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ASCENTS AND DESCENTS OF SEMISTAR OPERATIONS AND LOCALIZING SYSTEMS

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Abstract. We study ascents and descents of semistar operations and localizing systems for any extension domains.

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1. Introduction

Let D be an integral domain with quotient field K, and let $\bar{\mathrm{F}}(D)$ be the set of non-zero D-submodules of K. A mapping $\bar{\mathrm{F}}(D) \xrightarrow{\star} \bar{\mathrm{F}}(D)$, $E \longmapsto E^\star$ is called a semistar operation if, for every $x \in K \setminus \{0\}$ and $E, H \in \bar{\mathrm{F}}(D)$, $(xE)^\star = xE^\star$; $E \subset E^\star$; $(E^\star)^\star = E^\star$; and $E \subset H$ implies $E^\star \subset H^\star$. The set of semistar operations on D is denoted by $\mathrm{SStar}(D)$. Let T be an overring of D, that is, $T \subset K$. If we set $H^{\alpha(\star)} = H^\star$ for every $H \in \bar{\mathrm{F}}(T)$, $\alpha(\star)$ is a semistar operation on T, which is called the ascent of $\star$ to T. Let $\star'$ be a semistar operation on T. If we set $E^{\delta(\star')} = (ET)^{\star'}$ for every $E \in \bar{\mathrm{F}}(D)$, $\delta(\star')$ is a semistar operation on D, which is called the descent of $\star'$ to D. In this paper, for any extension domain T of D, we define ascents and descents of semistar operations and localizing systems, and study their basic properties.

Let $\mathcal{F}$ be a non-empty set of ideals of D with $\mathcal{F} \not\ni (0)$ which satisfies the following two conditions for every ideals I, J of D: If $I \in \mathcal{F}$ and $I \subset J$, then $J \in F$; If $I \in \mathcal{F}$ and $J :_D iD \in \mathcal{F}$ for every $i \in I$, then $J \in \mathcal{F}$. Then $\mathcal{F}$ is called a localizing system of D (P. Gabriel [3]). The set of localizing systems of D is denoted by $\mathrm{LS}(D)$. We refer to M. Fontana and J. Huckaba ([2]) for the following notions and terminologies. Thus, let $\mathrm{F}(D)$ be the set of non-zero submodules G of K such that $dG \subset D$ for some $d \in D \setminus \{0\}$, and let $\mathrm{f}(D)$ be the set of elements of $\mathrm{F}(D)$ which is finitely generated over D. Let $\star$ be a semistar operation on D. Then $\mathcal{F}^\star = \{I \mid I$ is a non-zero ideal of D with $I^\star \ni 1\}$ is a localizing system of D. Let $\mathcal{F}$ be a localizing system of D. Then the mapping $E \longmapsto E^{\star_\mathcal{F}} = \cup\{(E :_K I) \mid I \in \mathcal{F}\}$ is a semistar operation on D. The semistar operation $E \longmapsto \cup\{F^\star \mid F \in \mathrm{f}(D)$ with $F \subset E\}$ is denoted by $\star_f$. If $\star = \star_f$, $\star$ is called finite type. A localizing system $\mathcal{F}$ is called finite type if, for every $I \in \mathcal{F}$, $\mathcal{F}$ contains a finitely generated ideal J of D such that $J \subset I$.

Let T be the polynomial ring $D[X]$ of X over D. For every semistar operation $\star$ on T, E. Houston, S. Malik and J. Mott [4] and A. Okabe and R. Matsuda [5] defined the semistar operation $E \longmapsto (ET)^\star \cap K$ on D, which we denote by $\delta(\star)$. For every localizing system $\mathcal{F}$ of D, G. Picozza [8] defined the localizing system $\{J \mid J$ is an ideal of T with $J \supset I$ for some $I \in \mathcal{F}\}$ of T, which we denote by $\alpha(\mathcal{F})$. For every localizing system $\mathcal{F}$ of T, A. Okabe [6] defined the localizing system $\{I \mid I$ is an ideal of D such that $IT \in \mathcal{F}\}$ of D, which we denote by $\delta(\mathcal{F})$. In this paper, for *any* extension domain T of D, we define ascents and descents of semistar operations and localizing systems, and study their basic properties. This paper consists of three sections. Section 1 is an introduction, Section 2 is definitions of ascents and descents, and Section 3 is basic properties of ascents and descents.

2. Definitions of Ascents and Descents

Let D be a domain with $\mathrm{q}(D) = K$, and let T be *any* extension domain with $\mathrm{q}(T) = L$. In this section, we give definitions of ascents and descents of semistar operations and localizing systems.

Proposition 2.1. (cf., [4, Proposition 2.1] and [5, Proposition 35]) *Let $\star$ be a semistar operation on T. For every $E \in \bar{\mathrm{F}}(D)$, set $E^{\delta(\star)} = (ET)^\star \cap K$. Then $\delta(\star)$ is a semistar operation on D.*

Proof. The only condition which is not trivial is $\delta(\star)\delta(\star) = \delta(\star)$. For every $E \in \bar{\mathrm{F}}(D)$, we have the following: $(E^{\delta(\star)})^{\delta(\star)} = (((ET)^\star \cap K)T)^\star \cap K \subset ((ET)^\star T)^\star \cap K = ((ET)T)^\star \cap K = (ET)^\star \cap K = E^{\delta(\star)}$. □

$\delta(\star)$ is called the descent of $\star$ to D, and is also denoted by $\delta_{T/D}(\star)$.

Remark 2.2. Let T be an overring of D, and let $\star$ be a semistar operation on T. Then, for every $E \in \bar{\mathrm{F}}(D)$, we have $E^{\delta(\star)} = (ET)^\star$.

Proposition 2.3. (cf., [8, Proposition 3.1]) *Let $\mathcal{F}$ be a localizing system of D. Set $\alpha(\mathcal{F}) = \{J \mid J$ is an ideal of T with $J \supset I$ for some $I \in \mathcal{F}\}$.*

(1) $\alpha(\mathcal{F})$ is a localizing system of T.

(2) $\alpha(\mathcal{F}) = \{J \mid J$ is an ideal of T with $J \cap D \in \mathcal{F}\}$.

Proof. The only condition which needs a proof is: If J' is a non-zero ideal of T, and if $J \in \alpha(\mathcal{F})$ such that $(J' :_T j) \in \alpha(\mathcal{F})$ for every $j \in J$, then $J' \in \alpha(\mathcal{F})$.

Since $J \in \alpha(\mathcal{F})$, we have $J \cap D \in \mathcal{F}$. Let $j_0 \in J \cap D$. Since $(J' :_T j_0) \in \alpha(\mathcal{F})$, we have $(J' :_D j_0) \in \mathcal{F}$, and $((J' \cap D) :_D j_0) \in \mathcal{F}$. Therefore $J' \cap D \in \mathcal{F}$. It follows that $J' \in \alpha(\mathcal{F})$. □

$\alpha(\mathcal{F})$ is called the ascent of $\mathcal{F}$ to T, and is also denoted by $\alpha_{T/D}(\mathcal{F})$.

Proposition 2.4. *For every localizing system $\mathcal{F}$ of T, set $\delta(\mathcal{F}) := \mathcal{F}^{\delta(\star_{\mathcal{F}})}$. Then*

$$\delta(\mathcal{F}) = \{I \mid I \text{ is an ideal of } D \text{ with } IT \in \mathcal{F}\}.$$

Proof. Let $I \in \mathcal{F}^{\delta(\star_{\mathcal{F}})}$. Then $I^{\delta(\star_{\mathcal{F}})} \ni 1$, and hence $(IT)^{\star_{\mathcal{F}}} \ni 1$. There is $J \in \mathcal{F}$ such that $J \subset IT$, hence $IT \in \mathcal{F}$. The reverse inclusion is similar. □

$\delta(\mathcal{F})$ is called the descent of $\mathcal{F}$ to D, and is also denoted by $\delta_{T/D}(\mathcal{F})$.

Remark 2.5. (cf., [6, Lemma 32]) Let $T = D[X]$. Then

$$\delta(\mathcal{F}) = \{J \cap D \mid J \text{ is an ideal of } T \text{ with } (J \cap D)T \in \mathcal{F}\}.$$

Let $\star_1, \star_2$ be semistar operations on D with $E^{\star_1} \subset E^{\star_2}$ for every $E \in \bar{\mathrm{F}}(D)$. Then we denote $\star_1 \leq \star_2$.

Lemma 2.6. *Let $\star$ be a semistar operation on D. Then there is an extension domain T which satisfies the following two conditions:*

(1) There is a semistar operation $\star'$ on T such that $\delta(\star') \geq \star$.

(2) Every semistar operation $\star'$ on T satisfies $\delta(\star') \geq \star$.

Proof. Set $T := K$. Clearly, T satisfies the conditions (1) and (2). □

The mapping $E \longmapsto E$ from $\bar{\mathrm{F}}(D)$ to $\bar{\mathrm{F}}(D)$ is a semistar operation on D which is calld the d-semistar operation, and is denoted by d_D or by d. Similarly, we may define the e-semistar operation e_D on D: $E^{e_D} = K$ for every $E \in \bar{\mathrm{F}}(D)$. The localizing system $\mathcal{F}^{e_D} = \{I \mid I \text{ is a non-zero ideal of } D\}$ of D is called the trivial localizing system of D.

Proposition 2.7. *Let $\star$ be a semistar operation on D. Then there is a semistar operation $\star'$ on T such that $\delta(\star') \geq \star$. Let $\{\star_\lambda \mid \lambda \in \Lambda\}$ be the set of semistar operations $\star'$ on T such that $\delta(\star') \geq \star$. Then the mapping $\bar{\mathrm{F}}(T) \longrightarrow \bar{\mathrm{F}}(T)$, $H \longmapsto \cap_\lambda H^{\star_\lambda}$ is a semistar operation on T.*

Proof. Let e_T be the e-semistar operation on T. Then $\delta(e_T) \geq \star$. That the mapping $H \longmapsto \cap_\lambda H^{\star_\lambda}$ is a semistar operation on T follows from D.D. Anderson and D.F. Anderson [1, Lemma 1]. □

The semistar operation $H \longmapsto \cap_\lambda H^{\star_\lambda}$ on T is called the ascent of $\star$, and is denoted by $\alpha(\star)$, or by $\alpha_{T/D}(\star)$.

Proposition 2.8. *Let T be an overring of D. Let $\star$ be a semistar operation on D. Then, for every $H \in \bar{\mathrm{F}}(T)$, we have $H^{\alpha(\star)} = H^\star$.*

Proof. If we set $H^{\star_{\lambda_0}} = H^\star$ for every $H \in \bar{\mathrm{F}}(T)$, then $\star_{\lambda_0}$ is a semistar operation on T with $\delta(\star_{\lambda_0}) \geq \star$. Let $\{\star_\lambda \mid \lambda \in \Lambda\}$ be the set of semistar operations $\star'$ on T with $\delta(\star') \geq \star$. Then $(ET)^{\star_\lambda} \supset E^\star$ for every $E \in \bar{\mathrm{F}}(D)$. Since $\bar{\mathrm{F}}(T) \subset \bar{\mathrm{F}}(D)$, we have $H^{\star_\lambda} = (HT)^{\star_\lambda} \supset H^\star = H^{\star_{\lambda_0}}$ for every $H \in \bar{\mathrm{F}}(T)$. Hence $\alpha(\star) = \star_{\lambda_0}$. $\square$

Example 2.9. (1) $\alpha(d_D) = d_T$.

(2) If T is an overring of D, then $\alpha(e_D) = e_T$.

(3) If $T = D[X]$, then $\alpha(e_D) : H \longmapsto HK$.

(4) Let T be an overring of D. If $\mathcal{F}$ is the trivial localizing system of D, then $\alpha(\mathcal{F})$ is the trivial localizing system of T.

(5) If T is an overring of D, then $E^{\delta(d_T)} = ET$ for every $E \in \bar{\mathrm{F}}(D)$.

(6) If $T = D[X]$, then $\delta(d_T) = d_D$.

(7) If $\mathcal{F}$ is the trivial localizing system of T, then $\delta(\mathcal{F})$ is the trivial localizing system of D.

Proof. The proofs for (1), (2), (4), (5), (6), (7) are immediate.

(3) Set $H^\star = HK$ for every $H \in \bar{\mathrm{F}}(T)$. Then $\star$ is a semistar operation on T. Easily, we have $\delta(\star) \geq e_D$. Let $\star'$ be a semistar operation on T with $\delta(\star') \geq e_D$. If $\star' \geq \star$, we may conclude that $\alpha(e_D) = \star$. Since $D^{\delta(\star')} \supset D^{e_D}$, we have $T^{\star'} \supset K$. For every $H \in \bar{\mathrm{F}}(T)$, we have $H^{\star'} = (HT)^{\star'} = (HT^{\star'})^{\star'} \supset (HK)^{\star'} \supset HK = H^\star$. That is, $\star' \geq \star$. $\square$

3. Basic Properties of Ascents and Descents

In this section, we study basic properties of ascents and descents of semistar operations and localizing systems.

Proposition 3.1. *(1) Let $\star_1, \star_2$ be semistar operations on D with $\star_1 \leq \star_2$. Then $\alpha(\star_1) \leq \alpha(\star_2)$.*

(2) Let $\mathcal{F}_1, \mathcal{F}_2$ be localizing systems of T with $\mathcal{F}_1 \subset \mathcal{F}_2$. Then $\delta(\mathcal{F}_1) \subset \delta(\mathcal{F}_2)$.

(3) (cf., [6, Proposition 23 (5)]) *Let $\star_1, \star_2$ be semistar operations on T with $\star_1 \leq \star_2$. Then $\delta(\star_1) \leq \delta(\star_2)$.*

(4) Let $\mathcal{F}_1, \mathcal{F}_2$ be localizing systems of D with $\mathcal{F}_1 \subset \mathcal{F}_2$. Then $\alpha(\mathcal{F}_1) \subset \alpha(\mathcal{F}_2)$.

Proof. (1) Let $\{\star' \in \mathrm{SStar}(T) \mid \delta(\star') \geq \star_1\} = \{\star_\lambda \mid \lambda \in \Lambda\}$ and $\{\star' \in \mathrm{SStar}(T) \mid \delta(\star') \geq \star_2\} = \{\star_\sigma \mid \sigma \in \Sigma\}$. Then $\{\star_\lambda \mid \lambda\} \supset \{\star_\sigma \mid \sigma\}$. By the definition of $\alpha(\star_1)$ and $\alpha(\star_2)$, we have $\alpha(\star_2) \geq \alpha(\star_1)$.

(2) If $I \in \delta(\mathcal{F}_1)$, then $IT \in \mathcal{F}_1$, hence $IT \in \mathcal{F}_2$, and hence $I \in \delta(\mathcal{F}_2)$.

(3) For every $E \in \bar{\mathrm{F}}(D)$, we have $E^{\delta(\star_1)} = (ET)^{\star_1} \cap K \subset (ET)^{\star_2} \cap K = E^{\delta(\star_2)}$.

(4) If $J \in \alpha(\mathcal{F}_1)$, then $J \supset I$ for some $I \in \mathcal{F}_1$. Since $I \in \mathcal{F}_2$, we have $J \in \alpha(\mathcal{F}_2)$. □

Proposition 3.2. *(1) For every localizing system $\mathcal{F}$ of T, we have $\alpha(\delta(\mathcal{F})) \subset \mathcal{F}$.*

(2) For every localizing system $\mathcal{F}$ of D, we have $\delta(\alpha(\mathcal{F})) \supset \mathcal{F}$.

(3) For every semistar operation $\star$ on D, we have $\delta(\alpha(\star)) \geq \star$.

(4) For every semistar operation $\star$ on T, we have $\alpha(\delta(\star)) \leq \star$.

Proof. (1) Let $J \in \alpha(\delta(\mathcal{F}))$. Then $J \cap D \in \delta(\mathcal{F})$, hence $(J \cap D)T \in \mathcal{F}$. Since $J \supset (J \cap D)T$, we have $J \in \mathcal{F}$. Hence $\alpha(\delta(\mathcal{F})) \subset \mathcal{F}$.

(2) Let $I \in \mathcal{F}$. Since $I \subset (IT) \cap D$, we have $(IT) \cap D \in \mathcal{F}$. Hence $IT \in \alpha(\mathcal{F})$, and hence $I \in \delta(\alpha(\mathcal{F}))$. Therefore $\mathcal{F} \subset \delta(\alpha(\mathcal{F}))$.

(3) Let $\{\star_\lambda \mid \lambda \in \Lambda\}$ be the set of semistar operations $\star'$ on T such that $\delta(\star') \geq \star$. For every $E \in \bar{\mathrm{F}}(D)$, we have $E^{\delta(\alpha(\star))} = (ET)^{\alpha(\star)} \cap K = \cap_\lambda (ET)^{\star_\lambda} \cap K = \cap_\lambda E^{\delta(\star_\lambda)} \supset E^\star$. Hence $\delta(\alpha(\star)) \geq \star$.

(4) Set $\{\star' \in \mathrm{SStar}(T) \mid \delta(\star') \geq \delta(\star)\} = \{\star_\lambda \mid \lambda \in \Lambda\}$. Then $\star = \star_{\lambda_0}$ for some λ_0. Then, for every $H \in \bar{\mathrm{F}}(T)$, $H^{\alpha(\delta(\star))} = \cap_\lambda H^{\star_\lambda} \subset H^{\star_{\lambda_0}} = H^\star$. Hence $\alpha(\delta(\star)) \leq \star$. □

Proposition 3.3. *Let $D \subset T \subset R$ be domains.*

(1) For every semistar operation $\star$ on R, $(\delta_{T/D}\delta_{R/T})(\star) = \delta_{R/D}(\star)$.

(2) For every localizing system $\mathcal{F}$ on R, $(\delta_{T/D}\delta_{R/T})(\mathcal{F}) = \delta_{R/D}(\mathcal{F})$.

(3) For every semistar operation $\star$ on D, $(\alpha_{R/T}\alpha_{T/D})(\star) = \alpha_{R/D}(\star)$.

(4) For every localizing system $\mathcal{F}$ of D, $(\alpha_{R/T}\alpha_{T/D})(\mathcal{F}) = \alpha_{R/D}(\mathcal{F})$.

Proof. (3) Set $\star_1 = \alpha_{T/D}(\star)$, $\star_2 = \alpha_{R/T}(\star_1)$, and set $\star_3 = \alpha_{R/D}(\star)$. By Proposition 3.2 (3), we have $\delta_{R/T}(\star_2) \geq \star_1$ and $\delta_{T/D}(\star_1) \geq \star$. Then $\delta_{T/D}\delta_{R/T}(\star_2) \geq \delta_{T/D}(\star_1) \geq \star$ by Proposition 3.1(3). Hence $\delta_{R/D}(\star_2) \geq \star$ by Proposition 3.3 (1). Then $\star_2 \geq \star_3$ by the definition of $\star_3$. Similarly, we have $\star_3 \geq \star_2$, and hence $\star_2 = \star_3$.

The proofs of (1), (2) and (4) are straightforward. □

Proposition 3.4. *We have*

(1) (i) $\delta_{T/D}\alpha_{T/D}\delta_{T/D}(\star) = \delta_{T/D}(\star)$ for every $\star \in$ SStar(T).

(ii) $\alpha_{T/D}\delta_{T/D}\alpha_{T/D}(\star) = \alpha_{T/D}(\star)$ for every $\star \in$ SStar(D).

(2) (i) $\delta_{T/D}\alpha_{T/D}\delta_{T/D}(\mathcal{F}) = \delta_{T/D}(\mathcal{F})$ *for every* $\mathcal{F} \in LS(T)$.

(ii) $\alpha_{T/D}\delta_{T/D}\alpha_{T/D}(\mathcal{F}) = \alpha_{T/D}(\mathcal{F})$ *for every* $\mathcal{F} \in LS(D)$.

Proof. (1) (i) $\star \geq \alpha\delta(\star)$ by Proposition 3.2 (4). $\delta(\star) \geq \delta\alpha\delta(\star)$ by Proposition 3.1 (3). $\delta\alpha\delta(\star) \geq \delta(\star)$ by Proposition 3.2 (3). Hence $\delta\alpha\delta = \delta$. The proof of (ii) is similar.

The proof for (2) is similar. □

Proposition 3.5. *(1)* $SStar(T) \xrightarrow{\delta} SStar(D)$ *is an injection if and only if* $\alpha\delta = I$*, where* I *denotes the identity mapping. And then* $SStar(D) \xrightarrow{\alpha} SStar(T)$ *is a surjection.*

(2) $LS(T) \xrightarrow{\delta} LS(D)$ *is an injection if and only if* $\alpha\delta = I$*. And then* $LS(D) \xrightarrow{\alpha} LS(T)$ *is a surjection.*

(3) $SStar(D) \xrightarrow{\alpha} SStar(T)$ *is an injection if and only if* $\delta\alpha = I$*. And then* $SStar(T) \xrightarrow{\delta} SStar(D)$ *is a surjection.*

(4) $LS(D) \xrightarrow{\alpha} LS(T)$ *is an injection if and only if* $\delta\alpha = I$*. And then* $LS(T) \xrightarrow{\delta} LS(D)$ *is a surjection.*

Proof. (1) Assume that δ is an injection. We have $\delta\alpha\delta(\star) = \delta(\star)$ for every $\star \in \mathrm{SStar}(T)$ by Proposition 3.4 (1)(i). Hence $\alpha\delta(\star) = \star$. Therefore $\alpha\delta = I$ and α is a surjection.

Assume that $\alpha\delta = I$. Let $\delta(\star_1) = \delta(\star_2)$ for $\star_1, \star_2 \in \mathrm{SStar}(T)$. Then $\alpha\delta(\star_1) = \alpha\delta(\star_2)$, hence $\star_1 = \star_2$.

The proofs for (2), (3) and (4) are similar. □

Proposition 3.6. *(1)* (cf., [6, Proposition 35(2)]) *Let* $\mathcal{F}$ *be a localizing system of* T*. Then* $\alpha(\delta(\mathcal{F})) = \mathcal{F} \iff \mathcal{F} \in \alpha(LS(D)) \iff (J \cap D)T \in \mathcal{F}$ *for every* $J \in \mathcal{F}$.

(2) Let $\star$ *be a semistar operation on* T*. Then* $\alpha(\delta(\star)) = \star \iff \star \in \alpha(SStar(D))$ $\iff$ *For every* $\star' \in \alpha$ $(SStar(T))$ *with* $\delta(\star') \geq \delta(\star)$, $H^{\star'} \supset H^{\star}$ *for every* $H \in \bar{\mathrm{F}}(T)$.

(3) Let $\mathcal{F}$ *be a localizing system of* D*. Then* $\delta(\alpha(\mathcal{F})) = \mathcal{F} \iff \mathcal{F} \in \delta(LS(T))$ $\iff \{I \mid I$ *is an ideal of* D *with* $IT \cap D \in \mathcal{F}\} = \mathcal{F}$.

(4) Let $\star$ *be a semistar operation on* D*. Then* $\delta(\alpha(\star)) = \star \iff \star \in \delta(SStar(T))$.

Proof. (1) Let $\mathcal{F} = \alpha(\mathcal{F}_0)$ for some $\mathcal{F}_0 \in \mathrm{LS}(D)$. Then $\alpha\delta(\mathcal{F}) = \alpha\delta\alpha(\mathcal{F}_0) = \alpha(\mathcal{F}_0) = \mathcal{F}$. Let $J \in \mathcal{F}$. Since $J \in \alpha(\delta(\mathcal{F}))$, we have $J \cap D \in \delta(\mathcal{F})$. Hence $(J \cap D)T \in \mathcal{F}$. The reverse implication is similar.

(2) Let $\star = \alpha(\star_0)$ for some $\star_0 \in \mathrm{SStar}(D)$. Then $\alpha\delta(\star) = \alpha\delta\alpha(\star_0) = \star$. The definition of α completes the proof.

(3) We have

$\delta(\alpha(\mathcal{F})) = \{I \mid I$ is an ideal of D with $IT \in \alpha(\mathcal{F})\} = \{I \mid I$ is an ideal of D with $IT \cap D \in \mathcal{F}\}$.

(4) If $\star = \delta(\star_1)$ for some $\star_1 \in \mathrm{SStar}(T)$, then $\delta\alpha(\star) = \delta\alpha\delta(\star_1) = \delta(\star_1) = \star$. □

If T is a D-free module with free basis $\ni 1$, T is called free over D. Let X be a torsion-free abelian additive group. A subsemigroup $S \supsetneqq \{0\}$ of X is called a g-monoid.

An overring T of D is not free unless $T = D$.

Example 3.7. (1) $D[X]$ is free over D.

(2) Let S be a g-monoid. Then the semigroup ring $D[X; S]$ of S over D is free over D.

(3) Let K' be an extension field of K, let B be an algebraically independent subset of K' over K, and let $T = D[B]$. Then T is free over D.

Proposition 3.8. *(1)* (cf., [6, Proposition 34]) *If T is free, then $\delta(\alpha(\mathcal{F})) = \mathcal{F}$ for every $\mathcal{F} \in LS(D)$. Hence $LS(D) \xrightarrow{\alpha} LS(T)$ is an injection, and $LS(T) \xrightarrow{\delta} LS(D)$ is a surjection.*

(2) Assume that T is an overring of D. Then $\alpha(\delta(\star)) = \star$ for every $\star \in SStar(T)$. Hence $SStar(T) \xrightarrow{\delta} SStar(D)$ is an injection, and $SStar(D) \xrightarrow{\alpha} SStar(T)$ is a surjection.

Proof. (1) Let $I \in \delta(\alpha(\mathcal{F}))$. Then $IT \supset I_0$ for some $I_0 \in \mathcal{F}$. Since T is free over D, we have $I \supset I_0$, hence $I \in \mathcal{F}$. Proposition 3.5 (4) completes the proof.

(2) For every $H \in \bar{\mathrm{F}}(T)$, we have $H^{\alpha(\delta(\star))} = H^{\delta(\star)} = (HT)^\star = H^\star$ by Proposition 2.8. Proposition 3.5 (1) completes the proof. □

Proposition 3.9. *(1) Let $\mathcal{F}$ be a localizing system of T. Then $\star_{\delta(\mathcal{F})} \leq \delta(\star_{\mathcal{F}})$.*

(2) Let $\star$ be a semistar operation on T. Then $\mathcal{F}^{\delta(\star)} = \delta(\mathcal{F}^\star)$.

(3) Let $\star$ be a semistar operation on D. Then $\mathcal{F}^{\alpha(\star)} \supset \alpha(\mathcal{F}^\star)$.

Proof. (1) Let $E \in \bar{\mathrm{F}}(D)$. Then we have

$E^{\star_{\delta(\mathcal{F})}} = \{x \in K \mid$ There is an ideal I of D with $IT \in \mathcal{F}$ such that $xI \subset E\}$,

$E^{\delta(\star_\mathcal{F})} = \{x \in K \mid$ There is $J \in \mathcal{F}$ such that $xJ \subset ET\}$.

(2) We have

$\mathcal{F}^{\delta(\star)} = \{I \mid I$ is an ideal of D with $I^{\delta(\star)} \ni 1\} = \{I \mid I$ is an ideal of D with $(IT)^\star \ni 1\}$,

$\delta(\mathcal{F}^\star) = \{I \mid I$ is an ideal of D with $IT \in \mathcal{F}^\star\} = \{I \mid I$ is an ideal of D with $(IT)^\star \ni 1\}$.

(3) Let $\{\star' \in \mathrm{SStar}(T) \mid \delta(\star') \geq \star\} = \{\star_\lambda \mid \lambda \in \Lambda\}$. Then we have

$\mathcal{F}^{\alpha(\star)} = \{J \mid J$ is an ideal of T with $J^{\alpha(\star)} \ni 1\} = \{J \mid J$ is an ideal of T with $J^{\star_\lambda} \ni 1$ for every $\lambda\}$,

$\alpha(\mathcal{F}^\star) = \{J \mid J$ is an ideal of T with $J \supset I$ for some $I \in \mathcal{F}^\star\} = \{J \mid$ There is an ideal I of D with $I^\star \ni 1$ such that $J \supset I\}$. □

Proposition 3.10. *Let T be an overring of D. Let $\mathcal{F}$ be a localizing system of D. Then $\star_{\alpha(\mathcal{F})} = \alpha(\star_\mathcal{F})$.*

Proof. We use Proposition 2.8. For every $H \in \bar{\mathrm{F}}(T)$, we have

$H^{\star_{\alpha(\mathcal{F})}} = H^{\alpha(\star_\mathcal{F})} = \{x \in K \mid xI \subset H$ for some $I \in \mathcal{F}\}$. □

Proposition 3.11. *(1) For every $\star \in$ SStar(T), we have $\delta(\star)_f = \delta(\star_f)$.*

(2) For every $\mathcal{F} \in$ LS(D), we have $\alpha(\mathcal{F})_f \supset \alpha(\mathcal{F}_f)$.

(3) For every $\mathcal{F} \in$ LS(T), we have $\delta(\mathcal{F})_f \subset \delta(\mathcal{F}_f)$.

Proof. (1) For every $E \in \bar{\mathrm{F}}(D)$, we have

$E^{\delta(\star)_f} = \cup\{(FT)^\star \cap K \mid F \in \mathrm{f}(D)$ with $F \subset E\}$,

$E^{\delta(\star_f)} = \cup\{H^\star \mid H \in \mathrm{f}(T)$ with $H \subset ET\} \cap K$.

It follows that $E^{\delta(\star)_f} \subset E^{\delta(\star_f)}$. Conversely, let $0 \neq x \in E^{\delta(\star_f)}$. Then, there is $H \in \mathrm{f}(T)$ with $H \subset ET$ such that $x \in H^\star$. We have $H \subset (e_1, \cdots, e_n)T$ for some $e_1, \cdots, e_n \in E$. Set $F := (e_1, \cdots, e_n)D$. Then $x \in H^\star \cap K \subset (FT)^\star \cap K \subset E^{\delta(\star)_f}$, and hence $E^{\delta(\star_f)} \subset E^{\delta(\star)_f}$.

(2) We have

$\alpha(\mathcal{F})_f = \{J \mid$ There is $I \in \mathcal{F}$ and a finitely generated ideal J_1 of T with $J \supset J_1 \supset I\}$,

$\alpha(\mathcal{F}_f) = \{J \mid$ There is a finitely generated ideal $I \in \mathcal{F}$ with $J \supset I\}$.

(3) We have

$\delta(\mathcal{F})_f = \{I \mid$ There is a finitely generated ideal I_1 of D with $I_1T \in \mathcal{F}$ such that $I \supset I_1\}$,

$\delta(\mathcal{F}_f) = \{I \mid$ There is a finitely generated ideal $J \in \mathcal{F}$ such that $IT \supset J\}$. □

Proposition 3.12. *(1)* ([7, Proposition 3.2 (1)]) *If T is an overring of D, then $\delta(\star)_f = \delta(\star_f)$.*

(2) (cf., [8, Proposition 3.2]) *If $\mathcal{F}$ is of finite type on D, then $\alpha(\mathcal{F})$ is of finite type on T.*

(3) (cf., [6, Proposition 33]) *Let T be free over D. Then, for every $\mathcal{F} \in$ LS(T), we have $\delta(\mathcal{F}_f) = \delta(\mathcal{F})_f$.*

Proof. (2) Let $J \in \alpha(\mathcal{F})$. There is $I \in \mathcal{F}$ such that $I \subset J$. There is a finitely generated ideal $I_0 \in \mathcal{F}$ such that $I_0 \subset I$. Then I_0T is a finitely generated ideal of T with $I_0T \in \alpha(\mathcal{F})$ such that $I_0T \subset J$.

(3) Let $I \in \delta(\mathcal{F}_f)$. There is a finitely generated ideal $J = (t_1, \cdots, t_n)T \in \mathcal{F}$ such that $J \subset IT$. For every i, set $t_i = \sum x_{i\lambda} u_\lambda$, where $\{u_\lambda \mid \lambda \in \Lambda\}$ is a free basis of T over D and every $x_{i\lambda} \in D$. Set $(\cdots, x_{i\lambda}, \cdots)D = I_0$. Since $x_{i\lambda} \in I$ for every i, λ, we have $I \supset I_0$, and hence $J \subset I_0 T \in \mathcal{F}$. Proposition 3.11(3) completes the proof. □

References

[1] D.D. Anderson and D.F. Anderson, *Examples of star operations on integral domains*, Comm. Algebra, 18 (1990), 1621-1643.

[2] M. Fontana and J. Huckaba, Localizing Systems and Semistar Operations, Non Noetherian Commutative Ring Theory, Dordrecht, Kluwer Academic Publishers, 2000, 169-198.

[3] P. Gabriel, *Des Categories abeliennes*, Bull. Soc. Math. France 90 (1962), 323-448.

[4] E. Houston, S. Malik and J. Mott, *Characterizations of ⋆-multiplication domains*, Canad. Math. Bull. 27 (1984), 48-52.

[5] A. Okabe and R. Matsuda, *Semistar-operations on integral domains*, Math. J. Toyama Univ., 17 (1994), 1-21.

[6] A. Okabe, *On polynomial ascent and descent semistar operations on an integral domain*, Math. J. Ibaraki Univ., to appear.

[7] G. Picozza, *Star operations on overrings and semistar operations*, Comm. Algebra, 33 (2005), 2051-2073.

[8] G. Picozza, *A note on semistar Noetherian domains*, Houston J. Math., 33 (2007), 415-432.

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