2.11 (2) and 2.12 (1), the direct product $\prod_{j \in J} P_j$ is n-$\mathscr{X}$-flat. $\qquad \square$

We end this paper with a counterpart of [13, Proposition 6.5.1] such that we give a characterization of n-$\mathscr{X}$-coherent by n-$\mathscr{X}$-flat preenvelope. Here, the n-$\mathscr{X}$-flat preenvelopes are Enochs' $\mathscr{F}$-preenvelopes, where $\mathscr{F}$ is the class of all n-$\mathscr{X}$-flat modules (see Introduction). The proof of this result is analogous to the one of [13, Proposition 6.5.1]. So we need the following two results.

Lemma 2.14. [13, Lemma 5.3.12] *Let F and N be R-modules. Then, there is a cardinal number $\aleph_\alpha$ such that, for every homomorphism $f : N \to F$, there is a pure submodule P of F such that $f(N) \subset P$ and $\mathrm{Card}(P) \leq \aleph_\alpha$.*

Lemma 2.15. [13, Corollary 6.2.2] *Let $\mathscr{X}$ be a class of R-modules that is closed under direct products. Let M be an R-module with $\mathrm{Card}(M) = \aleph_\beta$. Suppose that there is a cardinal $\aleph_\alpha$ such that, for an R-module $F \in \mathscr{X}$ and a submodule N of F with $\mathrm{Card}(P) \leq \aleph_\beta$, there is a submodule P of F containing N with $P \in \mathscr{X}$ and $\mathrm{Card}(P) \leq \aleph_\alpha$. Then, M has an $\mathscr{X}$-preenvelope.*

Theorem 2.16. *Let $\mathscr{X}$ be a class of R-modules such that, for a positive integer $n \geq 1$, the subclass $\mathscr{X}_n$ of n-presented R-modules of $\mathscr{X}$ is not empty. Then, R is left n-$\mathscr{X}$-coherent if and only if every right R-module M has an n-$\mathscr{X}$-flat preenvelope.*

Proof. ($\Rightarrow$) Let M be a right R-module with $\mathrm{Card}(M) = \aleph_\beta$. From Lemma 2.14, there is a cardinal $\aleph_\alpha$ such that, for an n-$\mathscr{X}$-flat right R-module F and a submodule N of F with $\mathrm{Card}(P) \leq \aleph_\beta$, there is a pure submodule P of F containing N and $\mathrm{Card}(P) \leq \aleph_\alpha$. From Lemma 2.12 (1), P is n-$\mathscr{X}$-flat. Therefore, since the class of all n-$\mathscr{X}$-flat right R-modules is closed under direct products (by Theorem 2.6), Lemma 2.15 shows that M has an n-$\mathscr{X}$-flat preenvelope.

($\Leftarrow$) To prove that R is left n-$\mathscr{X}$-coherent, it is sufficient, by Theorem 2.6, to prove that every product of n-$\mathscr{X}$-flat right R-modules is n-$\mathscr{X}$-flat. Consider a family $(P_j)_{j \in J}$ of n-$\mathscr{X}$-flat right R-modules. By hypothesis, $\prod\limits_{j \in J} P_j$ has an n-$\mathscr{X}$-flat preenvelope $f : \prod\limits_{j \in J} P_j \to F$. Then, for each canonical projection $p_i : \prod\limits_{j \in J} P_j \to P_i$ with $i \in J$, there exists a homomorphism $p_i : F \to P_i$ such that $h_i f = p_i$. Now, consider the homomorphism $h = (h_j)_{j \in J} : F \to \prod\limits_{j \in J} P_j$ defined by $h(x) = (h_j(x))_{j \in J}$ for every $x \in F$. Then, for every $a = (a_j)_{j \in J} \in \prod\limits_{j \in J} P_j$, we have:

$$hf(a) = (h_j(f(a)))_{j \in J} = (p_j(a))_{j \in J} = a.$$

This means that $hf = 1_{\Pi P_j}$. Then, $\prod\limits_{j \in J} P_j$ is a direct summand of F. Therefore, by Lemma 2.7, $\prod\limits_{j \in J} P_j$ is n-$\mathscr{X}$-flat. $\qquad \square$

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