International Electronic Journal of Algebra
Volume 7 (2010) 140-151

ON GENERALIZED RELATIVE COMMUTATIVITY DEGREE OF A FINITE GROUP

A. K. Das and R. K. Nath

Received: 12 October 2009; Revised: 15 December 2009
Communicated by A. Çiğdem Özcan

Abstract. Given any two subgroups H and K of a finite group G, and an element $g \in G$, the aim of this article is to study the probability that the commutator of an arbitrarily chosen pair of elements (one from H and the other from K) equals g.

Mathematics Subject Classification (2000): 20C15, 20D60, 20P05
Keywords: commutativity degree, commutators, group characters

1. Introduction

Throughout this paper G denotes a finite group, H and K two subgroups of G, and g an element of G. In [1], Erfanian et al. have considered the probability $\Pr(H, G)$ for an element of H to commute with an element of G. On the other hand, in [8], Pournaki et al. have studied the probability $\Pr_g(G)$ that the commutator of an arbitrarily chosen pair of group elements equals g (a generalization of this notion can be found also in [7]). The main object of this paper is to further generalize these notions and study the probability that the commutator of a randomly chosen pair of elements (one from H and the other from K) equals g. In other words, we study the ratio

$$\Pr_g(H, K) = \frac{|\{(x, y) \in H \times K : xyx^{-1}y^{-1} = g\}|}{|H||K|}, \qquad (1)$$

and further extend some of the results obtained [1] and [8]. In the final section, with H normal in G, we also develop and study a character theoretic formula for $\Pr_g(H, G)$, which generalizes the formula for $\Pr_g(G)$ given in ([8], Theorem 2.1). In the process we generalize a classical result of Frobenius (see [2]).

Note that if $H = K = G$ then $\Pr_g(H, K) = \Pr_g(G)$, which coincides with the usual commutativity degree $\Pr(G)$ of G if we take $g = 1$, the identity element of G. It may be recalled (see, for example, [3]) that $\Pr(G) = \frac{k(G)}{|G|}$ where $k(G)$ denotes the number of conjugacy classes of G. On the other hand, if $K = G$ and $g = 1$ then $\Pr_g(H, K) = \Pr(H, G)$.

2. Some basic properties and a computing formula

Let $[H,K]$ denote the subgroup of G generated by the commutators $[x,y] = xyx^{-1}y^{-1}$ with $x \in H$ and $y \in K$. Also, for brevity, let us write $\Pr_1(H,K) = \Pr(H,K)$. Clearly,

$$\Pr(H,K) = 1 \iff [H,K] = \{1\},$$
$$\text{and} \quad \Pr_g(H,K) = 0 \iff g \notin \{[x,y] : x \in H, y \in K\}.$$

It is also easy to see that if $C_K(x) = \{1\}$ for all $x \in H - \{1\}$ then

$$\Pr(H,K) = \frac{1}{|H|} + \frac{1}{|K|} - \frac{1}{|H||K|}. \tag{2}$$

The following proposition says that $\Pr_g(H,K)$ is not very far from being symmetric with respect to H and K.

Proposition 2.1. $\Pr_g(H,K) = \Pr_{g^{-1}}(K,H)$. *However, if* $g^2 = 1$, *or if* $g \in H \cup K$ *(for example, when* H *or* K *is normal in* G*), we have* $\Pr_g(H,K) = \Pr_g(K,H) = \Pr_{g^{-1}}(H,K)$.

Proof. The first part follows from the fact that $[x,y]^{-1} = [y,x]$. On the other hand, for the second part, it is enough to note that if $g \in H$ then $(x,y) \mapsto (y^{-1}, yxy^{-1})$, and if $g \in K$ then $(x,y) \mapsto (xyx^{-1}, x^{-1})$ define bijective maps between the sets $\{(x,y) \in H \times K : [x,y] = g\}$ and $\{(y,x) \in K \times H : [y,x] = g\}$. □

$\Pr_g(H,K)$ respects the Cartesian product in the following sense.

Proposition 2.2. *Let* G_1 *and* G_2 *be two finite groups with subgroups* $H_1, K_1 \subseteq G_1$ *and* $H_2, K_2 \subseteq G_2$. *Let* $g_1 \in G_1$ *and* $g_2 \in G_2$. *Then,*

$$\Pr_{(g_1,g_2)}(H_1 \times H_2, K_1 \times K_2) = \Pr_{g_1}(H_1,K_1)\Pr_{g_2}(H_2,K_2).$$

Proof. It is enough to note that for all $x_1, y_1 \in G_1$ and for all $x_2, y_2 \in G_2$ we have $[(x_1,x_2),(y_1,y_2)] = ([x_1,y_1],[x_2,y_2])$. □

We now derive a computing formula which plays a key role in the study of $\Pr_g(H,K)$.

Theorem 2.3.

$$\Pr_g(H,K) = \frac{1}{|H||K|} \sum_{\substack{x \in H \\ g^{-1}x \in \mathrm{C}\ell_K(x)}} |C_K(x)| = \frac{1}{|H|} \sum_{\substack{x \in H \\ g^{-1}x \in \mathrm{C}\ell_K(x)}} \frac{1}{|\mathrm{C}\ell_K(x)|},$$

where $\mathrm{C}\ell_K(x) = \{yxy^{-1} : y \in K\}$, *the* K*-conjugacy class of* x.

Proof. We have $\{(x,y) \in H \times K : xyx^{-1}y^{-1} = g\} = \underset{x\in H}{\cup}\{x\} \times T_x$, where $T_x = \{y \in K : [x,y] = g\}$. Note that, for any $x \in H$, we have

$$T_x \neq \phi \Longleftrightarrow g^{-1}x \in \mathrm{C}\ell_K(x).$$

Let $T_x \neq \phi$ for some $x \in H$. Fix an element $y_0 \in T_x$. Then, $y \mapsto gy_0^{-1}y$ defines a one to one correspondence between the set T_x and the coset $gC_K(x)$. This means that $|T_x| = |C_K(x)|$.

Thus, we have

$$|\{(x,y) \in H \times K : xyx^{-1}y^{-1} = g\}| = \sum_{x\in H}|T_x| = \sum_{\substack{x\in H \\ g^{-1}x\in \mathrm{C}\ell_K(x)}} |C_K(x)|.$$

The first equality in the theorem now follows from (1).

For the second equality, consider the action of K on G by conjugation. Then, for all $x \in G$, we have

$$|\,\mathrm{C}\ell_K(x)| = |\,\mathrm{orb}(x)| = |K : \mathrm{stab}(x)| = \frac{|K|}{|C_K(x)|}. \qquad (3)$$

This completes the proof. □

As an immediate consequence, we have the following generalization of the well-known formula $\Pr(G) = \frac{k(G)}{|G|}$.

Corollary 2.4. *If H is normal in G then*

$$\Pr(H,K) = \frac{k_K(H)}{|H|},$$

where $k_K(H)$ is the number of K-conjugacy classes that constitute H.

Proof. Note that K acts on H by conjugation. The orbit of any element $x \in H$ under this action is given by $\mathrm{C}\ell_K(x)$, and so H is the disjoint union of these classes. Hence, we have

$$\Pr(H,K) = \frac{1}{|H|}\sum_{x\in H}\frac{1}{|\,\mathrm{C}\ell_K(x)|} = \frac{k_K(H)}{|H|};$$

noting that, for $g = 1$, the condition $g^{-1}x \in \mathrm{C}\ell_K(x)$ is superfluous. □

The Schur-Zassenhaus Theorem (see [6, page 125]) says that if H is a normal subgroup of G such that $\gcd(|H|, |G : H|) = 1$ then H has a complement in G. In particular, if H is a normal subgroup of G with $C_G(x) \subseteq H$ for all $x \in H - \{1\}$ then, using Sylow's theorems and the fact that nontrivial p-groups have nontrivial centers, we have $\gcd(|H|, |G : H|) = 1$. So, by the Schur-Zassenhaus Theorem, H has a complement in G. Such groups belong to a well-known class of groups called the Frobenius Groups; for example, the alternating group A_4, the dihedral groups of order $2n$ with n odd, the nonabelian groups of order pq where p and q are primes with $q|(p-1)$.

Proposition 2.5. *If H is an abelian normal subgroup of G with a complement K in G then*

$$\Pr_g(H,G) = \Pr_g(H,K).$$

Proof. Let $x \in H$. Since H is abelian, we have

$$C_{HK}(x) = \{hk : hkx = xhk\} = \{hk : kx = xk\} = HC_K(x).$$

Thus, $|C_{HK}(x)| = |H|\,|C_K(x)|$. Also, since H is abelian and normal, $\mathrm{C}\ell_K(x) = \mathrm{C}\ell_{HK}(x)$. Hence, from Theorem 2.3, it follows that

$$\Pr_g(H,G) = \frac{1}{|H|^2|K|} \sum_{\substack{x \in H \\ g^{-1}x \in \mathrm{C}\ell_{HK}(x)}} |C_{HK}(x)| = \Pr_g(H,K).$$

This completes the proof. □

Corollary 2.6. *If H is a normal subgroup of G with $C_G(x) = H$ for all $x \in H - \{1\}$ then*

$$\Pr_g(H,G) = \Pr_g(H,K),$$

where K is a complement of H in G. In particular,

$$\Pr(H,G) = \frac{1}{|H|} + \frac{|H|-1}{|G|}.$$

Proof. The first part follows from the discussion preceding the above proposition, and the second part follows from (2). □

3. Some bounds and inequalities

We begin with the inequality

$$\Pr(H,K) \geq \frac{|C_H(K)|}{|H||K|} + \frac{|C_K(H)|(|H| - |C_H(K)|)}{|H||K|},$$

which follows from (1) using the fact that

$$(C_H(K) \times K) \cup (H \times C_K(H)) \subseteq \{(x,y) \in H \times K : xyx^{-1}y^{-1} = 1\}.$$

On the other hand, we have

Proposition 3.1. *If $g \neq 1$ then*

(i) $\Pr_g(H,K) \neq 0 \Longrightarrow \Pr_g(H,K) \geq \dfrac{|C_H(K)||C_K(H)|}{|H||K|}$,

(ii) $\Pr_g(H, G) \neq 0 \Longrightarrow \Pr_g(H, G) \geq \dfrac{2|H \cap Z(G)||C_G(H)|}{|H||G|}$,

(iii) $\Pr_g(G) \neq 0 \Longrightarrow \Pr_g(G) \geq \dfrac{3}{|G : Z(G)|^2}$.

Proof. Let $g = [x, y]$ for some $(x, y) \in H \times K$. Since $g \neq 1$, we have $x \notin C_H(K)$ and $y \notin C_K(H)$. Consider the left coset $T_{(x,y)} = (x, y)(C_H(K) \times C_K(H))$ of $C_H(K) \times C_K(H)$ in $H \times K$. Clearly, $|T_{(x,y)}| = |C_H(K)||C_K(H)|$, and $[a, b] = g$ for all $(a, b) \in T_{(x,y)}$. This proves part (i).

Similarly, part (ii) follows considering the two disjoint cosets $T_{(x,y)}$ and $T_{(x,yx)}$ with $K = G$, while part (iii) follows considering the three disjoint cosets $T_{(x,y)}$, $T_{(xy,x)}$, and $T_{(x,yx)}$ with $H = K = G$. □

As a generalization of Proposition 5.1 of [8], we have

Proposition 3.2.

$$\Pr_g(H, K) \leq \Pr(H, K),$$

with equality if and only if $g = 1$.

Proof. By Theorem 2.3, we have

$$\begin{aligned}\Pr_g(H, K) &= \frac{1}{|H||K|} \sum_{\substack{x \in H \\ g^{-1}x \in \mathrm{C}\ell_K(x)}} |C_K(x)| \\ &\leq \frac{1}{|H||K|} \sum_{x \in H} |C_K(x)| = \Pr(H, K).\end{aligned}$$

Clearly, the equality holds if and only if $g^{-1}x \in \mathrm{C}\ell_K(x)$ for all $x \in H$, that is, if and only if $g = 1$. □

The following is an improvement to Proposition 5.2 of [8].

Proposition 3.3. *Let* p *be the smallest prime dividing* $|G|$, *and* $g \neq 1$. *Then,*

$$\Pr_g(H, K) \leq \frac{|H| - |C_H(K)|}{p|H|} < \frac{1}{p}.$$

Proof. Without any loss, we may assume that $C_H(K) \neq H$. Let $x \in H$ be such that $g^{-1}x \in \mathrm{C}\ell_K(x)$. Then, since $g \neq 1$, we have $x \notin C_H(K)$ and $|\,\mathrm{C}\ell_K(x)| > 1$. But $|\,\mathrm{C}\ell_K(x)|$ is a divisor of $|K|$, and hence of $|G|$. Therefore, $|\,\mathrm{C}\ell_K(x)| \geq p$. Hence, by Theorem 2.3, we have

$$\Pr_g(H, K) \leq \frac{1}{|H|} \sum_{\substack{x \in H \\ g^{-1}x \in \mathrm{C}\ell_K(x)}} \frac{1}{p} \leq \frac{|H| - |C_H(K)|}{p|H|} < \frac{1}{p}.$$

This completes the proof. □

$\Pr(H, K)$ is monotonic in the following sense.

Proposition 3.4. *If $K_1 \subseteq K_2$ are subgroups of G then*

$$\Pr(H, K_1) \geq \Pr(H, K_2),$$

with equality if and only if $\mathrm{C}\ell_{K_1}(x) = \mathrm{C}\ell_{K_2}(x)$ for all $x \in H$.

Proof. Clearly, $\mathrm{C}\ell_{K_1}(x) \subseteq \mathrm{C}\ell_{K_2}(x)$ for all $x \in H$. So, by Theorem 2.3, we have

$$\Pr(H, K_1) = \frac{1}{|H|} \sum_{x \in H} \frac{1}{|\,\mathrm{C}\ell_{K_1}(x)|} \geq \frac{1}{|H|} \sum_{x \in H} \frac{1}{|\,\mathrm{C}\ell_{K_2}(x)|} = \Pr(H, K_2).$$

The condition for equality is obvious. □

Since $\Pr(H, K) = \Pr(K, H)$, it follows from the above proposition that if $H_1 \subseteq H_2$ and $K_1 \subseteq K_2$ are subgroups of G then

$$\Pr(H_1, K_1) \geq \Pr(H_2, K_2).$$

Proposition 3.5. *If $K_1 \subseteq K_2$ are subgroups of G then*

$$\Pr(H, K_2) \geq \frac{1}{|K_2 : K_1|} \left(\Pr(H, K_1) + \frac{|K_2| - |K_1|}{|H||K_1|} \right),$$

with equality if and only if $C_H(x) = \{1\}$ for all $x \in K_2 - K_1$.

Proof. By Theorem 2.3, We have

$$\begin{aligned} \Pr(H, K_2) &= \frac{1}{|H||K_2|} \left(\sum_{x \in K_1} |C_H(x)| + \sum_{x \in K_2 - K_1} |C_H(x)| \right) \\ &\geq \frac{1}{|K_2 : K_1|} \left(\Pr(H, K_1) + \frac{|K_2| - |K_1|}{|H||K_1|} \right). \end{aligned}$$

Clearly, the equality holds if and only if

$$\sum_{x \in K_2 - K_1} (|C_H(x)| - 1) = 0,$$

that is, if and only if $C_H(x) = \{1\}$ for all $x \in K_2 - K_1$. □

In general, $\Pr_g(H, K)$ is not monotonic. For example, if $G = S_3$, $g = (123)$, $H =< (12) >$, $K_1 =< (1) >$, $K_2 =< (13) >$, and $K_3 = S_3$, then

$$\Pr_g(H, K_1) = 0 \lneq \Pr_g(H, K) = \frac{1}{4} \gneq \Pr_g(H, K) = \frac{1}{6}.$$

However, we have

Proposition 3.6. *If $H_1 \subseteq H_2$ and $K_1 \subseteq K_2$ are subgroups of G then*

$$\Pr_g(H_1, K_1) \leq |H_2 : H_1||K_2 : K_1|\Pr_g(H_2, K_2),$$

with equality if and only if

$$\begin{aligned} &g^{-1}x \notin \mathrm{C}\ell_{K_2}(x) \ \text{ for all } \ x \in H_2 - H_1, \\ &g^{-1}x \notin \mathrm{C}\ell_{K_2}(x) - \mathrm{C}\ell_{K_1}(x) \ \text{ for all } \ x \in H_1, \\ \text{and} \quad &C_{K_1}(x) = C_{K_2}(x) \ \text{ for all } \ x \in H_1 \ \text{ with } \ g^{-1}x \in \mathrm{C}\ell_{K_1}(x). \end{aligned}$$

In particular, for $g = 1$, the condition for equality reduces to $H_1 = H_2$, and $K_1 = K_2$.

Proof. By Theorem 2.3, we have

$$\begin{aligned} |H_1||K_1|\Pr_g(H_1, K_1) &= \sum_{\substack{x \in H_1 \\ g^{-1}x \in \mathrm{C}\ell_{K_1}(x)}} |C_{K_1}(x)| \\ &\leq \sum_{\substack{x \in H_2 \\ g^{-1}x \in \mathrm{C}\ell_{K_2}(x)}} |C_{K_2}(x)| = |H_2||K_2|\Pr_g(H_2, K_2). \end{aligned}$$

The condition for equality follows immediately. □

Using Propostion 3.2, we have

Corollary 3.7.

$$\Pr_g(H, G) \leq |G : H| \Pr(G),$$

with equality if and only if $g = 1$ and $H = G$.

The following theorem generalizes Theorem 3.5 of [1].

Theorem 3.8. *Let p be the smallest prime dividing $|G|$. Then*

$$\begin{aligned} \Pr(H, K) &\geq \frac{|C_H(K)|}{|H|} + \frac{p(|H| - |X_H| - |C_H(K)|) + |X_H|}{|H||K|}, \\ \text{and } \ \Pr(H, K) &\leq \frac{(p-1)|C_H(K)| + |H|}{p|H|} - \frac{|X_H|(|K| - p)}{p|H||K|}, \end{aligned}$$

where $X_H = \{x \in H : C_K(x) = 1\}$. Moreover, in each of these bounds, H and K can be interchanged.

Proof. If $[H, K] = 1$ then there is nothing to prove, as in that case $C_H(K) = H$, and X_H equals H or an empty set according as K is trivial or nontrivial.

On the other hand, if $[H,K]\neq 1$ then $X_H\cap C_H(K)=\phi$, and so

$$\begin{aligned}\sum_{x\in H}|C_K(x)| &= \sum_{x\in X_H}|C_K(x)| + \sum_{x\in C_H(K)}|C_K(x)| + \sum_{x\in H-(X_H\cup C_H(K))}|C_K(x)| \\ &= |X_H| + |K||C_H(K)| + \sum_{x\in H-(X_H\cup C_H(K))}|C_K(x)|.\end{aligned}$$

But, for all $x\in H-(X_H\cup C_H(K))$, we have $\{1\}\neq C_K(x)\neq K$, which means that $p\leq C_K(x)\leq \frac{|K|}{p}$. Hence, using Theorem 2.3, we get the required bounds for $\Pr(H,K)$. The final statement of the theorem follows from the fact that $\Pr(H,K)=\Pr(K,H)$. □

As a consequence we have

Corollary 3.9. *Let $[H,K]\neq\{1\}$. If p is the smallest prime divisor of $|G|$ then*

$$\Pr(H,K)\leq\frac{2p-1}{p^2}.$$

In particular, $\Pr(H,K)\leq\frac{3}{4}$.

Proof. Since, $[H,K]\neq\{1\}$, we have $K\neq\{1\}$ and $C_H(K)\neq H$. So, $|K|\geq p$ and $|C_H(K)|\leq\frac{|H|}{p}$. Therefore, by Theorem 3.8, we have

$$\Pr(H,K)\leq\frac{(p-1)|C_H(K)|+|H|}{p|H|}\leq\frac{\frac{p-1}{p}+1}{p}=\frac{2p-1}{p^2}\leq\frac{3}{4},$$

since $p\geq 2$. □

One can see that the above bound is best possible. For example, consider two non-commuting elements a and b of order p in a nonabelian group G with p as the smallest prime dividing $|G|$. Then, using (2) for $H=<a>$ and $K=<b>$, we have $\Pr(H,K)=\frac{2p-1}{p^2}$.

Proposition 3.10. *Let $\Pr(H,K)=\frac{2p-1}{p^2}$ for some prime p. Then, p divides $|G|$. If p happens to be the smallest prime divisor of $|G|$ then*

$$\frac{H}{C_H(K)}\cong\mathbb{Z}_p\cong\frac{K}{C_K(H)},\ \ \textit{and hence,}\ \ H\neq K.$$

In particular, $\frac{H}{C_H(K)}\cong\mathbb{Z}_2\cong\frac{K}{C_K(H)}$ *if* $\Pr(H,K)=\frac{3}{4}$.

Proof. The first part follows from the definition of $\Pr(H,K)$.

For the second part, by Theorem 3.8, we have

$$\begin{aligned}\frac{2p-1}{p^2} &\leq \frac{(p-1)|C_H(K)|+|H|}{p|H|}\\ \Longrightarrow |H:C_H(K)| &\leq p.\end{aligned}$$

Since p is the smallest prime divisor of $|G|$, it follows that $|H : C_H(K)| = p$, whence $\frac{H}{C_H(K)} \cong \mathbb{Z}_p$; noting that $C_H(K) \neq H$ because $\Pr(H, K) = \frac{2p-1}{p^2} \neq 1$. Similarly, we have $\frac{K}{C_K(H)} \cong \mathbb{Z}_p$. Since $\frac{H}{Z(H)}$ is never cyclic unless trivial, we have $H \neq K$.

The third part follows from the first two parts. □

4. A character theoretic formula

In this section, all results are under the assumption that H is normal in G. Let $\zeta(g)$ denote the number of solutions $(x, y) \in H \times G$ of the equation $[x, y] = g$. Thus, by (1),

$$\Pr_g(H, G) = \frac{\zeta(g)}{|H||G|}. \tag{4}$$

Our quest for a character theoretic formula for $\Pr_g(H, G)$ starts with the following lemma.

Lemma 4.1. *$\zeta(g)$ defines a class function on G.*

Proof. It is enough to note that, for each $a \in G$, the map $(x, y) \mapsto (axa^{-1}, aya^{-1})$ defines a one to one correspondence between the sets $\{(x, y) \in H \times G : [x, y] = g\}$ and $\{(x, y) \in H \times G : [x, y] = aga^{-1}\}$. □

Thus, we have

$$\zeta(g) = \sum_{\chi \in \mathrm{Irr}(G)} \langle \zeta, \chi \rangle \chi(g), \tag{5}$$

where $\mathrm{Irr}(G)$ denotes the set of all irreducible complex characters of G, and $\langle\, , \rangle$ denotes the inner product.

We now prove the main result of this section.

Theorem 4.2.

$$\zeta(g) = \sum_{\chi \in \mathrm{Irr}(G)} \frac{|H| \langle \chi_{_H}, \chi_{_H} \rangle}{\chi(1)} \chi(g),$$

where $\chi_{_H}$ denotes the restriction of χ to H.

Proof. Let $\chi \in \mathrm{Irr}(G)$, and let Φ_χ be a representation of G which affords χ. Then, as in the proof of Theorem 1 of [9], we have, by Schur's lemma (see [4], Lemma 2.25),

$$\sum_{y \in G} \Phi_\chi(yx^{-1}y^{-1}) = \frac{|G|}{\chi(1)} \chi(x^{-1})\, \mathrm{I}_\chi,$$

where $x \in H$ and I_χ is the identity matrix of size $\chi(1)$. Multiplying both sides by $\Phi_\chi(x)$, and summing over all $x \in H$, we get

$$\sum_{(x,y) \in H \times G} \Phi_\chi(xyx^{-1}y^{-1}) = \frac{|G|}{\chi(1)} \sum_{x \in H} \Phi_\chi(x) \chi(x^{-1}).$$

Taking trace, we have

$$\sum_{(x,y)\in H\times G} \chi(xyx^{-1}y^{-1}) = \frac{|G|}{\chi(1)} \sum_{x\in H} \chi(x)\chi(x^{-1})$$

$$\Longrightarrow \sum_{g\in G} \chi(g)\zeta(g) = \frac{|G|}{\chi(1)} \sum_{x\in H} \chi_{_H}(x)\overline{\chi_{_H}(x)}$$

$$\Longrightarrow \langle \chi, \zeta\rangle = \frac{|H|\langle \chi_{_H}, \chi_{_H}\rangle}{\chi(1)}.$$

Hence, in view of (5), the theorem follows. □

In particular, we have

Corollary 4.3. *ζ is a character of G,*

Proof. It is enough to show that $\chi(1)$ divides $|H|\langle \chi_{_H}, \chi_{_H}\rangle$ for every $\chi \in \mathrm{Irr}(G)$. With notations same as in the proof of the above theorem, consider the algebra homomorphism $\omega_\chi : \mathbf{Z}(\mathbb{C}[G]) \longrightarrow \mathbb{C}$ given by $\Phi(z) = \omega_\chi(z)\,\mathrm{I}_\chi$ for all $z \in \mathbf{Z}(\mathbb{C}[G])$. Since H is normal in G, there exist $x_1, x_2, \ldots, x_r \in H$ such that $H = \underset{1\leq i\leq r}{\cup} \mathrm{C}\ell_G(x_i)$. Let $K_i = \sum_{x\in \mathrm{C}\ell_G(x_i)} x$, the class sum corresponding to $\mathrm{C}\ell_G(x_i)$, $1 \leq i \leq r$. By ([4, Theorem 3.7] and the preceding discussion), $\omega_\chi(K_i)$ is an algebraic integer with

$$\omega_\chi(K_i) = \frac{\chi(x_i)|\,\mathrm{C}\ell_G(x_i)|}{\chi(1)},\ 1 \leq i \leq r.$$

Therefore, it follows that

$$|H|\langle \chi_{_H}, \chi_{_H}\rangle = \sum_{x\in H} \chi(x)\chi(x^{-1}) = \sum_{1\leq i\leq r} |\,\mathrm{C}\ell_G(x_i)|\chi(x_i)\chi(x_i^{-1})$$

$$= \sum_{1\leq i\leq r} \chi(1)\omega_\chi(K_i)\chi(x_i^{-1}).$$

Thus,

$$\frac{|H|\langle \chi_{_H}, \chi_{_H}\rangle}{\chi(1)} = \sum_{1\leq i\leq r} \omega_\chi(K_i)\chi(x_i^{-1}),$$

which is an algebraic integer, and hence, an integer. This completes the proof. □

In view of (4), the following character theoretic formula for $\mathrm{Pr}_g(H, G)$ can be easily derived from Theorem 4.2.

$$\mathrm{Pr}_g(H, G) = \frac{1}{|G|} \sum_{\chi\in \mathrm{Irr}(G)} \langle \chi_{_H}, \chi_{_H}\rangle \frac{\chi(g)}{\chi(1)}. \tag{6}$$

This formula enables us to strengthen Corollary 3.7 as follows.

Proposition 4.4. *If $g \in G'$, the commutator subgroup of G, then*
$$\left|\Pr_g(H,G) - \frac{1}{|G'|}\right| \leq |G:H| \left(\Pr(G) - \frac{1}{|G'|}\right).$$

Proof. For all $\chi \in \operatorname{Irr}(G)$, with $\chi(1) = 1$, we have $\langle \chi_H, \chi_H \rangle = 1$ and $G' \subseteq \ker \chi$. Also, $|G : G'|$ equals the number of linear characters of G. Therefore, by (6),
$$\Pr_g(H,G) = \frac{1}{|G'|} + \frac{1}{|G|} \sum_{\substack{\chi \in \operatorname{Irr}(G) \\ \chi(1) \neq 1}} \langle \chi_H, \chi_H \rangle \frac{\chi(g)}{\chi(1)}.$$
Since $|\chi(g)| \leq \chi(1)$ for all $\chi \in \operatorname{Irr}(G)$, we have, using Lemma 2.29 of [4],
$$\begin{aligned}\left|\Pr_g(H,G) - \frac{1}{|G'|}\right| &\leq \sum_{\substack{\chi \in \operatorname{Irr}(G) \\ \chi(1) \neq 1}} \langle \chi_H, \chi_H \rangle \\ &\leq \frac{1}{|G|} \left(|\operatorname{Irr}(G)| - |G:G'| \right) |G:H| \\ &= |G:H| \left(\Pr(G) - \frac{1}{|G'|} \right).\end{aligned}$$
This completes the proof. □

In particular, we have

Corollary 4.5. *If G' contains a non-commutator (an element which is not a commutator) then* $\Pr(G) \geq \frac{2}{|G'|}$.

Proof. The corollary follows by choosing a non-commutator $g \in G'$, and putting $H = G$. □

As a consequence, we have the following result which is closely related to the subject matter of [5].

Proposition 4.6. *If $|G'| < p^2 + 1$, where p is the smallest prime divisor of $|G|$, then every element of G' is a commutator.*

Proof. It is well-known that
$$\Pr(G) \leq \frac{1}{|G'|} \left[1 + \frac{|G'| - 1}{p^2} \right],$$
which, in fact, can be derived from the inequality
$$|G| = \sum_{\chi \in \operatorname{Irr}(G)} \chi(1)^2 \geq |G:G'| + p^2[k(G) - |G:G'|].$$
Hence, if G' contains a non-commutator, it follows, using Corollary 4.5, that $|G'| \geq p^2 + 1$. □

References

[1] A. Erfanian, R. Rezaei and P. Lescot, *On the Relative Commutativity Degree of a Subgroup of a Finite Group*, Comm. Algebra, 35 (12) (2007), 4183–4197.

[2] F. G. Frobenius, Über Gruppencharaktere, Gesammelte Abhandlungen Bände III, p. 1–37 (J. P. Serre, ed.), Springer-Verlag, Berlin, 1968.

[3] W. H. Gustafson, *What is the probability that two group elements commute?*, Amer. Math. Monthly, 80 (1973), 1031–1034.

[4] I. M. Isaacs, Character Theory of Finite Groups, Dover Publications, Inc., New YorK, 1994.

[5] L. Kappe and R. F. Morse, On commutators in groups, (English summary) Groups St. Andrews 2005. Vol. 2, 531–558, London Math. Soc. Lecture Note Ser., 340, Cambridge Univ. Press, Cambridge, 2007.

[6] H. Kurzweil and B. Stellmacher, The Theory of Finite Groups, An Introduction, Universitext, Springer, New York-Berlin-Heidelberg, 2004.

[7] R. K. Nath and A. K. Das, *On generalized commutativity degree of a finite group*, to appear in Rocky Mountain J. Math.

[8] M. R. Pournaki and R. Sobhani, *Probability that the commutator of two group elements is equal to a given element*, J. Pure and Applied Algebra, 212 (2008), 727–734.

[9] T. Tambour, *The number of solutions of some equations in finite groups and a new proof of Itô's theorem*, Comm. Algebra, 28 (11) (2000), 5353–5361.

A. K. Das and **R. K. Nath**
Department of Mathematics,
North-Eastern Hill University,
Permanent Campus, Shillong-793022,
Meghalaya, India.
e-mails: akdas@nehu.ac.in (A. K. Das)
rajatkantinath@yahoo.com (R. K. Nath)